Handling Non-Convexities: 
Entrepreneurship and Financial Frictions

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1 Model Setup

This note describes a model of entrepreneurship and financial frictions, similar to that in Buera and Shin (2013) and Cagetti and De Nardi (2006). In particular, occupational choice in combination with financial friction introduces a non-convexity in individuals optimization problems. We also introduce a second additional non-convexity, namely a choice between operating two technologies. Achdou et al. (2014) consider a version with aggregate shocks and examine its business cycle implications (here we focus on the model with idiosyncratic shocks only).

- Individuals’ preferences
  \[ \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \]
- Workers earn labor income \( wz^\theta \) where \( \theta \geq 0 \)
- Entrepreneurs choose technology, maximize profits
- Two technologies productive (\( p \)) and unproductive (\( u \))

\[ y_u = F_u(z, k, \ell) = zB_u k^\alpha \ell^\beta \]
\[ y_p = F_p(z, k, \ell) = zB_p((k - f_k)^+)((\ell - f_\ell)^+)^\beta \]

- \( B_p > B_u \), but per-period overhead cost \( f_k, f_\ell \)
- Notation: for any scalar \( x \), \( x^+ = \max\{x, 0\} \)
- \( F_p \) non-concave in \( k \) and \( \ell \)
- \( z \): idiosyncratic shock, diffusion process

- Collateral constraints

\[ k \leq \lambda a, \quad \lambda \geq 1. \]
• Income maximization: occupation and technology choice

\[ M(a, z, A; w, r) = \max \{ wz^\theta, \Pi_u(a, z, A; w, r), \Pi_p(a, z, A; w, r) \} \]

\[ \Pi_j(a, z, A; w, r) = \max_{k \leq \lambda_a} F_j(z, A, k, \ell) - (r + \delta)k - w\ell, \quad j = p, u. \]

• Individuals solve

\[
\max_{\{c_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.} \\
da_t = [M(a_t, z_t, A_t; w_t, r_t) + r_t a_t - c_t] dt \\
dz_t = \mu(z_t) dt + \sigma(z_t) dW_t
\]

\[ a_t \geq 0 \]

• Optimal capital and labor choices corresponding to the productive technology

\[ k_p(a, z; w, r) = \min \left\{ \lambda a, (zAB_p)^{1 \over 1-\alpha - \beta} \left( {\alpha \over \rho + \delta} \right)^{1-\beta \over 1-\alpha - \beta} \left( {\beta \over w} \right)^{\beta \over 1-\alpha - \beta} + f_k \right\} \]

\[ \ell_p(a, z; w, r) = \left( {\beta zAB_p \over w} \right)^{1 \over \beta - \alpha} k_p(a, z; w, r)^{\alpha \over 1-\beta} + f_\ell \]

and a similar expression for optimal capital and labor choices corresponding to the unproductive technology.

• Representative firm

\[ Y_c = F_c(A, K_c, L_c) = AB_cK_c^nL_c^{1-n}, \]

2 Equilibrium Conditions

Individual optimization and evolution of distribution

\[
\rho v(a, z, t) = \max_c u(c) + \partial_a v(a, z, t)[M(a, z; w(t), r(t)) + r(t)a - c] \\
+ \partial_z v(a, z, t)\mu(z) + 1 \over 2 \partial_{zz} v(a, z, t)\sigma^2(z) + \partial_t v(a, z, t)
\]

(1)

\[
\partial_t g(a, z, t) = - \partial_a [s(a, z, t)g(a, z, t)] - \partial_z [\mu(z)g(a, z, t)] + 1 \over 2 \partial_{zz} [\sigma^2(z)g(a, z, t)],
\]

(2)

\[
s(a, z, t) = M(a, z; w(t), r(t)) + r(t)a - c(a, z, t)
\]

(3)
Public firms

\[ r(t) = \partial_K F_c(A, K_c(t), L_c(t)) - \delta, \quad w(t) = \partial_L F_c(A, K_c(t), L_c(t)) \]  \hspace{1cm} (4)

Capital market clearing:

\[ K_c(t) + \int k_u(a, z; w(t), r(t)) 1_{\{\Pi_u > \max\{\Pi_p, wz^\theta\}\}} g(a, z, t) daz \]
\[ + \int k_p(a, z; w(t), r(t)) 1_{\{\Pi_p > \max\{\Pi_u, wz^\theta\}\}} g(a, z, t) daz \]
\[ = \int ag(a, z, t) daz \]  \hspace{1cm} (5)

Labor market clearing:

\[ L_c(t) + \int \ell_u(a, z; w(t), r(t)) 1_{\{\Pi_u > \max\{\Pi_p, wz^\theta\}\}} g(a, z, t) daz \]
\[ + \int \ell_p(a, z; w(t), r(t)) 1_{\{\Pi_p > \max\{\Pi_u, wz^\theta\}\}} g(a, z, t) daz \]
\[ = \int z^\theta 1_{\{wz^\theta > \max\{\Pi_u, \Pi_p\}\}} g(a, z, t) daz \]  \hspace{1cm} (6)

Given initial condition \( g_0(a, z) \), the two PDEs (1), (2) together with (3) and the equilibrium conditions (4), (5) and (6) fully characterize equilibrium.

3 Numerical Solution

The algorithm for solving the HJB equation (1) and the Kolmogorov Forward equation (2) are nearly identical to that used for solving the Huggett model with a diffusion process described in section 5 here [link]. The algorithm is then a simple bisection algorithm on the equilibrium interest rate.

3.1 Algorithm for Steady State

Use a bisection algorithm for \( r_{\min} \leq r \leq r_{\max} \). Given an initial guesses \( r_0 \) for \( \ell = 0, 1, 2, \ldots \) follow

1. given \( r_\ell \), find \( \xi_\ell = K_c/L_c \) from (4)

2. given \( \xi_\ell \) find \( w \) from (4)
3. given \( r_\ell \) and \( w_\ell \), solve the HJB equation

4. find \( K_{c,\ell} \) from (5)

5. Compute \( L_{c,\ell} = K_{c,\ell}/\xi_\ell \) and compute “excess labor demand”

\[
D_\ell = L_c(t) + \int \ell_u(a, z; w(t), r(t))1_{\{\Pi_u > \max\{\Pi_p, wz^\theta\}\}}g(a, z, t)dadz \\
+ \int \ell_p(a, z; w(t), r(t))1_{\{\Pi_p > \max\{\Pi_u, wz^\theta\}\}}g(a, z, t)dadz - \int z^\theta 1_{\{wz^\theta > \max\{\Pi_u, \Pi_p\}\}}g(a, z, t)dadz
\]

6. Update \( r_\ell \): if \( D_\ell > 0 \), choose \( r_{\ell+1} < r_\ell \) and vice versa.

4 Results

Figure 1 plots the saving and consumption policy functions. The policy functions can be non-monotonic. Figure 2 plots the wealth distribution. The wealth distribution has a fat right tail as in Cagetti and De Nardi (2006).

Figure 1: Saving and Consumption Policy Functions

References


Figure 2: Wealth Distribution