HANK

Heterogeneous Agent New Keynesian Models

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HANK: Heterogeneous Agent New Keynesian models

- Framework for quantitative analysis of aggregate shocks and macroeconomic policy
- Three building blocks
  1. Uninsurable idiosyncratic income risk
  2. Nominal price rigidities
  3. Assets with different degrees of liquidity
- Today: Transmission mechanism for conventional monetary policy

How monetary policy works in RANK

- Total consumption response to a drop in real rates

$$C \text{ response} = \underbrace{\text{direct response to } r}_{>95\%} + \underbrace{\text{indirect effects due to } Y}_{<5\%}$$

- Direct response is everything, pure intertemporal substitution

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- However, data suggest:

  1. Low sensitivity of C to r
  2. Sizable sensitivity of C to Y
  3. Micro sensitivity vastly heterogeneous, depends crucially on household balance sheets

How monetary policy works in HANK

- Once matched to micro data, HANK delivers realistic:
  - wealth distribution: small direct effect
  - MPC distribution: large indirect effect (depending on ΔY)

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	direct response to r	indirect effects due to Y
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HANK	<1/3	>2/3

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RANK: >95%	RANK: <5%
HANK: <1/3	HANK: >2/3

- Overall effect depends crucially on fiscal response, unlike in RANK where Ricardian equivalence holds

Why does this difference matter?

Suppose Central Bank wants to stimulate C

RANK view:

- sufficient to influence the path for real rates $\{r_t\}$
- household intertemporal substitution does the rest

HANK view:

- must rely heavily on GE feedbacks to boost hh labor income
- through fiscal policy reaction and/or an investment boom
- responsiveness of C to i is, to a larger extent, out of CB's control

Literature and contribution

Combine two workhorses of modern macroeconomics:

- New Keynesian models Gali, Gertler, Woodford
- Bewley models Aiyagari, Bewley, Huggett

Closest existing work:

1. New Keynesian models with limited heterogeneity

   Campell-Mankiw, Gali-LopezSalido-Valles, Iacoviello, Bilbiie, Challe-Matheron-Ragot-Rubio-Ramirez, Broer-Hansen-Krusell-Öberg

   - micro-foundation of spender-saver behavior

2. Bewley models with sticky prices

   Oh-Reis, Guerrieri-Lorenzoni, Ravn-Sterk, Gornemann-Kuester-Nakajima, DenHaan-Rendal-Riegler, Bayer-Luetticke-Pham-Tjaden, McKay-Reis, McKay-Nakamura-Steinsson, Huo-RiosRull, Werning, Luetticke

   - assets with different liquidity Kaplan-Violante
   - new view of individual earnings risk Guvenen-Karahan-Ozkan-Song

- Continuous time approach Achdou-Han-Lasry-Lions-Moll

HANK

Building blocks

Households

- Face uninsured idiosyncratic labor income risk
- Consume and supply labor
- Hold two assets: liquid and illiquid

Firms

- Monopolistically competitive intermediate-good producers
- Quadratic price adjustment costs à la Rotemberg (1982)

Government

- Issues liquid debt, spends, taxes

Monetary Authority

- Sets nominal rate on liquid assets based on a Taylor rule

Households

$$\max_{\{c_t, \ell_t, \quad\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-(\rho+\lambda)t} u(c_t, \ell_t) dt \quad \text{s.t.}$$

$$\dot{b}_t = r^b(b_t) b_t + w z_t \ell_t \qquad\qquad - c_t$$

$$z_t = \text{ some Markov process}$$

$$b_t \geq -\underline{b}$$

- c_t: non-durable consumption
- b_t: liquid assets
- z_t: individual productivity
- ℓ_t: hours worked
-

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- Γ: income from firm ownership
- no housing – see working paper

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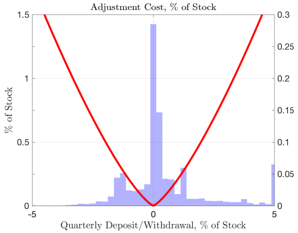
Households

- Adjustment cost function

$$\chi(d, a) = \chi_0 |d| + \chi_1 \left| \frac{d}{\max\{a, \underline{a}\}} \right|^{\chi_2} \max\{a, \underline{a}\}$$

- Linear component implies inaction region
- Convex component implies finite deposit rates



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- Recursive solution of hh problem consists of:
  1. consumption policy function $c(a, b, z; w, r^a, r^b)$
  2. deposit policy function $d(a, b, z; w, r^a, r^b)$
  3. labor supply policy function $\ell(a, b, z; w, r^a, r^b)$

  $\Rightarrow$ joint distribution of households $\mu(da, db, dz; w, r^a, r^b)$

Firms

Representative competitive final goods producer:

$$Y = \left(\int_0^1 y_j^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad \Rightarrow \quad y_j = \left(\frac{p_j}{P} \right)^{-\varepsilon} Y$$

Monopolistically competitive intermediate goods producers:

- Technology: $y_j = Z k_j^\alpha n_j^{1-\alpha} \quad \Rightarrow \quad m = \frac{1}{Z} \left(\frac{r}{\alpha} \right)^\alpha \left(\frac{w}{1-\alpha} \right)^{1-\alpha}$
- Set prices subject to quadratic adjustment costs:

$$\Theta \left(\frac{\dot{p}}{p} \right) = \frac{\theta}{2} \left(\frac{\dot{p}}{p} \right)^2 Y$$

Exact NK Phillips curve:

$$\left(r^a - \frac{\dot{Y}}{Y} \right) \pi = \frac{\varepsilon}{\theta} (m - \bar{m}) + \dot{\pi}, \quad \bar{m} = \frac{\varepsilon - 1}{\varepsilon}$$

Determination of illiquid return, distribution of profits

- Illiquid assets = part capital, part equity

$$a = k + qs$$

  - k: capital, pays return $r - \delta$
  - s: shares, price q, pay dividends $\omega\Pi = \omega(1 - m)Y$

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$$\frac{\omega\Pi + \dot{q}}{q} = r - \delta := r^a$$

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$$\Gamma = (1-\omega)\frac{z}{\bar{z}}\Pi$$

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- Set $\omega = \alpha$ (capital share) $\Rightarrow$ neutralize countercyclical markups

$$\text{total illiquid flow} = rK + \omega\Pi = \alpha mY + \omega(1-m)Y = \alpha Y$$

$$\text{total liquid flow} = wL + (1-\omega)\Pi = (1-\alpha)Y$$

Monetary authority and government

- Taylor rule
$$i = \bar{r}^b + \phi\pi + \epsilon, \quad \phi > 1$$
with $r^b := i - \pi$ (Fisher equation), $\epsilon =$ innovation ("MIT shock")

- Progressive tax on labor income:
$$\tilde{T}\left(wz\ell + \Gamma\right) = -T + \tau \times \left(wz\ell + \Gamma\right)$$

- Government budget constraint (in steady state)
$$G - r^b B^g = \int \tilde{T} d\mu$$

- Transition? Ricardian equivalence fails $\Rightarrow$ this matters!

Summary of market clearing conditions

- Liquid asset market

$$B^h + B^g = 0$$

- Illiquid asset market

$$A = K + q$$

- Labor market

$$N = \int z\ell(a, b, z) d\mu$$

- Goods market:

$$Y = C + I + G + \chi + \Theta + \text{borrowing costs}$$

Solution Method

Solution Method (from Achdou-Han-Lasry-Lions-Moll)

- Solving het. agent model = solving PDEs
  1. Hamilton-Jacobi-Bellman equation for individual choices
  2. Kolmogorov Forward equation for evolution of distribution
- Many well-developed methods for analyzing and solving these
  - simple but powerful: finite difference method
  - codes: `http://www.princeton.edu/~moll/HACTproject.htm`
- Apparatus is very general: applies to any heterogeneous agent model with continuum of atomistic agents
  1. heterogeneous households (Aiyagari, Bewley, Huggett,...)
  2. heterogeneous producers (Hopenhayn,...)
- can be extended to handle aggregate shocks (Krusell-Smith,...)

Computational Advantages relative to Discrete Time

1. Borrowing constraints only show up in boundary conditions
    - FOCs always hold with "$=$"
2. "Tomorrow is today"
    - FOCs are "static", compute by hand: $c^{-\gamma} = V_b(a, b, y)$
3. Sparsity
    - solving Bellman, distribution = inverting matrix
    - but matrices very sparse ("tridiagonal")
    - reason: continuous time $\Rightarrow$ one step left or one step right
4. Two birds with one stone
    - tight link between solving (HJB) and (KF) for distribution
    - matrix in discrete (KF) is transpose of matrix in discrete (HJB)
    - reason: diff. operator in (KF) is adjoint of operator in (HJB)

Parameterization

Three key aspects of parameterization

1. Measurement and partition of asset categories into: ▸ 50 shades of K
    - Liquid (cash, bank accounts + government/corporate bonds)
    - Illiquid (equity, housing)

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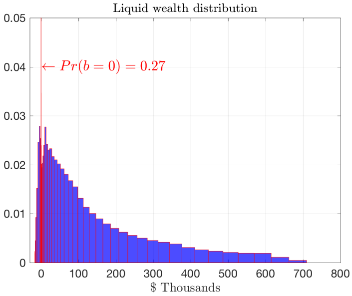
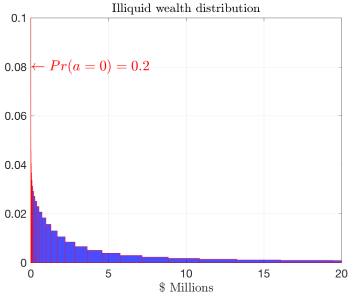
- Production side: standard calibration of NK models
- Standard separable preferences: $u(c, \ell) = \log c - \frac{1}{2}\ell^2$

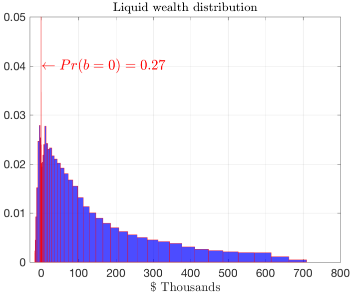
Model matches key feature of U.S. wealth distribution

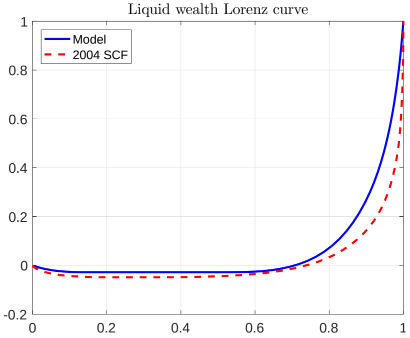




	Data	Model
Mean illiquid assets (rel to GDP)	2.920	2.920
Mean liquid assets (rel to GDP)	0.260	0.268
Poor hand-to-mouth	10%	9%
Wealthy hand-to-mouth	20%	18%

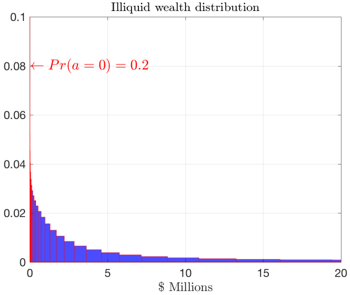
Wealth distributions: Liquid wealth

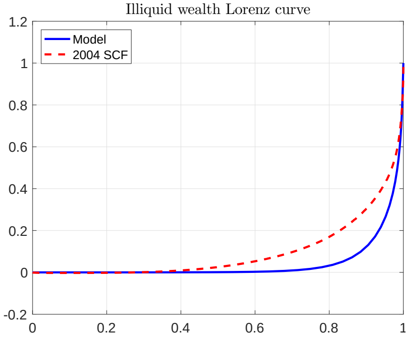




- Top 10% share: SCF 2004: 86%, Model: 73%
- Top 1% share: SCF 2004: 47%, Model: 16%
- Gini coefficient: SCF 2004: 0.98, Model: 0.85

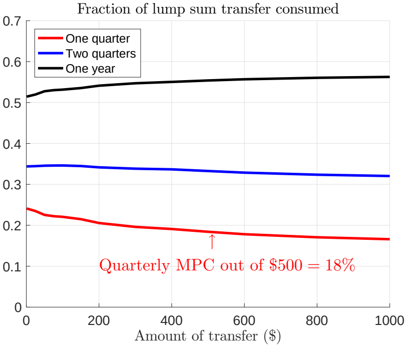
Wealth distributions: Illiquid wealth

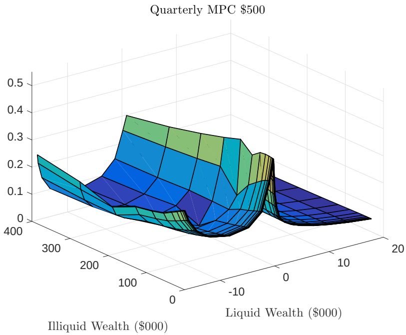




- Top 10% share: SCF 2004: 70%, Model: 87%
- Top 1% share: SCF 2004: 33%, Model: 40%
- Gini coefficient: SCF 2004: 0.81, Model: 0.82

Model generates high and heterogeneous MPCs





Description		Value	Target / Source
Preferences			
λ	Death rate	1/180	Av. lifespan 45 years
γ	Risk aversion	1	
φ	Frisch elasticity (GHH)	0.5	
ψ	Disutility of labor	27	Av. hours worked equal to 1/3
ρ	Discount rate (pa)	4.7%	Internally calibrated
Production			
ε	Demand elasticity	10	Profit share 10 %
α	Capital share	0.33	
δ	Depreciation rate (p.a.)	10%	
θ	Price adjustment cost	100	Slope of Phillips curve, $\varepsilon/\theta = 0.1$
Government			
τ	Proportional labor tax	0.25	
T	Lump sum transfer (rel GDP)	0.075	40% hh with net govt transfer
$\bar{g}$	Govt debt to annual GDP	0.26	government budget constraint
Monetary Policy			
ϕ	Taylor rule coefficient	1.25	
r^b	Steady state real liquid return (pa)	2%	
Housing			
r^h	Net housing return (pa)	1.5%	Kaplan and Violante (2014)
Illiquid Assets			
r^a	Illiquid asset return (pa)	6.5%	Equilibrium outcome
Borrowing			
r^{borr}	Borrowing rate (pa)	8.4%	Internally calibrated
$\underline{b}$	Borrowing limit	-0.42	1 $\times$ quarterly labor inc

Results

Transmission of monetary policy shock to C

Innovation $\epsilon < 0$ to the Taylor rule: $i = \bar{r}^b + \phi\pi + \epsilon$

- All experiments: $\epsilon_0 = -0.0025$, i.e. -1% annualized

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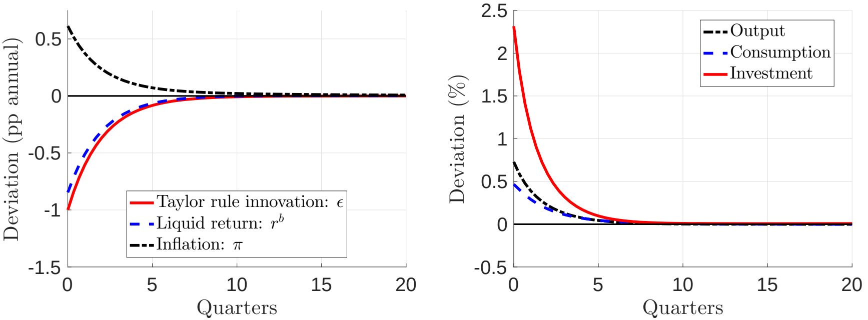

Transmission of monetary policy shock to C

$$dC_0 = \underbrace{\int_0^\infty \frac{\partial C_0}{\partial r_t^b} dr_t^b dt}_{\text{direct}} + \underbrace{\int_0^\infty \left[\frac{\partial C_0}{\partial r_t^a} dr_t^a + \frac{\partial C_0}{\partial w_t} dw_t + \frac{\partial C_0}{\partial T_t} dT_t \right] dt}_{\text{indirect}}$$

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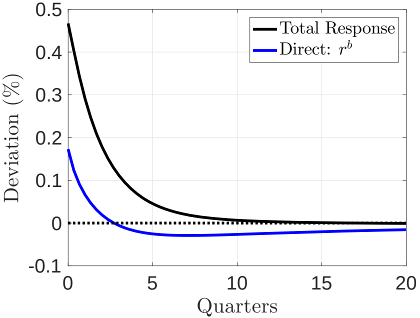
✓

Intertemporal substitution and income effects from $r^b \downarrow$



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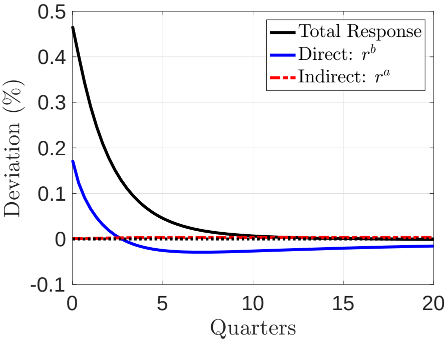
✓

Portfolio reallocation effect from $r^a - r^b \uparrow$



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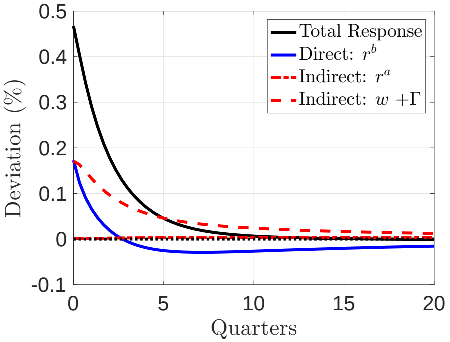
Labor demand channel from $w \uparrow$



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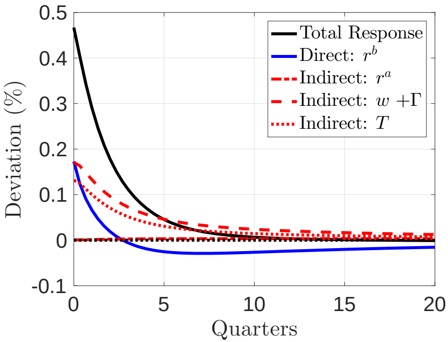
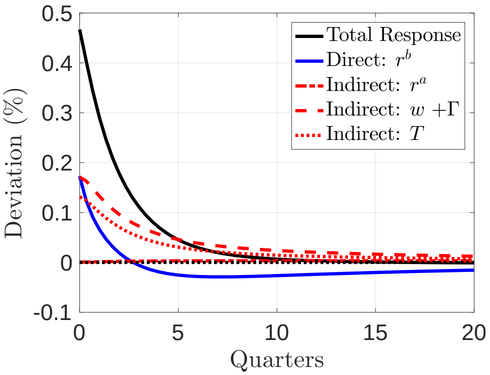
✓

Fiscal adjustment: $T \uparrow$ in response to $\downarrow$ in interest payments on B

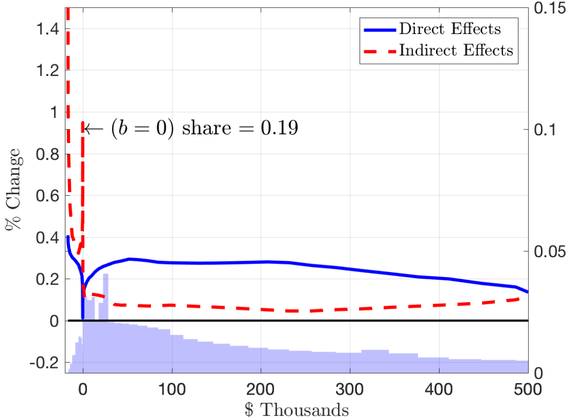


Transmission of monetary policy shock to C

$$dC_0 = \underbrace{\int_0^\infty \frac{\partial C_0}{\partial r_t^b} dr_t^b dt}_{19\%} + \underbrace{\int_0^\infty \left[\frac{\partial C_0}{\partial r_t^a} dr_t^a + \frac{\partial C_0}{\partial w_t} dw_t + \frac{\partial C_0}{\partial T_t} dT_t\right] dt}_{81\%}$$

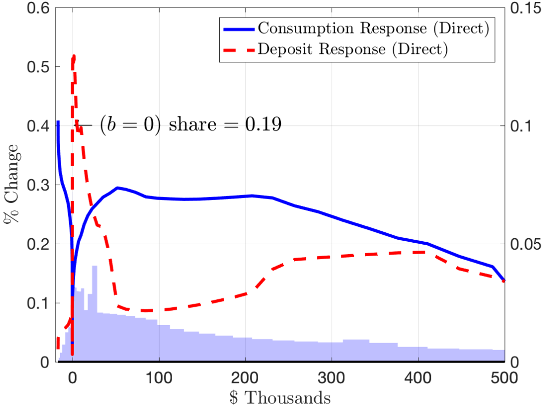


Monetary transmission across liquid wealth distribution



- Total change = c-weighted sum of (direct + indirect) at each b

Why small direct effects?



- Intertemporal substitution: (+) for non-HtM
- Income effect: (-) for rich households
- Portfolio reallocation: (-) for those with low but > 0 liquid wealth

Role of fiscal response in determining total effect

	T adjusts (1)	G adjusts (2)	B^g adjusts (3)
Elasticity of C_0 to r^b	-2.21	-2.07	-1.48
Share of Direct effects:	19%	22%	46%

- Fiscal response to lower interest payments on debt:
  - T adjusts: stimulates AD through MPC of HtM households
  - G adjusts: translates 1-1 into AD
  - B^g adjusts: no initial stimulus to AD from fiscal side

Monetary transmission in RANK and HANK

$$\Delta C \quad = \quad \text{direct response to } r \quad + \quad \text{indirect GE response}$$

	direct response to r	indirect GE response
RANK	90%	10%
HANK	2/3	1/3

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- RANK view:
  - High sensitivity of C to r: intertemporal substitution
  - Low sensitivity of C to Y: the RA is a PIH consumer
- HANK view:
  - Low sensitivity to r: income effect of wealthy offsets int. subst.
  - High sensitivity to Y: sizable share of hand-to-mouth agents

  $\Rightarrow$ **Q:** Is Fed less in control of C than we thought?

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  $\Rightarrow$ **Q:** Is Fed less in control of C than we thought?
- Work in progress: perturbation methods $\Rightarrow$ estimation, inference

Numerical Solution of HJB Equations

Finite Difference Methods

- See `http://www.princeton.edu/~moll/HACTproject.htm`
- Explain using neoclassical growth model, easily generalized to heterogeneous agent models

$$\rho v(k) = \max_c \; u(c) + v'(k)(F(k) - \delta k - c)$$

- Functional forms

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad F(k) = k^\alpha$$

- Use finite difference method
  - Two MATLAB codes

    `http://www.princeton.edu/~moll/HACTproject/HJB_NGM.m`

    `http://www.princeton.edu/~moll/HACTproject/HJB_NGM_implicit.m`

Barles-Souganidis

- There is a well-developed theory for numerical solution of HJB equation using finite difference methods
- Key paper: Barles and Souganidis (1991), "Convergence of approximation schemes for fully nonlinear second order equations
  `https://www.dropbox.com/s/vhw5qqrczw3dvw3/barles-souganidis.pdf?dl=0`
- Result: finite difference scheme "converges" to unique viscosity solution under three conditions
  1. monotonicity
  2. consistency
  3. stability
- Good reference: Tourin (2013), "An Introduction to Finite Difference Methods for PDEs in Finance."

Finite Difference Approximations to $v'(k_i)$

- Approximate $v(k)$ at I discrete points in the state space, k_i, $i = 1, ..., I$. Denote distance between grid points by Δk.
- Shorthand notation

$$v_i = v(k_i)$$

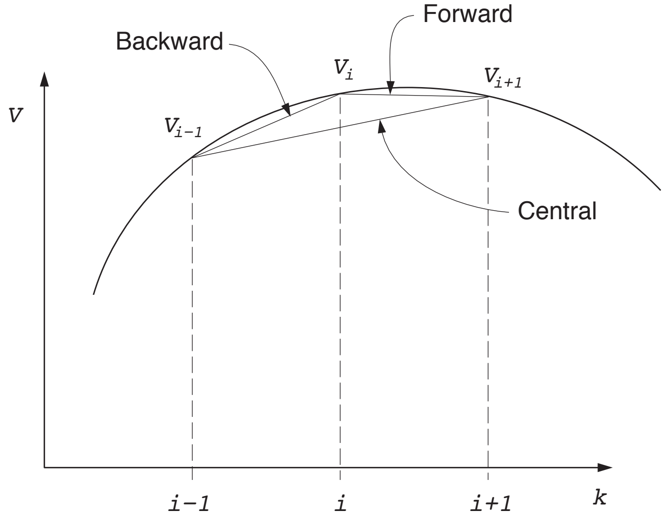
- Need to approximate $v'(k_i)$.
- Three different possibilities:

$$v'(k_i) \approx \frac{v_i - v_{i-1}}{\Delta k} = v'_{i,B} \qquad \text{backward difference}$$

$$v'(k_i) \approx \frac{v_{i+1} - v_i}{\Delta k} = v'_{i,F} \qquad \text{forward difference}$$

$$v'(k_i) \approx \frac{v_{i+1} - v_{i-1}}{2\Delta k} = v'_{i,C} \qquad \text{central difference}$$

Finite Difference Approximations to $v'(k_i)$



Finite Difference Approximation

FD approximation to HJB is

$$\rho v_i = u(c_i) + v_i'[F(k_i) - \delta k_i - c_i] \qquad (*)$$

where $c_i = (u')^{-1}(v_i')$, and v_i' is one of backward, forward, central FD approximations.

Two complications:

1. which FD approximation to use? "Upwind scheme"
2. $(*)$ is extremely non-linear, need to solve iteratively: "explicit" vs. "implicit method"

My strategy for next few slides:

- what works
- slides on my website: why it works (Barles-Souganidis)

Which FD Approximation?

- Which of these you use is extremely important
- Best solution: use so-called "upwind scheme." Rough idea:
  - forward difference whenever drift of state variable positive
  - backward difference whenever drift of state variable negative

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- In our example: define

$$s_{i,F} = F(k_i) - \delta k_i - (u')^{-1}(v'_{i,F}), \quad s_{i,B} = F(k_i) - \delta k_i - (u')^{-1}(v'_{i,B})$$

- Approximate derivative as follows

$$v'_i = v'_{i,F}\mathbf{1}_{\{s_{i,F}>0\}} + v'_{i,B}\mathbf{1}_{\{s_{i,B}<0\}} + \bar{v}'_i\mathbf{1}_{\{s_{i,F}<0<s_{i,B}\}}$$

where $\mathbf{1}_{\{\cdot\}}$ is indicator function, and $\bar{v}'_i = u'(F(k_i) - \delta k_i)$.

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where $\mathbf{1}_{\{\cdot\}}$ is indicator function, and $\bar{v}'_i = u'(F(k_i) - \delta k_i)$.

- Where does $\bar{v}'_i$ term come from? Answer:
  - since v is concave, $v'_{i,F} < v'_{i,B}$ (see figure) $\Rightarrow s_{i,F} < s_{i,B}$
  - if $s'_{i,F} < 0 < s'_{i,B}$, set $s_i = 0 \Rightarrow v'(k_i) = u'(F(k_i) - \delta k_i)$, i.e. we're at a steady state.

Sparsity

- Discretized HJB equation is

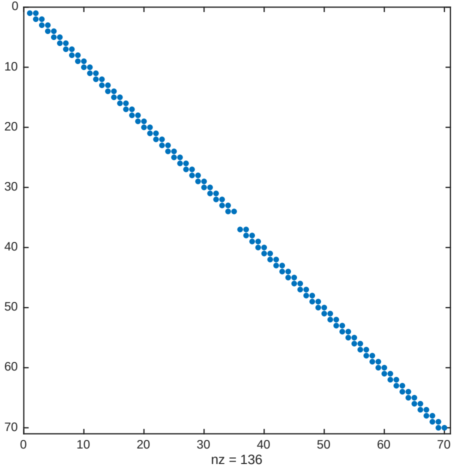
$$\rho v_i = u(c_i) + \frac{v_{i+1} - v_i}{\Delta k} s_{i,F}^+ + \frac{v_i - v_{i-1}}{\Delta k} s_{i,B}^-$$

- Notation: for any x, $x^+ = \max\{x, 0\}$ and $x^- = \min\{x, 0\}$
- Can write this in matrix notation

$$\rho v = u + \mathbf{A} v$$

where $\mathbf{A}$ is $I \times I$ (I= no of grid points) and looks like...

Visualization of **A** (output of `spy(A)` in Matlab)



The matrix $\mathbf{A}$

- FD method approximates process for k with discrete Poisson process, $\mathbf{A}$ summarizes Poisson intensities
  - entries in row i:

$$\begin{bmatrix} \underbrace{-\frac{s_{i,B}^{-}}{\Delta k}}_{\text{inflow}_{i-1} \geq 0} & \underbrace{\frac{s_{i,B}^{-}}{\Delta k} - \frac{s_{i,F}^{+}}{\Delta k}}_{\text{outflow}_{i} \leq 0} & \underbrace{\frac{s_{i,F}^{+}}{\Delta k}}_{\text{inflow}_{i+1} \geq 0} \end{bmatrix} \begin{bmatrix} v_{i-1} \\ v_i \\ v_{i+1} \end{bmatrix}$$

  - negative diagonals, positive off-diagonals, rows sum to zero:
  - tridiagonal matrix, very sparse
- $\mathbf{A}$ depends on v (nonlinear problem). Next: iterative method...

Iterative Method

- Idea: Solve FOC for given v^n, update v^{n+1} according to

$$\frac{v_i^{n+1} - v_i^n}{\Delta} + \rho v_i^n = u(c_i^n) + (v^n)'(k_i)(F(k_i) - \delta k_i - c_i^n) \qquad (*)$$

- Algorithm: Guess $v_i^0, i = 1, ..., I$ and for $n = 0, 1, 2, ...$ follow
  1. Compute $(v^n)'(k_i)$ using FD approx. on previous slide.
  2. Compute c^n from $c_i^n = (u')^{-1}[(v^n)'(k_i)]$
  3. Find v^{n+1} from $(*)$.
  4. If v^{n+1} is close enough to v^n: stop. Otherwise, go to step 1.
- See `http://www.princeton.edu/~moll/HACTproject/HJB_NGM.m`
- Important parameter: Δ = step size, cannot be too large ("CFL condition").
- Pretty inefficient: I need 5,990 iterations (though quite fast)

Efficiency: Implicit Method

- Efficiency can be improved by using an "implicit method"

$$\frac{v_i^{n+1} - v_i^n}{\Delta} + \rho v_i^{n+1} = u(c_i^n) + (v_i^{n+1})'(k_i)[F(k_i) - \delta k_i - c_i^n]$$

- Each step n involves solving a linear system of the form

$$\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^{n+1} = u + \mathbf{A}_n v^{n+1}$$
$$\left((\rho + \tfrac{1}{\Delta})\mathbf{I} - \mathbf{A}_n\right) v^{n+1} = u + \tfrac{1}{\Delta} v^n$$

- but $\mathbf{A}_n$ is super sparse $\Rightarrow$ super fast
- See `http://www.princeton.edu/~moll/HACTproject/HJB_NGM_implicit.m`

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- but $\mathbf{A}_n$ is super sparse $\Rightarrow$ super fast
- See `http://www.princeton.edu/~moll/HACTproject/HJB_NGM_implicit.m`
- In general: implicit method preferable over explicit method
  1. stable regardless of step size Δ
  2. need much fewer iterations
  3. can handle many more grid points

Implicit Method: Practical Consideration

- In Matlab, need to explicitly construct **A** as sparse to take advantage of speed gains
- Code has part that looks as follows

```
X = -min(mub,0)/dk;
Y = -max(muf,0)/dk + min(mub,0)/dk;
Z = max(muf,0)/dk;
```

- Constructing full matrix – slow

```
for i=2:I-1
    A(i,i-1) = X(i);
    A(i,i) = Y(i);
    A(i,i+1) = Z(i);
end
A(1,1)=Y(1); A(1,2) = Z(1);
A(I,I)=Y(I); A(I,I-1) = X(I);
```

- Constructing sparse matrix – fast

```
A =spdiags(Y,0,I,I)+spdiags(X(2:I),-1,I,I)+spdiags([0;Z(1:I-1)],1,I,I);
```

Intermediate good firm pricing problem

$$\max_{\{p_t\}_{t\geq 0}} \int_0^\infty e^{-r^a t} \left\{ \Pi_t(p_t) - \Theta_t \left(\frac{\dot{p}_t}{p_t} \right) \right\} dt \quad \text{s.t.}$$

$$\Pi(p) = \left(\frac{p}{P} - m \right) \left(\frac{p}{P} \right)^{-\varepsilon} Y$$

$$m = \frac{1}{Z} \left(\frac{r}{\alpha} \right)^{\alpha} \left(\frac{w}{1-\alpha} \right)^{1-\alpha}$$

$$\Theta(\pi) = \frac{\theta}{2} \pi^2 Y$$

Back to firms

Fifty shades of K

	Liquid	Illiquid	Total
Non-productive	Household deposits net of revolving debt Corp & Govt bonds $B^h = 0.26$		0.26
Productive		Indirectly held equity Directly held equity Noncorp bus equity net housing net durables	2.92 K
Total	$-B^g = 0.26$	$A = 2.92$	3.18

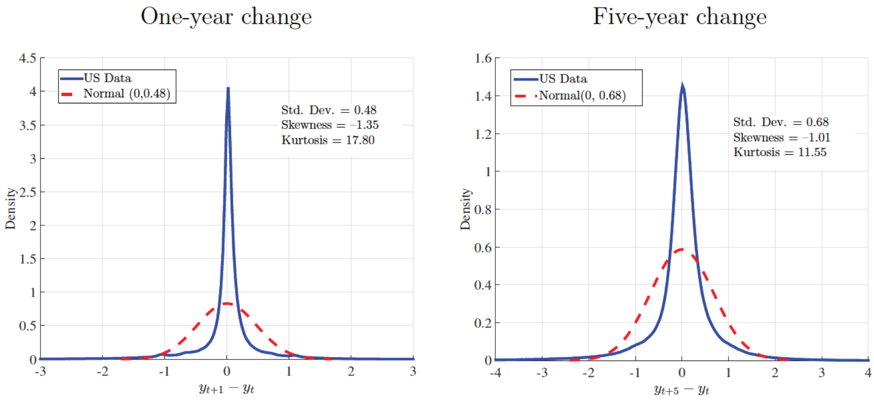
- Quantities are multiples of annual GDP
- Sources: Flow of Funds and SCF 2004
- Working paper: part of housing, durables = unproductive illiquid assets

back

Continuous time earnings dynamics

- Literature provides little guidance on statistical models of high frequency earnings dynamics
- Key challenge: inferring within-year dynamics from annual data
- Higher order moments of annual changes are informative
- Target key moments of one 1-year and 5-year labor earnings growth from SSA data
- Model generates a thick right tail for earnings levels

Leptokurtic earnings changes (Guvenen et al)



Two-component jump-drift process

- Flow earnings ($y = wzl$) modeled as sum of two components:

$$\log y_t = y_{1t} + y_{2t}$$

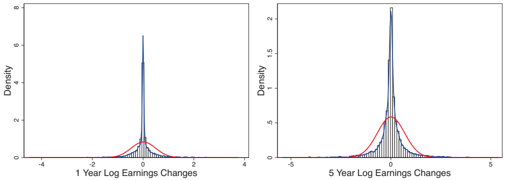
- Each component is a jump-drift with:
  - mean-reverting drift: $-\beta y_{it} dt$
  - jumps with arrival rate: λ_i, drawn from $\mathcal{N}(0, \sigma_i)$
- Estimate using SMM aggregated to annual frequency
- Choose six parameters to match eight moments:

Model distribution of earnings changes

Moment	Data	Model	Moment	Data	Model
Variance: annual log earns	0.70	0.70	Frac 1yr change $< 10\%$	0.54	0.56
Variance: 1yr change	0.23	0.23	Frac 1yr change $< 20\%$	0.71	0.67
Variance: 5yr change	0.46	0.46	Frac 1yr change $< 50\%$	0.86	0.85
Kurtosis: 1yr change	17.8	16.5			
Kurtosis: 5yr change	11.6	12.1			

Transitory component: $\hat{\lambda}_1 = 0.08, \quad \hat{\beta}_1 = 0.76, \quad \hat{\sigma}_1 = 1.74$

Persistent component: $\hat{\lambda}_2 = 0.007, \quad \hat{\beta}_2 = 0.009, \quad \hat{\sigma}_2 = 1.53$



▸ back

Adjustment cost function calibration

$$\chi(d, a) = \chi_0 |d| + \chi_1 \left| \frac{d}{\max\{a, \underline{a}\}} \right|^{\chi_2} \max\{a, \underline{a}\}$$

- Choose χ_0, χ_1, χ_2 together with:
  - discount rate ρ
  - borrowing intermediation wedge: $\kappa^b = r^{bor} - r^b$

to match five features of the household wealth distribution

Moment	Data	Model
Mean illiquid assets (rel to GDP)	2.920	2.920
Mean liquid assets (rel to GDP)	0.260	0.263
Poor hand-to-mouth	10%	12%
Wealthy hand-to-mouth	20%	17%
Fraction borrowing	15%	14%

Monetary Policy in Benchmark NK Models

Monetary Policy in Benchmark NK Models

Goal:

- Introduce decomposition of C response to r change

Setup:

- Prices and wages perfectly rigid $= 1$, GDP=labor $= Y_t$
- Households: CRRA(γ), income Y_t, interest rate r_t

$$\Rightarrow \quad C_t(\{r_s, Y_s\}_{s\geq 0})$$

- Monetary policy: sets time path $\{r_t\}_{t\geq 0}$, special case

$$r_t = \rho + e^{-\eta t}(r_0 - \rho), \quad \eta > 0 \tag{$*$}$$

- Equilibrium: $C_t(\{r_s, Y_s\}_{s\geq 0}) = Y_t$
- Overall effect of monetary policy

$$-\frac{d \log C_0}{d r_0} = \frac{1}{\gamma\eta}$$

Monetary Policy in RANK

- Decompose C response by totally differentiating $C_0(\{r_t, Y_t\}_{t\geq 0})$

$$dC_0 = \underbrace{\int_0^\infty \frac{\partial C_0}{\partial r_t} dr_t dt}_{\text{direct response to } r} + \underbrace{\int_0^\infty \frac{\partial C_0}{\partial Y_t} dY_t dt}_{\text{indirect effects due to } Y}.$$

- In special case (∗)

$$-\frac{d \log C_0}{dr_0} = \frac{1}{\gamma\eta}\left[\underbrace{\frac{\eta}{\rho+\eta}}_{\text{direct response to } r} + \underbrace{\frac{\rho}{\rho+\eta}}_{\text{indirect effects due to } Y}\right].$$

- Reasonable parameterizations ⇒ very small indirect effects, e.g.
  - $\rho = 0.5\%$ quarterly
  - $\eta = 0.5$, i.e. quarterly autocorr $e^{-\eta} = 0.61$

$$\Rightarrow \quad \frac{\eta}{\rho+\eta} = 99\%, \qquad \frac{\rho}{\rho+\eta} = 1\%$$

What if some households are hand-to-mouth?

- "Spender-saver" or Two-Agent New Keynesian (TANK) model
- Fraction Λ are HtM "spenders": $C_t^{sp} = Y_t$
- Decomposition in special case $(*)$

$$-\frac{d \log C_0}{d r_0} = \frac{1}{\gamma\eta}\left[\underbrace{(1-\Lambda)\frac{\eta}{\rho+\eta}}_{\text{direct response to } r} + \underbrace{(1-\Lambda)\frac{\rho}{\rho+\eta} + \Lambda}_{\text{indirect effects due to } Y}\right].$$

- $\Rightarrow$ indirect effects $\approx \Lambda = 20\text{-}30\%$

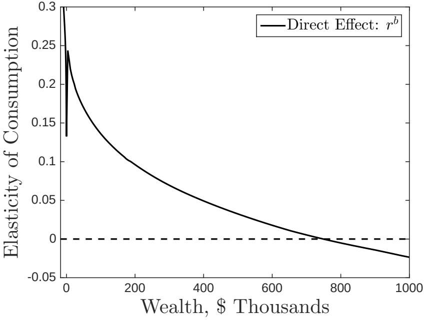
What if there are assets in positive supply?

- Govt issues debt B to households sector
- Fall in r_t implies a fall in interest payments of $(r_t - \rho)\, B$
- Fraction λ^T of income gains transferred to spenders
- Initial consumption restponse in special case $(*)$

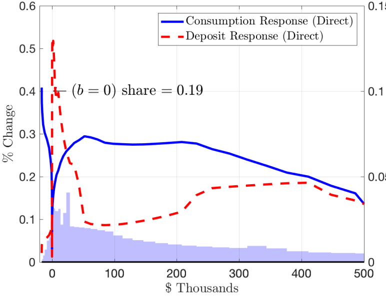
$$-\frac{d \log C_0}{d r_0} = \frac{1}{\gamma\eta} \quad + \quad \underbrace{\frac{\lambda^T}{1-\lambda}\frac{B}{\bar{Y}}}_{\text{fiscal redistribution channel}} \quad .$$

- Interaction between non-Ricardian households and debt in positive net supply matters for overall effect of monetary policy

Comparison to One-Asset Model



(a) One-Asset Model



(b) Two-Asset Model