Heterogeneous Agent Models in Continuous Time
Part I

Benjamin Moll
Princeton

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What this mini-course is about

• Many interesting questions require thinking about distributions
  • Why are income and wealth so unequally distributed?
  • Is there a trade-off between inequality and economic growth?
  • What are the forces that lead to the concentration of economic activity in a few very large firms?

• Modeling distributions is hard
  • closed-form solutions are rare
  • computations are challenging

• Goal: teach you some new methods that make progress on this
  • solving heterogeneous agent model = solving PDEs
  • main difference to existing continuos-time literature: handle models for which closed-form solutions do not exist

• based on joint work with Yves Achdou, SeHyoun Ahn, Jiequn Han, Greg Kaplan, Pierre-Louis Lions, Jean-Michel Lasry, Gianluca Violante, Tom Winberry
Solving het. agent model = solving PDEs

• More precisely: a system of two PDEs
  1. Hamilton-Jacobi-Bellman equation for individual choices
  2. Kolmogorov Forward equation for evolution of distribution

• Many well-developed methods for analyzing and solving these
  • Codes: http://www.princeton.edu/~moll/HACTproject.htm

• Apparatus is very general: applies to any heterogeneous agent model with continuum of atomistic agents
  1. heterogeneous households (Aiyagari, Bewley, Huggett,...)
  2. heterogeneous producers (Hopenhayn,...)

• can be extended to handle aggregate shocks (Krusell-Smith,...)
Outline

Lecture 1

1. Refresher: HJB equations
2. Textbook heterogeneous agent model
3. Tools
   • viscosity solutions
   • numerical solution of HJB equations

Lecture 2

1. Analysis and numerical solution of heterogeneous agent model
2. models with non-convexities (Skiba)
3. models with multiple assets (HANK)
4. perturbation methods for handling aggregate uncertainty (starring Tom Winberry)
Computational Advantages relative to Discrete Time

1. **Borrowing constraints only show up in boundary conditions**
   - FOCs always hold with “=”

2. “Tomorrow is today”
   - FOCs are “static”, compute by hand: \( c^{-\gamma} = v_a(a, z) \)

3. **Sparsity**
   - solving Bellman, distribution = inverting matrix
   - but matrices very sparse (“tridiagonal”)
   - reason: continuous time \( \Rightarrow \) one step left or one step right

4. **Two birds with one stone**
   - tight link between solving (HJB) and (KF) for distribution
   - matrix in discrete (KF) is transpose of matrix in discrete (HJB)
   - reason: diff. operator in (KF) is adjoint of operator in (HJB)
Real Payoff: extends to more general setups

- non-convexities
- multiple assets
- aggregate shocks
- focus of lecture 2
What you’ll be able to do at end of this course

• Joint distribution of income and wealth in Aiyagari model
What you’ll be able to do at end of this course

- Experiment: effect of one-time redistribution of wealth
What you’ll be able to do at end of this course

Video of convergence back to steady state

https://www.dropbox.com/s/op5u2n1lfmmr2o/distribution_tax.mp4?dl=0
Review: HJB Equations
Hamilton-Jacobi-Bellman Equation: Some “History”

- Aside: why called “dynamic programming”?

- Bellman: “Try thinking of some combination that will possibly give it a pejorative meaning. It’s impossible. Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities.” [http://en.wikipedia.org/wiki/Dynamic_programming#History](http://en.wikipedia.org/wiki/Dynamic_programming#History)
Hamilton-Jacobi-Bellman Equations

- Pretty much all deterministic optimal control problems in continuous time can be written as

\[ \nu(x_0) = \max_{\{\alpha(t)\}_{t \geq 0}} \int_0^\infty e^{-\rho t} h(x(t), \alpha(t)) \, dt \]

subject to the law of motion for the state

\[ \dot{x}(t) = f(x(t), \alpha(t)) \quad \text{and} \quad \alpha(t) \in A \]

for \( t \geq 0 \), \( x(0) = x_0 \) given.

- \( \rho \geq 0 \): discount rate
- \( x \in X \subseteq \mathbb{R}^m \): state vector
- \( \alpha \in A \subseteq \mathbb{R}^n \): control vector
- \( h : X \times A \rightarrow \mathbb{R} \): instantaneous return function
Example: Neoclassical Growth Model

\[ v(k_0) = \max_{\{c(t)\}_{t \geq 0}} \int_0^\infty e^{-\rho t} u(c(t)) \, dt \]

subject to

\[ \dot{k}(t) = F(k(t)) - \delta k(t) - c(t) \]

for \( t \geq 0, \; k(0) = k_0 \) given.

- Here the state is \( x = k \) and the control \( \alpha = c \)
- \( h(x, \alpha) = u(\alpha) \)
- \( f(x, \alpha) = F(x) - \delta x - \alpha \)
Generic HJB Equation

• How to analyze these optimal control problems? Here: “cookbook approach”

• **Result:** the value function of the generic optimal control problem satisfies the Hamilton-Jacobi-Bellman equation

\[
\rho v(x) = \max_{\alpha \in A} h(x, \alpha) + v'(x) \cdot f(x, \alpha)
\]

• In the case with more than one state variable \( m > 1 \), \( v'(x) \in \mathbb{R}^m \) is the gradient of the value function.

• Sometimes people (especially mathematicians) also write

\[
\rho v(x) = H(x, v'(x))
\]

\[
H(x, p) := \max_{\alpha \in A} h(x, \alpha) + p \cdot f(x, \alpha).
\]

• mathematicians call \( H \) the “Hamiltonian”
Example: Neoclassical Growth Model

• “cookbook” implies:

\[ \rho v(k) = \max_c u(c) + v'(k)(F(k) - \delta k - c) \]

• Proceed by taking first-order conditions etc

\[ u'(c) = v'(k) \]

• Derivation from discrete time Bellman equation
Poisson Uncertainty

• Easy to extend this to stochastic case. Simplest case: two-state Poisson process

• Example: RBC Model. Production is $Z_t F(k_t)$ where $Z_t \in \{Z_1, Z_2\}$ Poisson with intensities $\lambda_1, \lambda_2$

• Result: HJB equation is

$$\rho v_i(k) = \max_c u(c) + v'_i(k)[Z_i F(k) - \delta k - c] + \lambda_i[v_j(k) - v_i(k)]$$

for $i = 1, 2, j \neq i$.

• Derivation similar as before.
A Textbook Heterogeneous Agent Model
Households

are heterogeneous in their wealth $a$ and income $y$, solve

$$\max_{c_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.}$$

$$da_t = (y_t + r_t a_t - c_t) dt$$

$$y_t \in \{y_1, y_2\} \text{ Poisson with intensities } \lambda_1, \lambda_2$$

$$a_t \geq a$$

- $c_t$: consumption
- $u$: utility function, $u' > 0$, $u'' < 0$.
- $\rho$: discount rate
- $r_t$: interest rate
- $a > -\infty$: borrowing limit e.g. if $a = 0$, can only save later: carries over to $y_t = \text{general diffusion process.}$
Stationary Equilibrium

Bonds in zero net supply

\[ 0 = S(r) = \int ag_1(a) \, da + \int ag_2(a) \, da \]  
\text{(EQ)}

\[ \rho v_i(a) = \max_c u(c) + v'_i(a)(y_i + ra - c) + \lambda_i(v_j(a) - v_i(a)) \]  
\text{(HJB)}

\[ 0 = - \frac{d}{da} [s_i(a)g_i(a)] - \lambda_i g_i(a) + \lambda_j g_j(a), \]  
\text{(KF)}

\[ s_i(a) = y_i + ra - c_i(a), \quad c_i(a) = (u')^{-1}(v'_i(a)) \]

• The two PDEs (HJB) and (KF) together with (EQ) fully characterize stationary equilibrium

Derivation of (HJB) (KF)
Transition Dynamics

\[ 0 = S(r) = \int ag_1(a, t) da + \int ag_2(a, t) da \]  \hspace{1cm} (EQ)

\[ \rho v_i(a, t) = \max_c u(c) + \partial_a v_i(a, t)(y_i + r(t)a - c) \]
\[ + \lambda_i(v_j(a, t) - v_i(a, t)) + \partial_t v_i(a, t), \]  \hspace{1cm} (HJB)

\[ \partial_t g_i(a, t) = -\partial_a [s_i(a, t)g_i(a, t)] - \lambda_i g_i(a, t) + \lambda_j g_j(a, t), \]  \hspace{1cm} (KF)

\[ s_i(a, t) = y_i + r(t)a - c_i(a, t), \quad c_i(a, t) = (u')^{-1}(\partial_a v_i(a, t)) \]

• Given initial condition \( g_{i,0}(a) \), the two PDEs (HJB) and (KF) together with (EQ) fully characterize equilibrium.

• Notation: for any function \( f \), \( \partial_x f \) means \( \frac{\partial f}{\partial x} \)
Viscosity Solutions: A Primer
Viscosity Solutions

For our purposes, useful for two reasons:

1. problems with kinks, e.g. coming from non-convexities
2. problems with “state constraints”
   - borrowing constraints
   - computations with bounded domain

Things to remember

1. viscosity solution ⇒ no concave kinks (convex kinks are allowed)
2. constrained viscosity solution: “boundary inequalities”
3. uniqueness: HJB equations have unique viscosity solution
Viscosity Solutions: Definition

Consider HJB for generic optimal control problem

\[ \rho \nu(x) = \max_{\alpha \in A} \left\{ h(x, \alpha) + \nu'(x)f(x, \alpha) \right\} \]  

(HJB)

- Next slide: definition of viscosity solution
- Basic idea: \( \nu \) may have kinks i.e. may not be differentiable
- replace \( \nu'(x) \) at point where it does not exist (because of kink in \( \nu \)) with derivative of smooth function \( \phi \) touching \( \nu \)
- two types of kinks: concave and convex \( \Rightarrow \) two conditions
  - concave kink: \( \phi \) touches \( \nu \) from above
  - convex kink: \( \phi \) touches \( \nu \) from below
- Remark: definition allows for concave kinks. In a few slides: these never arise in maximization problems
Convex and Concave Kinks

Convex Kink (Supersolution)

Concave Kink (Subsolution)
Definition: A viscosity solution of (HJB) is a continuous function $v$ such that the following hold:

1. (Subsolution) If $\phi$ is any smooth function and if $v - \phi$ has a local maximum at point $x^*$ ($v$ may have a concave kink), then

$$\rho v(x^*) \leq \max_{\alpha \in A} \left\{ h(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha) \right\}$$

2. (Supersolution) If $\phi$ is any smooth function and if $v - \phi$ has a local minimum at point $x^*$ ($v$ may have a convex kink), then

$$\rho v(x^*) \geq \max_{\alpha \in A} \left\{ h(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha) \right\}.$$
A few remarks, terminology

- If \( v \) is differentiable at \( x^* \), then
  - local max or min of \( v - \phi \) implies \( v'(x^*) = \phi'(x^*) \)
  - sub- and supersolution conditions \( \Rightarrow \) viscosity solution of (HJB) is just classical solution

- If a continuous function \( v \) satisfies condition 1 (but not necessarily 2) we say that it is a “viscosity subsolution”

- Conversely, if \( v \) satisfies condition 2 (but not necessarily 1), we say that it is a “viscosity supersolution”

- \( \Leftrightarrow \) a continuous function \( v \) is a “viscosity solution” if it is both a “viscosity subsolution” and a “viscosity supersolution.”

- “subsolution” and “supersolution” come from \( \leq 0 \) and \( \geq 0 \)

- “viscosity” is in honor of the “method of vanishing viscosity”: add Brownian noise and \( \to 0 \) (movements in viscous fluid)
Viscosity Solutions: Intuition

• Consider discrete time Bellman:
  \[ v(x) = \max_{\alpha} h(x, \alpha) + \beta v(x'), \quad x' = f(x, \alpha) \]

• Think about it as an operator \( T \) on function \( v \)
  \[ (Tv)(x) = \max_{\alpha} h(x, \alpha) + \beta v(f(x, \alpha)) \]

• Solution = fixed point: \( T v = v \)

• Intuitive property of \( T \) ("monotonicity")
  \[ \phi(x) \leq v(x) \quad \forall x \quad \Rightarrow \quad (T\phi)(x) \leq (Tv)(x) \quad \forall x \quad (*) \]

• Intuition: if my continuation value is higher, I’m better off

• **Viscosity solution is exactly same idea**

• Key idea: sidestep non-differentiability of \( v \) by using "monotonicity"

• Mathematicians call "monotonicity" in (*) "comparison principle"
Viscosity Solutions: Heuristic Derivation

- Time periods of length $\Delta$. Consider HJB equation
  \[ v(x_t) = \max_{\alpha} \Delta h(x_t, \alpha) + (1 - \rho \Delta) v(x_{t+\Delta}) \]  
  s.t.
  \[ x_{t+\Delta} = \Delta f(x_t, \alpha) + x_t \]

- Suppose $v$ is not differentiable at $x^*$ and has a convex kink
  - problem: when taking $\Delta \to 0$, pick up derivative $v'$
  - solution: replace continuation value with smooth function $\phi$
Viscosity Solutions: Heuristic Derivation

- For now: consider \( \phi \) such that \( \phi(x^*) = v(x^*) \)
  - local min of \( v - \phi \) and \( \phi(x^*) = v(x^*) \Rightarrow v(x) > \phi(x), x \neq x^* \)

- Then for \( x_t = x^* \), we have

\[
v(x_t) \geq \max_{\alpha} \Delta h(x_t, \alpha) + (1 - \rho \Delta)\phi(x_{t+\Delta})
\]

- Subtract \( (1 - \rho \Delta)\phi(x_t) \) from both sides and use \( \phi(x_t) = v(x_t) \)

\[
\Delta \rho v(x_t) \geq \max_{\alpha} \Delta h(x_t, \alpha) + (1 - \rho \Delta)(\phi(x_{t+\Delta}) - \phi(x_t))
\]

- Dividing by \( \Delta \) and letting \( \Delta \to 0 \) yields the supersolution condition

\[
\rho v(x_t) \geq \max_{\alpha} h(x_t, \alpha) + \phi'(x_t)f(x_t, \alpha)
\]
Viscosity Solutions: Heuristic Derivation

- Turns out this works for any $\phi$ such that $v - \phi$ has local min at $x^*$
  - define $\kappa = v(x^*) - \phi(x^*)$. Then $v(x^*) = \phi(x^*) + \kappa$ and
    $$v(x_t) \geq \max_{\alpha} \Delta h(x_t, \alpha_t) + (1 - \rho\Delta)(\phi(x_{t+\Delta}) + \kappa)$$
  - subtract $(1 - \rho\Delta)(\phi(x_t) + \kappa)$ from both sides
    $$\Delta \rho v(x_t) \geq \max_{\alpha} \Delta h(x_t, \alpha) + (1 - \rho\Delta)(\phi(x_{t+\Delta}) - \phi(x_t))$$
  - rest is the same...

- Derivation of subsolution condition exactly symmetric
Viscosity + maximization \(\Rightarrow\) no concave kinks

**Proposition**

The viscosity solution of the HJB equation corresponding to

\[
\rho v(x) - \max_{\alpha \in A} \left\{ h(x, \alpha) + v'(x) f(x, \alpha) \right\} = 0
\]

only admits convex (downward) kinks, but not concave (upward) kinks.

- think: not special to HJB, true for any \( v(x) = \max_y f(x, y) \)
- opposite would be true for minimization problem
Constrained Viscosity Soln: Boundary Inequalities

• How handle “state constraints”?
  • borrowing constraints
  • computations with bounded domain
• Example: growth model with state constraint

\[ v(k_0) = \max_{\{c(t)\}_{t \geq 0}} \int_0^\infty e^{-\rho t} u(c(t)) \, dt \quad \text{s.t.} \]
\[ \dot{k}(t) = F(k(t)) - \delta k(t) - c(t) \]
\[ k(t) \geq k_{\min} \quad \text{all } t \geq 0 \]

• purely pedagogical: constraint will never bind if \( k_{\min} < \text{st.st.} \)
• HJB equation

\[ \rho v(k) = \max_c \left\{ u(c) + v'(k)(F(k) - \delta k - c) \right\} \quad \text{(HJB)} \]

• Key question: how impose \( k \geq k_{\min} \)?
Example: Growth Model with Constraint

- **Result:** if \( v \) is differentiable at \( k_{\text{min}} \), it needs to satisfy
  \[
  v'(k_{\text{min}}) \geq u'(F(k_{\text{min}}) - \delta k_{\text{min}})
  \]  
  (BI)

- **Intuition:**
  - \( v'(k_{\text{min}}) \) is such that if \( k(t) = k_{\text{min}} \) then \( \dot{k}(t) \geq 0 \)
  - if \( v \) is differentiable, the FOC still holds at the constraint
    \[
    u'(c(k_{\text{min}})) = v'(k_{\text{min}})
    \]  
    (FOC)
  - for constraint not to be violated, need
    \[
    F(k_{\text{min}}) - \delta k_{\text{min}} - c(k_{\text{min}}) \geq 0
    \]  
    (*)
  - (FOC) and (*) \( \Rightarrow \) (BI).

- Next: rigorous derivation from “constrained viscosity solution”
Constrained Viscosity Soln: Rigorous Derivation

• Consider variant of generic maximization problem

\[ v(x) = \max_{\{\alpha(t)\}_{t \geq 0}} \int_0^\infty e^{-\rho t} h(x(t), \alpha(t)) dt \quad \text{s.t.} \]
\[ \dot{x}(t) = f(x(t), \alpha(t)), \quad x(0) = x \]
\[ x(t) \geq x_{\min} \quad \text{all } t \geq 0 \]

• HJB equation

\[ \rho v(x) = \max_{\alpha \in A} \left\{ h(x, \alpha) + v'(x)f(x, \alpha) \right\} \quad \text{(HJB)} \]

• Key question: how impose \( x \geq x_{\min} \)?

  • natural: \( v \) has to be a viscosity solution for \( x > x_{\min} \)
  • but what about at \( x = x_{\min} \)?
Constrained Viscosity Soln: Rigorous Derivation

• **Definition:** a constrained viscosity solution of (HJB) is a continuous function $v$ such that
  1. $v$ is a viscosity solution (i.e. both sub- and supersolution) for all $x > x_{\text{min}}$
  2. $v$ is a supersolution at $x = x_{\text{min}}$: if $\phi$ is any smooth function and if $v - \phi$ has a local minimum at point $x_{\text{min}}$, then
     \[ \rho v(x_{\text{min}}) \geq \max_{\alpha \in A} \left\{ h(x_{\text{min}}, \alpha) + \phi'(x_{\text{min}}) f(x_{\text{min}}, \alpha) \right\} \] (*)

• (*) functions as boundary condition, or rather “boundary inequality”

• More can be said in case $v$ is differentiable: next slide
Constrained Viscosity Soln: Rigorous Derivation

- Use Hamiltonian formulation:
  \[ H(x, p) := \max_{\alpha \in A} \ h(x, \alpha) + p f(x, \alpha) \]

- Note:
  \[ \partial_p H(x, v'(x)) = f(x_{\min}, \alpha^*(v'(x_{\min}))) = \text{optimal drift} \]

- Condition 2 becomes: if \( \phi \) is any smooth function and if \( v - \phi \) has a local minimum at point \( x_{\min} \), then
  \[ \rho v(x_{\min}) \geq H(x_{\min}, \phi'(x_{\min})) \] (**)  

- **Corollary**: if \( v \) is differentiable at \( x_{\min} \), then (**) is equivalent to
  \[ \partial_p H(x_{\min}, v'(x_{\min})) = \text{optimal drift at } x_{\min} \geq 0 \]

- **Proof**

- Again intuitive: \( v'(x_{\min}) \) is such that \( \dot{x}(t) \geq 0 \) if \( x(t) = x_{\min} \)
Example: Growth Model with Bounded Domain

• Consider again HJB equation for growth model

\[ \rho v(k) = \max_c \left\{ u(c) + v'(k)(F(k) - \delta k - c) \right\} \]

• For numerical solution, want to impose

\[ k_{\text{min}} \leq k(t) \leq k_{\text{max}} \quad \text{all } t \]

• How can we ensure this?

• Answer: impose two boundary inequalities

\[ v'(k_{\text{min}}) \geq u'(F(k_{\text{min}}) - \delta k_{\text{min}}) \]

\[ v'(k_{\text{max}}) \leq u'(F(k_{\text{max}}) - \delta k_{\text{max}}) \]
Uniqueness of Viscosity Solution

- **Theorem 1**: HJB equation has unique viscosity solution
- **Theorem 2**: viscosity solution equals value function, i.e. solution to “sequence problem”
- My intuition for **Theorem 1** in one-dimensional case
  - ODE has unique solution given one boundary condition
  - two boundary inequalities = one boundary condition
- but much more powerful: generalizes to \( n \) dimensions, kinks etc
Some general, somewhat philosophical thoughts

- MAT 101 way (“first-order ODE needs one boundary condition”) is **not** the right way to think about HJB equations

- these equations have very special structure which one should exploit when analyzing and solving them

- Particularly true for computations (next)

- Important: all results/algorithms apply to problems with more than one state variable, i.e. it doesn’t matter whether you solve ODEs or PDEs
Numerical Solution of HJB Equations
Finite Difference Methods

- See http://www.princeton.edu/~moll/HACTproject.htm

- Explain using neoclassical growth model, easily generalized to heterogeneous agent models

\[ \rho \nu(k) = \max_c u(c) + \nu'(k)(F(k) - \delta k - c) \]

- Functional forms

\[ u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad F(k) = k^\alpha \]

- Use finite difference method

- Two MATLAB codes

  http://www.princeton.edu/~moll/HACTproject/HJB_NGM.m

  http://www.princeton.edu/~moll/HACTproject/HJB_NGM_implicit.m
Barles-Souganidis

• There is a well-developed theory for numerical solution of HJB equation using finite difference methods

• Key paper: Barles and Souganidis (1991), “Convergence of approximation schemes for fully nonlinear second order equations”  

• **Result:** finite difference scheme “converges” to unique viscosity solution under three conditions
  
  1. monotonicity
  2. consistency
  3. stability

Finite Difference Approximations to $v'(k_i)$

- Approximate $v(k)$ at $I$ discrete points in the state space, $k_i, i = 1, \ldots, I$. Denote distance between grid points by $\Delta k$.

- Shorthand notation
  
  $$v_i = v(k_i)$$

- Need to approximate $v'(k_i)$.

- Three different possibilities:

  
  $$v'(k_i) \approx \frac{v_i - v_{i-1}}{\Delta k} = v_{i,B} \quad \text{backward difference}$$

  $$v'(k_i) \approx \frac{v_{i+1} - v_i}{\Delta k} = v_{i,F} \quad \text{forward difference}$$

  $$v'(k_i) \approx \frac{v_{i+1} - v_{i-1}}{2\Delta k} = v_{i,C} \quad \text{central difference}$$
Finite Difference Approximations to $v'(k_i)$
Finite Difference Approximation

FD approximation to HJB is

$$\rho v_i = u(c_i) + v'_i[F(k_i) - \delta k_i - c_i]$$

where $c_i = (u')^{-1}(v'_i)$, and $v'_i$ is one of backward, forward, central FD approximations.

Two complications:

1. which FD approximation to use? “Upwind scheme”
2. $(*)$ is extremely non-linear, need to solve iteratively: “explicit” vs. “implicit method”

My strategy for next few slides:

- first: what works
- afterwards: why it works (Barles-Souganidis)
Which FD Approximation?

- Which of these you use is extremely important
- Best solution: use so-called “upwind scheme.” Rough idea:
  - forward difference whenever drift of state variable positive
  - backward difference whenever drift of state variable negative
- In our example: define
  \[ s_{i,F} = F(k_i) - \delta k_i - (u')^{-1}(v'_{i,F}), \quad s_{i,B} = F(k_i) - \delta k_i - (u')^{-1}(v'_{i,B}) \]
- Approximate derivative as follows
  \[ v'_i = v'_{i,F} \mathbf{1}_{\{s_{i,F}>0\}} + v'_{i,B} \mathbf{1}_{\{s_{i,B}<0\}} + \bar{v}'_i \mathbf{1}_{\{s_{i,F}<0<s_{i,B}\}} \]
  where \( \mathbf{1}_{\{\cdot\}} \) is indicator function, and \( \bar{v}'_i = u'(F(k_i) - \delta k_i) \).
- Where does \( \bar{v}'_i \) term come from? Answer:
  - since \( v \) is concave, \( v'_{i,F} < v'_{i,B} \) (see figure) \( \Rightarrow s_{i,F} < s_{i,B} \)
  - if \( s'_{i,F} < 0 < s'_{i,B} \), set \( s_i = 0 \) \( \Rightarrow v'(k_i) = u'(F(k_i) - \delta k_i) \), i.e. we’re at a steady state.
Sparsity

- Discretized HJB equation is

\[ \rho v_i = u(c_i) + \frac{v_{i+1} - v_i}{\Delta k} s_{i,F}^+ + \frac{v_i - v_{i-1}}{\Delta k} s_{i,B}^- \]

- Notation: for any \( x \), \( x^+ = \max\{x, 0\} \) and \( x^- = \min\{x, 0\} \)

- Can write this in matrix notation

\[ \rho v = u + Av \]

where \( A \) is \( I \times I \) (\( I \) = no of grid points) and looks like...
Visualization of $\mathbf{A}$ (output of $\text{spy}(\mathbf{A})$ in Matlab)
The matrix $A$

- FD method approximates process for $k$ with discrete Poisson process, $A$ summarizes Poisson intensities
  
  - entries in row $i$:

    $$
    \begin{bmatrix}
    -\frac{s_{i,B}}{\Delta k} & \frac{s_{i,B}}{\Delta k} & \frac{s_{i,F}}{\Delta k} \\
    \text{inflow}_{i-1} \geq 0 & \text{outflow}_i \leq 0 & \text{inflow}_{i+1} \geq 0
    \end{bmatrix}
    \begin{bmatrix}
    v_{i-1} \\
    v_i \\
    v_{i+1}
    \end{bmatrix}
    $$

  - negative diagonals, positive off-diagonals, rows sum to zero:
  - tridiagonal matrix, very sparse

- $A$ depends on $\nu$ (nonlinear problem). Next: iterative method...
Iterative Method

• Idea: Solve FOC for given $v^n$, update $v^{n+1}$ according to

$$\frac{v_{i}^{n+1} - v_{i}^{n}}{\Delta} + \rho v_{i}^{n} = u(c_{i}^{n}) + (v^{n})'(k_{i})(F(k_{i}) - \delta k_{i} - c_{i}^{n}) \quad (*)$$

• Algorithm: Guess $v_{i}^{0}$, $i = 1, \ldots, l$ and for $n = 0, 1, 2, \ldots$ follow
  1. Compute $(v^{n})'(k_{i})$ using FD approx. on previous slide.
  2. Compute $c^{n}$ from $c_{i}^{n} = (u')^{-1}[(v^{n})'(k_{i})]$
  3. Find $v^{n+1}$ from $(*)$.
  4. If $v^{n+1}$ is close enough to $v^{n}$: stop. Otherwise, go to step 1.

• See http://www.princeton.edu/~moll/HACTproject/HJB_NGM.m

• Important parameter: $\Delta =$ step size, cannot be too large (“CFL condition”).

• Pretty inefficient: I need 5,990 iterations (though quite fast)
Efficiency: Implicit Method

- Efficiency can be improved by using an “implicit method”

\[
\frac{v_i^{n+1} - v_i^n}{\Delta} + \rho v_i^{n+1} = u(c_i^n) + (v_i^{n+1})'(k_i)[F(k_i) - \delta k_i - c_i^n]
\]

- Each step \( n \) involves solving a linear system of the form

\[
\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^{n+1} = u + A_n v^{n+1}
\]

\[
((\rho + \frac{1}{\Delta})I - A_n) v^{n+1} = u + \frac{1}{\Delta} v^n
\]

- but \( A_n \) is super sparse \( \Rightarrow \) super fast

- See http://www.princeton.edu/~moll/HACTproject/HJB_NGM_implicit.m

- In general: implicit method preferable over explicit method
  1. stable regardless of step size \( \Delta \)
  2. need much fewer iterations
  3. can handle many more grid points
Implicit Method: Practical Consideration

• In Matlab, need to explicitly construct $A$ as sparse to take advantage of speed gains

• Code has part that looks as follows

  $X = -\min(mub,0)/dk$;
  $Y = -\max(muf,0)/dk + \min(mub,0)/dk$;
  $Z = \max(muf,0)/dk$;

• Constructing full matrix – slow

  for $i=2:I-1$
    $A(i,i-1) = X(i);$  
    $A(i,i) = Y(i);$  
    $A(i,i+1) = Z(i);$  
  end
  $A(1,1)=Y(1)$;  $A(1,2) = Z(1)$;
  $A(I,I)=Y(I)$;  $A(I,I-1) = X(I)$;

• Constructing sparse matrix – fast

  $A = \text{spdiags}(Y,0,I,I)+\text{spdiags}(X(2:I),-1,I,I)+\text{spdiags}([0;Z(1:I-1)],1,I,I)$;
Why this works? Barles-Souganidis

• Here: version with one state variable, but generalizes

• Can write any HJB equation with one state variable as

\[ 0 = G(k, v(k), v'(k), v''(k)) \]  \hspace{1cm} (G)

• Corresponding FD scheme

\[ 0 = S(\Delta k, k_i, v_i; v_{i-1}, v_{i+1}) \]  \hspace{1cm} (S)

• Growth model

\[
G(k, v(k), v'(k), v''(k)) = \rho v(k) - \max_c u(c) + v'(k)(F(k) - \delta k - c)
\]

\[
S(\Delta k, k_i, v_i; v_{i-1}, v_{i+1}) = \rho v_i - u(c_i) - \frac{v_{i+1} - v_i}{\Delta k}(F(k_i) - \delta k_i - c_i)^+ \\
- \frac{v_i - v_{i-1}}{\Delta k}(F(k_i) - \delta k_i - c_i)^-
\]
1. **Monotonicity**: the numerical scheme is monotone, that is $S$ is non-increasing in both $v_{i-1}$ and $v_{i+1}$

2. **Consistency**: the numerical scheme is consistent, that is for every smooth function $v$ with bounded derivatives

\[
S (\Delta k, k_i, v(k_i); v(k_{i-1}), v(k_{i+1})) \rightarrow G(v(k), v'(k), v''(k))
\]

as $\Delta k \rightarrow 0$ and $k_i \rightarrow k$.

3. **Stability**: the numerical scheme is stable, that is for every $\Delta k > 0$, it has a solution $v_i$, $i = 1, .., I$ which is uniformly bounded independently of $\Delta k$. 
Why this works? Barles-Souganidis

Theorem (Barles-Souganidis)
If the scheme satisfies the monotonicity, consistency and stability conditions 1 to 3, then as $\Delta k \to 0$ its solution $v_i, i = 1, \ldots, l$ converges locally uniformly to the unique viscosity solution of (G)

- Note: “convergence” here has nothing to do with iterative algorithm converging to fixed point
- Instead: convergence of $v_i$ as $\Delta k \to 0$. More momentarily.
Intuition for Monotonicity

- Write \((S)\) as
  \[
  \rho v_i = \tilde{S}(\Delta k, k_i, v_i; v_{i-1}, v_{i+1})
  \]

- For example, in growth model
  \[
  \tilde{S}(\Delta k, k_i, v_i; v_{i-1}, v_{i+1}) = u(c_i) + \frac{v_{i+1} - v_i}{\Delta k} (F(k_i) - \delta k_i - c_i)^+ \\
  + \frac{v_i - v_{i-1}}{\Delta k} (F(k_i) - \delta k_i - c_i)^-
  \]

- Monotonicity: \(\tilde{S} \uparrow\) in \(v_{i-1}, v_{i+1}\) \(\iff\) \(S \downarrow\) in \(v_{i-1}, v_{i+1}\)

- Intuition: if my continuation value at \(i - 1\) or \(i + 1\) is larger, I must be at least as well off (i.e. \(v_i\) on LHS must be at least as high)
Checking the Monotonicity Condition in Growth Model

• Recall upwind scheme:

\[ S(\Delta k, k_i, v_i; v_{i-1}, v_{i+1}) = \rho v_i - u(c_i) - \frac{v_{i+1} - v_i}{\Delta k}(F(k_i) - \delta k_i - c_i)^+ \]

\[ - \frac{v_i - v_{i-1}}{\Delta k}(F(k_i) - \delta k_i - c_i)^- \]

• Can check: satisfies monotonicity: \( S \) is indeed non-increasing in both \( v_{i-1} \) and \( v_{i+1} \)

• \( c_i \) depends on \( v_i \)'s but doesn’t affect monotonicity due to envelope condition
Meaning of “Convergence”

Convergence is about $\Delta k \rightarrow 0$. What, then, is content of theorem?

- have a system of $l$ non-linear equations $S (\Delta k, k, v_i; v_{i-1}, v_{i+1}) = 0$
- need to solve it somehow
- Theorem guarantees that solution (for given $\Delta k$) converges to solution of the HJB equation ($G$) as $\Delta k$.

Why does iterative scheme work? Two interpretations:

1. **Newton method** for solving system of non-linear equations ($S$)
2. **Iterative scheme** $\Leftrightarrow$ **solve (HJB) backward in time**

\[
\frac{v_i^{n+1} - v_i^n}{\Delta} + \rho v_i^n = u(c_i^n) + (v^n)'(k_i)(F(k_i) - \delta k_i - c_i^n)
\]

in effect sets $v(k, T) = \text{initial guess}$ and solves

\[
\rho v(k, t) = \max_c u(c) + \partial_k v(k, t)(F(k) - \delta k - c) + \partial_t v(k, t)
\]

backwards in time. $v(k) = \lim_{t \rightarrow -\infty} v(k, t)$. 

Relation to Kushner-Dupuis “Markov-Chain Approx”

- There’s another common method for solving HJB equation: “Markov Chain Approximation Method”
  - effectively: convert to discrete time, use value fn iteration
- FD method not so different: also converts things to “Markov Chain”
  \[ \rho v = u + A v \]
- Connection between FD and MCAC
  - also shows how to exploit insights from MCAC to find FD scheme satisfying Barles-Souganidis conditions
- Another source of useful notes/codes: Frédéric Bonnans’ website
  http://www.cmap.polytechnique.fr/~bonnans/notes/edpfin/edpfin.html
Appendix
• Time periods of length $\Delta$

• Discount factor

$$\beta(\Delta) = e^{-\rho \Delta}$$

• Note that $\lim_{\Delta \to 0} \beta(\Delta) = 1$ and $\lim_{\Delta \to \infty} \beta(\Delta) = 0$.

• Discrete-time Bellman equation:

$$v(k_t) = \max_{c_t} \Delta u(c_t) + e^{-\rho \Delta} v(k_{t+\Delta}) \quad \text{s.t.}$$

$$k_{t+\Delta} = \Delta[F(k_t) - \delta k_t - c_t] + k_t$$
Derivation from Discrete-time Bellman

• For small $\Delta$ (will take $\Delta \to 0$), $e^{-\rho \Delta} \approx 1 - \rho \Delta$

  $$v(k_t) = \max_{c_t} \Delta u(c_t) + (1 - \rho \Delta) v(k_{t+\Delta})$$

• Subtract $(1 - \rho \Delta)v(k_t)$ from both sides

  $$\rho \Delta v(k_t) = \max_{c_t} \Delta u(c_t) + (1 - \Delta \rho)[v(k_{t+\Delta}) - v(k_t)]$$

• Divide by $\Delta$ and manipulate last term

  $$\rho v(k_t) = \max_{c_t} u(c_t) + (1 - \Delta \rho) \frac{v(k_{t+\Delta}) - v(k_t)}{k_{t+\Delta} - k_t} \frac{k_{t+\Delta} - k_t}{\Delta}$$

Take $\Delta \to 0$

  $$\rho v(k_t) = \max_{c_t} u(c_t) + v'(k_t) \dot{k}_t$$
• Work with CDF (in wealth dimension)

\[ G_i(a, t) = \Pr(\tilde{a}_t \leq a, \tilde{z}_t = z_i) \]

• Over time period of length \( \Delta \), wealth evolves as

\[ \tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta s_i(\tilde{a}_t) \]

• income switches from \( z_i \) to \( z_j \) with probability \( \Delta \lambda_i \)

• Fraction of people with wealth below \( a \) evolves as

\[
\Pr(\tilde{a}_{t+\Delta} \leq a, \tilde{z}_{t+\Delta} = z_i) = (1 - \Delta \lambda_i) \Pr(\tilde{a}_t \leq a - \Delta s_i(a), \tilde{z}_t = z_i) \\
+ \Delta \lambda_j \Pr(\tilde{a}_t \leq a - \Delta s_j(a), \tilde{z}_t = z_j)
\]

• Intuition: if have wealth \( < a - \Delta s_i(a) \) at \( t \), have wealth \( < a \) at \( t + \Delta \)
Derivation of Poisson KF Equation

• Subtracting $G_i(a, t)$ from both sides and dividing by $\Delta$

$$\frac{G_i(a, t + \Delta) - G_i(a, t)}{\Delta} = \frac{G_i(a - \Delta s_i(a), t) - G_i(a, t)}{\Delta}$$

$$- \lambda_i G_i(a - \Delta s_i(a), t) + \lambda_j G_j(a - \Delta s_j(a), t)$$

• Taking the limit as $\Delta \to 0$

$$\partial_t G_i(a, t) = -s_i(a)\partial_a G_i(a, t) - \lambda_i G_i(a, t) + \lambda_j G_j(a, t)$$

where we have used that

$$\lim_{\Delta \to 0} \frac{G_i(a - \Delta s_i(a), t) - G(a, t)}{\Delta} = \lim_{x \to 0} \frac{G(a - x, t) - G(a, t)}{x} s_i(a)$$

$$= -s_i(a)\partial_a G_i(a, t)$$

• Differentiate w.r.t. $a$ and use $g_i(a, t) = \partial_a G_i(a, t) \Rightarrow$ (KF).
Proof of State Constraint “Boundary Inequality”

• Recall condition 2: if \( \phi \) is any smooth function and if \( v - \phi \) has a local minimum at point \( x_{\text{min}} \), then

\[
\rho v(x_{\text{min}}) \geq H(x_{\text{min}}, \phi'(x_{\text{min}}))
\]

• If \( v \) is differentiable at \( x_{\text{min}} \), local min of \( v - \phi \) \( \Rightarrow (v - \phi)'(x_{\text{min}}) \geq 0 \)

\[
\Rightarrow \rho v(x_{\text{min}}) \geq H(x_{\text{min}}, p) \quad \text{all } p \leq v'(x_{\text{min}}) \quad (1)
\]

• Note: (1) true only at boundary \( x = x_{\text{min}} \)
  • interior: have \( v'(x) = \phi'(x) \Rightarrow \rho v(x) \geq H(x, p), p = v'(x) \)

• By continuity of \( v \) and \( v' \)

\[
\rho v(x_{\text{min}}) = H(x_{\text{min}}, v'(x_{\text{min}})) \quad (2)
\]

• Combining (1) and (2)

\[
H(x_{\text{min}}, v'(x_{\text{min}})) \geq H(x_{\text{min}}, p) \quad \text{all } p \leq v'(x_{\text{min}})
\]

• Equivalently \( \partial_p H(x_{\text{min}}, v'(x_{\text{min}})) \geq 0. \square \)