Mean Field Games in Economics
Part II

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GSSI Workshop, 31 August 2017, L’Aquila
Plan

Lecture 1

1. A benchmark MFG for macroeconomics: the Aiyagari-Bewley-Huggett (ABH) heterogeneous agent model
2. The ABH model with common noise (“Krusell-Smith”)
3. If time: some interesting extensions of the ABH model
   • the “wealthy hand-to-mouth” and marginal propensities to consume (MPCs)
   • present bias and self-control (economics meets psychology)

Lecture 2

1. Numerical solution of MFGs with common noise based on “When Inequality Matters for Macro…”
2. Other stuff…
Recall Stationary MFG, Aiyagari’s Variant

Functions $v$ and $g$ on $(a, \infty) \times (y, \bar{y})$ and scalar $r$ satisfy

$$\rho v = H(\partial_a v) + (wy + ra)\partial_a v + \mu(y)\partial_y v + \frac{\sigma^2(y)}{2}\partial_{yy} v$$  \hspace{1cm} (HJB)$$

where $H(p) := \max_{c \geq 0} \{u(c) - pc\}$, with state constraint $a \geq a$

and $0 = \partial_y v(a, y) = \partial_y v(a, \bar{y})$ all $a$

$$0 = -\partial_a((wy + ra + H'(\partial_a v))g) - \partial_y (\mu(y)g) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g)$$  \hspace{1cm} (FP)$$

$$1 = \int_0^\infty \int_a^\infty g \, da \, dy, \quad g \geq 0$$

$$r = e^Z \partial_K F(K, L) = \frac{1}{2} e^Z \sqrt{L/K}, \quad w = e^Z \partial_L F(K, L) = \frac{1}{2} e^Z \sqrt{K/L},$$

$$K = \int_0^\infty \int_a^\infty ag \, da \, dy, \quad L = \int_0^\infty \int_a^\infty yg \, da \, dy$$  \hspace{1cm} (EQ)$$

- Coupling through scalars $r$ and $w$ (prices) determined by (EQ)
- Algorithm: guess $(r, w)$, solve (HJB), solve (FP), check (EQ)
Macroeconomic MFGs with Common Noise

• **This is where the money is!**

• Can fit 90% of macroeconomics into this apparatus so any progress would be extremely valuable

• To understand setup consider Aiyagari (1994) with stochastic aggregate productivity, \( Z \), common to all firms

• First studied by
  
  • Per Krusell and Tony Smith (1998), ”Income and Wealth Heterogeneity in the Macroeconomy”, J of Political Economy
  

• Language: instead of "common noise" economists say "aggregate shocks" or "aggregate uncertainty"
Macroeconomic MFGs with Common Noise

- **Households:**

  \[
  \max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) \, dt \quad \text{s.t.}
  \]

  \[
  da_t = (w_t y_t + r_t a_t - c_t) \, dt
  \]

  \[
  dy_t = \mu(y_t) \, dt + \sigma(y_t) \, dW_t
  \]

  \[
  a_t \geq a
  \]

- **Firms:**

  \[
  \max_{K_t, L_t} \{ e^{Z_t} F(K_t, L_t) - r_t K_t - w_t L_t \} \]

  \[
  dZ_t = -\theta Z_t \, dt + \eta dB_t, \quad \text{common } B_t \text{ for all firms}
  \]

  \[
  \Rightarrow r_t = e^{Z_t} \partial_K F(K_t, L_t), \quad w_t = e^{Z_t} \partial_L F(K_t, L_t)
  \]

- **Equilibrium:**

  \[
  L_t = \int_{0}^{\infty} \int_{a}^{\infty} y g(a, y, t) \, da \, dy, \quad K_t = \int_{0}^{\infty} \int_{a}^{\infty} a g(a, y, t) \, da \, dy
  \]
Macroeconomic MFGs with Common Noise

• Households:

\[
\max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) \, dt \quad \text{s.t.}
\]
\[
da_t = (w_t y_t + r_t a_t - c_t) \, dt
\]
\[
dy_t = \mu(y_t) \, dt + \sigma(y_t) \, dW_t
\]
\[
a_t \geq a
\]

• Firms:

\[
\max_{K_t, L_t} \left\{ e^{Z_t} F(K_t, L_t) - r_t K_t - w_t L_t \right\}
\]
\[
dZ_t = -\theta Z_t \, dt + \eta dB_t, \quad \text{common } B_t \text{ for all firms}
\]
\[
\Rightarrow r_t = e^{Z_t} \partial_K F(K_t, L_t), \quad w_t = e^{Z_t} \partial_L F(K_t, L_t)
\]

• Equilibrium if restrict to stationary \(y\)-process with 1st moment = 1:

\[
L_t = 1, \quad K_t = \int_0^\infty \int_a^\infty ag(a, y, t) \, da \, dy
\]
MFG System with Common Noise

- both $g_t$ and $v_t$ are now random variables
- dynamic programming notation w.r.t. individual states only
- $\mathbb{E}_t$ is conditional expectation w.r.t. future $(g_t, Z_t)$

\[
\rho v_t(a, y) = H(\partial_a v_t(a, y)) + \partial_a v_t(a, y)(w_t y + r_t a) + \mu(y)\partial_y v_t(a, y) + \frac{\sigma^2(y)}{2}\partial_{yy} v_t(a, y) + \frac{1}{dt}\mathbb{E}_t [dv_t(a, y)],
\]

\[
\partial_t g_t(a, y) = -\partial_a [(w_t y + r_t a + H'(\partial_a v_t(a, y)))g_t(a, y)] - \partial_y (\mu(y)g_t(a, y)) + \frac{1}{2}\partial_{yy} (\sigma^2(y)g_t(a, y)),
\]

\[
w_t = \frac{1}{2}e^{Z_t} \sqrt{1/K_t}, \quad r_t = \frac{1}{2}e^{Z_t} \sqrt{K_t}, \quad K_t = \int a g_t(a, y) \, da \, dy
\]

\[
dZ_t = -\theta Z_t \, dt + \eta dB_t
\]

Note: $\frac{1}{dt}\mathbb{E}_t [dv_t]$ means $\lim_{s \downarrow 0} \mathbb{E}_t [v_{t+s} - v_t]/s$ – sorry if weird notation
Analogous System for Textbook MFG

• See Cardialaguet-Delarue-Lasry-Lions
  https://arxiv.org/abs/1509.02505

• Standard MFG with common noise $W_t$

$$dX_{i,t} = \ldots + \sqrt{2}dB_{i,t} + \sqrt{2\beta}dW_t$$

• See their equation (8) for MFG system with common noise:

$$
\begin{cases}
  dtu_t = \left\{- (1 + \beta)\Delta u_t + H(x, Du_t) - F(x, m_t) - \sqrt{2\beta}\text{div}(v_t)\right\}dt + v_t \cdot \sqrt{2\beta}dW_t \\
  in \ [0, T] \times \mathbb{T}^d, \\
  dtm_t = \left\{ (1 + \beta)\Delta m_t + \text{div}(m_tD_pH(m_t, Du_t)) \right\}dt - \text{div}(m_t\sqrt{2\beta}dW_t), \\
  in \ [0, T] \times \mathbb{T}^d, \\
  u_T(x) = G(x, m_T), \ m_0 = m(0), \ in \ \mathbb{T}^d
\end{cases}
$$

• “where the map $v_t$ is a random vector field that forces the solution $u_t$ of the backward equation to be adapted to the filtration generated by $(W_t)_{t \in [0, T]}$”

• Previous slide is my sloppy version of this for my particular model
Today

• A computational method for MFGs with common noise, based on “When Inequality Matters for Macro…”

• Idea: linearize MFG with common noise $Z_t$ around MFG without common noise $Z_t = 0$

• Works beautifully in practice and in many different applications

• But we have no idea about the underlying mathematics!

• $\Rightarrow$ Great problem for mathematicians

• Today: will do in terms of our specific example (Krusell-Smith)

• Good exercise for you: work this out for equation (8) in Cardialaguet-Delarue-Lasry-Lions
Warm-Up: Linearizing Economic Models

- Economists often solve dynamic economic models using linearization methods

- Explain in context of particularly basic macroeconomic model: “neoclassical growth model”
  - for the moment: no heterogeneity, “representative agent”

\[
\max_{\{c_t\}_{t \geq 0}} \int_{0}^{\infty} e^{-\rho t} u(c_t) \, dt \quad \text{s.t.} \quad \dot{k}_t = f(k_t) - c_t, \quad k_t \geq 0, \quad c_t \geq 0
\]

- \(c_t\): consumption
- \(u\): utility function, \(u' > 0, u'' < 0\)
- \(\rho\): discount rate
- \(k_t\): capital stock, \(k_0 = \bar{k}_0\) given
- \(f\): production function, \(f' > 0, f'' < 0, f'(\infty) < \rho < f'(0)\)

- Interpretation: a fictitious “social planner” decides how to allocate production \(f(k_t)\) between consumption \(c_t\) and investment \(\dot{k}_t\)
Warm-Up: Linearizing Economic Models

- You can obviously solve this problem numerically from the HJB equation: value function $v$ satisfies
  \[ \rho v(k) = \max_{c \geq 0} u(c) + v'(k)(f(k) - c) \quad \text{on} \quad (0, \infty) \]
- But suppose you don’t want to do this for some reason
  - e.g. don’t know finite difference methods
  - or want to know more about optimal $k_t$
- Can proceed as follows: differentiate HJB equation w.r.t. $k$
  \[ v''(k)(f(k) - c(k)) = (\rho - f'(k))v'(k) \]
- Define $\nu_t = v'(k_t)$, evaluate along characteristic $\dot{k}_t = f(k_t) - c_t$
  \[ \dot{\nu}_t = (\rho - f'(k_t))\nu_t \]
  \[ \dot{k}_t = f(k_t) - (u')^{-1}(\nu_t) \]
- $(\nu_t, k_t)$ satisfy two ODEs with initial condition $k_0 = \bar{k}_0$, and can also derive terminal condition: $\lim_{t \to \infty} e^{-\rho t} \nu_t k_t = 0$
Warm-Up: Linearizing Economic Models

- Recall \((\nu_t, k_t)\) satisfy two ODEs

\[
\begin{align*}
\dot{\nu}_t &= (\rho - f'(k_t))\nu_t \\
\dot{k}_t &= f(k_t) - (u')^{-1}(\nu_t)
\end{align*}
\]

(ODEs)

with \(k_0 = \bar{k}_0\), \(\lim_{t \to \infty} e^{-\rho t} \nu_t k_t = 0\)

(BOUNDARY)

- Unique stationary \((\nu^*, k^*)\) satisfying \(f'(k^*) = \rho, \nu^* = u'(f(k^*))\)

- To understand dynamics: first-order expansion around \((\nu^*, k^*)\)

\[
\begin{bmatrix}
\hat{\nu}_t \\
\hat{k}_t
\end{bmatrix}
\approx
\begin{bmatrix}
-\frac{1}{u''(c^*)} & -f''(k^*)\nu^* \\
-\frac{1}{u''(c^*)} & \rho
\end{bmatrix}
\begin{bmatrix}
\hat{\nu}_t \\
\hat{k}_t
\end{bmatrix},
\begin{bmatrix}
\hat{\nu}_t \\
\hat{k}_t
\end{bmatrix} := \begin{bmatrix}
\nu_t - \nu^* \\
k_t - k^*
\end{bmatrix}
\]

\(\mathbf{B}\)

- Easy to show: eigenvalues \((\lambda_1, \lambda_2)\) of \(\mathbf{B}\) are real, \(\lambda_1 < 0 < \lambda_2\)

\[
\Rightarrow \begin{bmatrix}
\hat{\nu}_t \\
\hat{k}_t
\end{bmatrix} \approx c_1 e^{\lambda_1 t} \phi_1 + c_2 e^{\lambda_2 t} \phi_2, \quad \phi_j \in \mathbb{R}^2 = \text{eigenvectors}
\]

- constants \((c_1, c_2)\) pinned down from (BOUNDARY) \(\Rightarrow\) need \(c_2 = 0\)
Warm-Up: Linearizing Economic Models

- Linearization strategy also works with common noise. Consider
\[
\max_{\{c_t\}_{t \geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.}
\]
\[
\dot{k}_t = e^{Z_t} f(k_t) - c_t, \quad dZ_t = -\theta Z_t dt + \eta dB_t = \text{common noise}
\]

- Value function \( v(k, Z) \). Differentiate with respect to \( k \):
\[
(\rho - e^Z f'(k)) \partial_k v = (e^Z f(k) - c(k, Z)) \partial_{kk} v - \theta Z \partial_k Z v + \frac{\eta^2}{2} \partial_k ZZ v
\]

- Define \( \nu_t := \partial_k v(k_t, Z_t) \). Then Ito’s formula yields:
\[
d\nu_t = b(k_t, Z_t) dt + \eta \partial_{kZ} v(k_t, Z_t) dB_t
\]
\[
b(k_t, Z_t) := (e^{Z_t} f(k_t) - c_t) \partial_{kk} v(k_t, Z_t) - \theta Z_t \partial Z v(k_t, Z_t) + \frac{\eta^2}{2} \partial_{kZ} ZZ v(k_t, Z_t)
\]
\[
\Rightarrow \quad \nu_{t+s} - \nu_t = \int_t^{t+s} b(k_u, Z_u) du + \eta \int_t^{t+s} \partial_{kZ} v(k_u, Z_u) dB_u
\]
expanding right-hand side terms
\[
\Rightarrow \quad \lim_{s \downarrow 0} \frac{1}{s} \mathbb{E}_t[\nu_{t+s} - \nu_t] = b(k_t, Z_t)
\]
Warm-Up: Linearizing Economic Models

- Recall
  \[(\rho - e^Z f'(k))\partial_k \nu = (e^Z f(k) - c(k, Z))\partial_{kk} \nu - \theta Z \partial_k Z \nu + \frac{\eta^2}{2} \partial_{kZ} Z \nu\]

- Evaluate along characteristic \((k_t, Z_t)\) using previous slide
  \[
  \mathbb{E}_t[d\nu_t] = (\rho - e^{Z_t} f'(k_t))dt
  
  dk_t = e^{Z_t} f(k_t) - (u')^{-1}(\nu_t)
  
  dZ_t = -\theta Z_t dt + \eta dB_t
  \]
  with \(k_0 = \bar{k}_0, Z_0 = \bar{Z}_0\) and a terminal condition for \(\nu_t\) (in expect.)

- Expansion around stationary point w/o common noise \((\nu^*, k^*, 0):\)
  \[
  \begin{bmatrix}
  \mathbb{E}_t[d\hat{\nu}_t] \\
  d\hat{k}_t \\
  dZ_t
  \end{bmatrix} 
  \approx \begin{bmatrix}
  \hat{\nu}_t \\
  \hat{k}_t \\
  Z_t
  \end{bmatrix} \begin{bmatrix}
  0 \\
  \eta
  \end{bmatrix} dB_t,
  \begin{bmatrix}
  \hat{\nu}_t \\
  \hat{k}_t \\
  Z_t
  \end{bmatrix} = \begin{bmatrix}
  \nu_t - \nu^* \\
  k_t - k^* \\
  Z_t - 0
  \end{bmatrix}
  \]

- Can show: \(\mathcal{B} \in \mathbb{R}^{3 \times 3}\) has real eigenvalues \(\lambda_1 \leq \lambda_2 < 0 < \lambda_3 \Rightarrow\)
  system of SDEs has unique sol’n satisfying boundary conditions

- Impulse response functions (IRFs): \((\hat{\nu}_t, \hat{k}_t, Z_t), t \geq 0\) after \(dB_0 = 1\)
IRF to A Technological Shock
IRF to A Technological Shock

Output

Consumption

Investment

Capital

Hours worked

Labor productivity (Wages)
A good fit with estimated shocks
Real Business Cycle (RBC) Model

• Aside: this model (neoclassical growth model + common noise in productivity $Z_t$) with addition of hours worked choice is called the “Real Business Cycle” (RBC) model

• fits aggregate data surprisingly well

• Finn Kydland and Ed Prescott got a Nobel prize for it

• what’s a negative “technology shock”? Do we suddenly forget how to produce stuff?

• one example is oil price shock, but technology shocks probably a bit of a stretch
Summary of Linearization Method

1. Compute **stationary point without common noise**

2. Compute **first-order Taylor expansion** around stationary point without common noise

3. Solve linear stochastic **differential equations**
Key idea: same strategy in MFG with common noise

1. Compute **stationary MFG without common noise**

2. Compute **first-order Taylor expansion** around stationary MFG without common noise

3. Solve linear stochastic **differential equations**
MFG System with Common Noise

Recall MFG System with Common Noise

\[ \rho v_t(a, y) = H(\partial_a v_t(a, y)) + \partial_a v_t(a, y)(w_t y + r_t a) \]

\[ + \mu(y) \partial_y v_t(a, y) + \frac{\sigma^2(y)}{2} \partial_{yy} v_t(a, y) + \frac{1}{dt} \mathbb{E}_t [dv_t(a, y)], \]

(HJB)

\[ \partial_t g_t(a, y) = - \partial_a [(w_t y + r_t a + H'(\partial_a v_t(a, y)))g_t(a, y)] \]

\[ - \partial_y (\mu(y)g_t(a, y)) + \frac{1}{2} \partial_{yy} (\sigma^2(y)g_t(a, y)), \]

(KF)

\[ w_t = \frac{1}{2} e^{Z_t} \sqrt{1/K_t}, \quad r_t = \frac{1}{2} e^{Z_t} \sqrt{K_t}, \quad K_t = \int ag_t(a, y) \, da \, dy \]

\[ dZ_t = -\theta Z_t \, dt + \eta dB_t \]
Linearization and Discretization: Which Order?

• Numerical solution method has two components
  • linearization (first-order Taylor expansion) around MFG without common noise
  • discretization of \((v, g)\) via finite difference method

• What we do:
  1. discretization
  2. linearization

  Reason: don’t understand linearized infinite-dimensional system

• What one probably should do:
  1. linearization
  2. discretization

  i.e. analyze linearized infinite-dimensional system before discretizing and putting on computer
Interesting Exercise

• Start with equation (8) in Cardialaguet-Delarue-Lasry-Lions
  https://arxiv.org/abs/1509.02505

\[
\begin{cases}
  d_t u_t = \left\{ -(1 + \beta) \Delta u_t + H(x, Du_t) - F(x, m_t) - \sqrt{2\beta} \text{div}(v_t) \right\} dt + v_t \cdot \sqrt{2\beta} dW_t \\
  & \text{in } [0, T] \times \mathbb{T}^d, \\
  d_t m_t = \left[ (1 + \beta) \Delta m_t + \text{div}(m_tD \phi H(m_t, Du_t)) \right] dt - \text{div}(m_t \sqrt{2\beta} dW_t), \\
  & \text{in } [0, T] \times \mathbb{T}^d, \\
  u_T(x) = G(x, m_T), \ m_0 = m_{(0)}, & \text{in } \mathbb{T}^d
\end{cases}
\]

• Linearize this system around stationary MFG with $\beta = 0$

\[
\begin{cases}
  0 = -\Delta u + H(x, Du) & \text{in } \mathbb{T}^d \\
  0 = -\Delta m + \text{div}(mD \phi H(x, Du)) & \text{in } \mathbb{T}^d
\end{cases}
\]
Linearization: Three Steps

1. Compute stationary MFG without common noise

2. Compute first-order Taylor expansion around stationary MFG without common noise

3. Solve linear stochastic differential equations
Step 1: Compute stationary MFG w/o common noise

\[ \rho v = H(\partial_a v) + (wy + ra)\partial_a v + \mu(y)\partial_y v + \frac{\sigma^2(y)}{2} \partial_{yy} v \]  \hspace{1cm} (HJB*)

\[ 0 = -\partial_a((wy + ra + H'(\partial_a v))g) - \partial_y(\mu(y)g) + \frac{1}{2} \partial_{yy}(\sigma^2(y)g) \]  \hspace{1cm} (FP*)

\[ r = \frac{1}{2}\sqrt{1/K}, \quad w = \frac{1}{2}\sqrt{K}, \quad K = \int_0^\infty \int_{-\infty}^\infty agdady \]  \hspace{1cm} (EQ*)
Step 1: Compute stationary MFG w/o common noise

Compute using finite difference method, notation: $\partial_a v(a_i, y_j) \approx \partial_a v_{i,j}$

$$\rho v_{i,j} = H(\partial_a v_{i,j}) + (wy_j + ra_i)\partial_a v_{i,j} + \mu(y_j)\partial_y v_{i,j} + \frac{\sigma^2(y_j)}{2} \partial_{yy} v_{i,j} \quad \text{(HJB*)}$$

$$0 = -\partial_a ((wy + ra + H'(\partial_a v))g) - \partial_y (\mu(y)g) + \frac{1}{2} \partial_{yy} (\sigma^2(y)g) \quad \text{(FP*)}$$

$$r = \frac{1}{2} \sqrt{1/K}, \quad w = \frac{1}{2} \sqrt{K}, \quad K = \int_0^\infty \int_a^\infty ag \, dady \quad \text{(EQ*)}$$
Step 1: Compute stationary MFG w/o common noise

Compute using finite difference method, notation: \( \mathbf{v} = (v_{1,1}, \ldots, v_{I,J})^T \)

\[
\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + A(\mathbf{v}; \mathbf{p}) \mathbf{v}, \quad \mathbf{p} := (r, w)
\]  

\[
(HJB^*)
\]

\[
0 = - \partial_a ((wy + ra + H'(\partial_a v))g) - \partial_y(\mu(y)g) + \frac{1}{2} \partial_{yy}(\sigma^2(y)g) 
\]

\[
(FP^*)
\]

\[
r = \frac{1}{2} \sqrt{1/K}, \quad w = \frac{1}{2} \sqrt{K}, \quad K = \int_0^\infty \int_a^\infty agdady 
\]  

\[
(EQ^*)
\]
Step 1: Compute stationary MFG w/o common noise

Compute using finite difference method, notation: \( g = (g_{1,1}, \ldots, g_{l,j})^T \)

\[
\rho v = u(v) + A(v; p) v \quad \text{(HJB\textsuperscript{*})}
\]

\[
0 = A(v; p)^T g \quad \text{(FP\textsuperscript{*})}
\]

\[
r = \frac{1}{2} \sqrt{1/K}, \quad w = \frac{1}{2} \sqrt{K}, \quad K = \int_0^\infty \int_a^\infty agdady \quad \text{(EQ\textsuperscript{*})}
\]
Step 1: Compute stationary MFG w/o common noise

Compute using finite difference method

\[ \rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; \mathbf{p}) \mathbf{v} \quad \text{(HJB*)} \]

\[ 0 = \mathbf{A}(\mathbf{v}; \mathbf{p})^T \mathbf{g} \quad \text{(FP*)} \]

\[ \mathbf{p} = \mathbf{F}(\mathbf{g}) \quad \text{(EQ*)} \]
Linearization: Three steps

1. Compute stationary MFG without common noise
   - Yves’ finite difference method
   - stationary MFG reduces to sparse matrix equations

2. **Compute first-order Taylor expansion** around stationary MFG without common noise
   - use **automatic differentiation** routine

3. Solve linear stochastic **differential equation**
Step 2: Linearize discretized system w common noise

- Discretized system with common noise

\[
\rho v_t = u(v_t) + A(v_t;p_t) v_t + \frac{1}{dt} E_t [d v_t]
\]

\[
\frac{d g_t}{dt} = A(v_t;p_t)^T g_t
\]

\[
p_t = F(g_t; Z_t)
\]

\[
dZ_t = -\theta Z_t dt + \eta dB_t
\]
Step 2: Linearize discretized system w common noise

- Discretized system with common noise

\[ \rho v_t = u(v_t) + A(v_t;p_t) v_t + \frac{1}{dt} \mathbb{E}_t[dv_t] \]

\[ \frac{dg_t}{dt} = A(v_t;p_t)^T g_t \]

\[ p_t = F(g_t; Z_t) \]

\[ dZ_t = -\theta Z_t dt + \eta dB_t \]

- Structure basically the same as

\[ \mathbb{E}_t[d\nu_t] = (\rho - e^{Z_t} f'(k_t)) dt \]

\[ dk_t = e^{Z_t} f(k_t) - (u')^{-1}(\nu_t) \]

\[ dZ_t = -\theta Z_t dt + \eta dB_t \]

from warm-up exercise
Step 2: Linearize discretized system w common noise

- Discretized system with common noise

\[
\begin{align*}
\rho v_t &= u(v_t) + A(v_t; p_t) v_t + \frac{1}{d} E_t[dv_t] \\
\frac{dg_t}{dt} &= A(v_t; p_t) g_t \\
p_t &= F(g_t; Z_t) \\
dZ_t &= -\theta Z_t dt + \eta dB_t
\end{align*}
\]

- ... which we linearized as

\[
\begin{bmatrix}
E_t[d\hat{v}_t] \\
d\hat{k}_t \\
dZ_t
\end{bmatrix} \approx B \begin{bmatrix}
\hat{v}_t \\
\hat{k}_t \\
Z_t
\end{bmatrix} dt + 
\begin{bmatrix}
0 \\
0 \\
\eta
\end{bmatrix} dB_t,
\begin{bmatrix}
\hat{v}_t \\
\hat{k}_t \\
Z_t
\end{bmatrix} = 
\begin{bmatrix}
\nu_t - \nu^* \\
k_t - k^* \\
Z_t - 0
\end{bmatrix}
\]
Step 2: Linearize discretized system w common noise

- Discretized system with common noise

\[
\rho v_t = u(v_t) + A(v_t; p_t) v_t + \frac{1}{dt} \mathbb{E}_t[dv_t]
\]

\[
\frac{dg_t}{dt} = A(v_t; p_t)^T g_t
\]

\[
p_t = F(g_t; Z_t)
\]

\[
dZ_t = -\theta Z_t dt + \eta dB_t
\]

- \( \Rightarrow \) Linearize in analogous fashion (using automatic differentiation)

\[
\begin{bmatrix}
\mathbb{E}_t[d\hat{v}_t] \\
d\hat{g}_t \\
0 \\
dZ_t
\end{bmatrix} =
\begin{bmatrix}
B_{vv} & 0 & B_{vp} & 0 \\
B_{gv} & B_{gg} & B_{gp} & 0 \\
0 & B_{pg} & -I & B_{pZ} \\
0 & 0 & 0 & -\theta
\end{bmatrix}
\begin{bmatrix}
\hat{v}_t \\
\hat{g}_t \\
\hat{p}_t \\
Z_t
\end{bmatrix} dt +
\begin{bmatrix}
0 \\
0 \\
0 \\
\eta
\end{bmatrix} dB_t
\]
Step 2: Linearize discretized system w common noise

- Discretized system with common noise

\[
\rho v_t = u(v_t) + A(v_t; p_t) v_t + \frac{1}{dt} \mathbb{E}_t [d v_t]
\]
\[
\frac{d g_t}{d t} = A(v_t; p_t) \mathbb{g}_t
\]
\[
p_t = F(g_t; Z_t)
\]
\[
dZ_t = -\theta Z_t dt + \eta dB_t
\]

- Can simplify further by eliminating \( \hat{p}_t \)

\[
\begin{bmatrix}
\mathbb{E}_t [d \hat{v}_t] \\
\hat{g}_t \\
dZ_t
\end{bmatrix} =
\begin{bmatrix}
B_{v v} & B_{vp} B_{pg} & B_{vp} B_{pZ} \\
B_{gv} & B_{gg} + B_{gp} B_{pg} & B_{gp} B_{pZ} \\
0 & 0 & -\theta
\end{bmatrix}
\begin{bmatrix}
\hat{v}_t \\
\hat{g}_t \\
Z_t
\end{bmatrix} dt +
\begin{bmatrix}
0 \\
0 \\
\eta
\end{bmatrix} d B_t
\]

Only difference to \( (\hat{v}_t, \hat{k}_t, Z_t) \) system: dimensionality

- rep agent model: dimension 3
- MFG: \( 2 \times N + 1, N = I \times J \), e.g. = 2001 if \( I = 50, J = 20 \)
Linearization: Three steps

1. Compute stationary MFG without common noise
   - Yves’ finite difference method
   - stationary MFG reduces to sparse matrix equations

2. Compute first-order Taylor expansion around stationary MFG without common noise
   - use automatic differentiation routine

3. Solve linear stochastic differential equation
   - moderately-sized systems ⇒ can diagonalize system, compute eigenvalues (typically $N + 1$ are < 0)
   - large systems, e.g. two-asset model from Lecture 1
     ⇒ dimensionality reduction
Dimensionality Reduction in Step 3

- Use tools from engineering literature: “Model reduction”

- Approximate $N$-dimensional distribution by projecting onto $k$-dimensional subspace of $\mathbb{R}^N$ with $k \ll N$
  
  $$g_t \approx \gamma_1 t x_1 + \ldots + \gamma_k t x_k$$

- Adapt to problems with forward-looking decisions

- For details, see “When Inequality Matters for Macro...”
IRFs in Krusell & Smith Model

- Comparison of full distribution vs. $k = 1$ approximation

  $\implies$ recovers Krusell & Smith’s result: ok to work with 1D object
IRFs in Krusell & Smith Model

- Instead two-asset model from Lecture 1 requires $k = 300$
  $\implies$ not ok to work with 1D object
Our Method Is Fast, Accurate in Krusell & Smith Model

Our method is fast

<table>
<thead>
<tr>
<th></th>
<th>w/o Reduction</th>
<th>w/ Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady State</td>
<td>0.082 sec</td>
<td>0.082 sec</td>
</tr>
<tr>
<td>Linearize</td>
<td>0.021 sec</td>
<td>0.021 sec</td>
</tr>
<tr>
<td>Reduction</td>
<td>×</td>
<td>0.007 sec</td>
</tr>
<tr>
<td>Solve</td>
<td>0.14 sec</td>
<td>0.002 sec</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0.243 sec</td>
<td>0.112 sec</td>
</tr>
</tbody>
</table>

- JEDC comparison project (2010): fastest alternative \( \approx 7 \) minutes

Our method is accurate

<table>
<thead>
<tr>
<th></th>
<th>0.01%</th>
<th>0.1%</th>
<th>0.7%</th>
<th>1%</th>
<th>5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common noise ( \eta )</td>
<td>0.01%</td>
<td>0.1%</td>
<td>0.7%</td>
<td>1%</td>
<td>5%</td>
</tr>
<tr>
<td>Den Haan Error</td>
<td>0.000%</td>
<td>0.002%</td>
<td>0.053%</td>
<td>0.135%</td>
<td>3.347%</td>
</tr>
</tbody>
</table>

- JEDC comparison project: most accurate alternative \( \approx 0.16\% \)
Linearizing MFGs with Common Noise: Summary

- Method works beautifully in practice ...
- ... and in many applications
- But we don’t understand underlying mathematics
- Great problem for mathematicians!
- Again, from economists’ point of view, MFGs with common noise is where the money is
- Probably want to switch order:
  1. linearize ...
  2. ... then discretize and put on computer
Conclusion

• Mean field games extremely useful in economics...

• ... lots of exciting questions involve mean field type interactions...

• ... but mathematics often pretty challenging, at least for the average economist

• Potentially high payoff from mathematicians working on this!

• Questions? Come talk to me or shoot me an email moll@princeton.edu