Plan for Today’s Lecture

1. Numerical solution of Huggett model

2. Background: diffusion processes, HJB equations, Kolmogorov forward equations

   - Do income and wealth distribution matter for the macroeconomy?
   - better way (IMHO) of doing Krusell and Smith (1998)
Numerical Solution: Finite Difference Methods
Finite Difference Methods

- See http://www.princeton.edu/~moll/HACTproject.htm
- Explain HJB using neoclassical growth model, easily generalized to Huggett model

\[ \rho V(k) = \max_c U(c) + V'(k)[F(k) - \delta k - c] \]

- Functional forms

\[ U(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad F(k) = k^\alpha \]

- Use finite difference method. MATLAB code HJB_NGM.m
- Approximate \( V(k) \) at \( l \) discrete points in the state space, \( k_i, i = 1, ..., l \). Denote distance between grid points by \( \Delta k \).
- Shorthand notation

\[ V_i = V(k_i) \]
Finite Difference Approximations to $V'(k_i)$

- Need to approximate $V'(k_i)$.

- Three different possibilities:

\[
V'(k_i) \approx \frac{V_i - V_{i-1}}{\Delta k} = V'_{i,B} \quad \text{backward difference}
\]

\[
V'(k_i) \approx \frac{V_{i+1} - V_i}{\Delta k} = V'_{i,F} \quad \text{forward difference}
\]

\[
V'(k_i) \approx \frac{V_{i+1} - V_{i-1}}{2\Delta k} = V'_{i,C} \quad \text{central difference}
\]
Finite Difference Approximations to $V'(k_i)$
Finite Difference Approximation

• FD approximation to HJB is

\[ \rho V_i = U(c_i) + V'_i[F(k_i) - \delta k_i - c_i] \]  

(*)

where \( c_i = (U')^{-1}(V'_i) \), and \( V'_i \) is one of backward, forward, central FD approximations.

• Two complications:
  1. which FD approximation to use? “Upwind scheme”
  2. (*) is extremely non-linear, need to solve iteratively: “explicit” vs. “implicit method”
Which FD Approximation?

• Which of these you use it **extremely important**
• Correct way: use so-called **“upwind scheme.”** Rough idea:
  • **forward** difference whenever drift of state variable **positive**
  • **backward** difference whenever drift of state variable **negative**

• In our example: define
  \[ s_i, F = F(k_i) - \delta k_i - (U')^{-1}(V'_{i,F}), \quad s_i, B = F(k_i) - \delta k_i - (U')^{-1}(V'_{i,B}) \]

• Approximate derivative as follows
  \[ V'_i = V'_{i,F} 1\{s_i,F>0\} + V'_{i,B} 1\{s_i,B<0\} + \tilde{V}'_i 1\{s_i,F<0<s_i,B\} \]
  where \( 1\{\cdot\} \) is indicator function, and \( \tilde{V}'_i = U'(F(k_i) - \delta k_i) \).

• Where does \( \tilde{V}'_i \) term come from? Answer:
  • since \( V \) is concave, \( V'_{i,F} < V'_{i,B} \) (see figure) \( \Rightarrow s_{i,F} < s_{i,B} \)
  • if \( s'_{i,F} < 0 < s'_{i,B} \), set \( s_i = 0 \) \( \Rightarrow V'(k_i) = U'(F(k_i) - \delta k_i) \), i.e.
    we’re at a steady state.

• Note: importantly avoids using the points \( i = 0 \) and \( i = I + 1 \)
Iterative Method

- **Idea:** Solve FOC for given $V^n$, update $V^{n+1}$ according to
  \[
  \frac{V^{n+1}_i - V^n_i}{\Delta} + \rho V^n_i = U(c^n_i) + (V^n)'(k_i)[F(k_i) - \delta k_i - c^n_i] \quad (\ast)
  \]

- **Algorithm:** Guess $V^0_i$, $i = 1, \ldots, I$ and for $n = 0, 1, 2, \ldots$ follow
  1. Compute $(V^n)'(k_i)$ using FD approx. on previous slide.
  2. Compute $c^n$ from $c^n_i = (U')^{-1}[(V^n)'(k_i)]$
  3. Find $V^{n+1}$ from $(\ast)$.
  4. If $V^{n+1}$ is close enough to $V^n$: stop. Otherwise, go to step 1.

- See MATLAB code
  
  http://www.princeton.edu/~moll/HACTproject/HJB_NGM.m

- Important parameter: $\Delta = $ step size, cannot be too large
  (“CFL condition”).
Do we know this converges? **yes if**

1. upwind scheme done correctly
2. step size $\Delta$ not too large

References that prove this (hard to read... but easy to cite!):

Efficiency: Implicit Method

- Pretty inefficient: I need 5,990 iterations (though super fast)
- Efficiency can be improved by using an "implicit method"

\[
\frac{V_i^{n+1} - V_i^n}{\Delta} + \rho V_i^{n+1} = U(c_i^n) + (V_i^{n+1})'(k_i)[F(k_i) - \delta k_i - c_i^n]
\]

- Each step \( n \) involves solving a linear system of the form

\[
((\rho + \frac{1}{\Delta})I - A_n) V^{n+1} = U(c^n) + \frac{1}{\Delta} V^n
\]

- Advantage: step size \( \Delta \) can be much larger \( \Rightarrow \) more stable.
- Implicit method preferable in general.
The Magic of Upwind Schemes

• **No boundary conditions are needed!**

• How is that possible? Don’t we know from MAT 101 that solving a first order ODE requires a boundary condition?

• Answer: in effect, we used the steady state as a boundary condition:

$$V'(k^*) = U'(c^*)$$

$$V'(k_i) = U'(c^*), \quad k_i = k^*$$

$$V'(k_i) = \frac{V_{i+1} - V_i}{\Delta k}, \quad k_i < k^*$$

$$V'(k_i) = \frac{V_i - V_{i-1}}{\Delta k}, \quad k_i > k^*$$

• **But** we did it without having to actually calculate steady state (never use $F'(k^*) = \rho + \delta$ equation). Code chooses steady state by itself.

• Upwind magic **much more general**: works for all stochastic models – **do not even have steady state** – e.g. Huggett.
Deeper Reason: Unique Viscosity Solution

• basically there is a sense in which the HJB equation for the neoclassical growth model has a **unique** solution, **independent of boundary conditions**

• 3 Theorems (references two slides ahead)

  • **Theorem 1**: HJB equation has unique “viscosity solution”
  
  • **Theorem 2**: viscosity solution equals value function, i.e. solution to “sequence problem.”
  
  • **Theorem 3**: FD method converges to unique “viscosity solution”, upwind scheme picks “viscosity solution”

• Comments

  • “viscosity solution” = bazooka, not really needed, designed for non-differentiable PDEs
  
  • Important for our purposes: **uniqueness property**
  
  • typically, viscosity solution = “nice”, concave solution. Definition ⇒ doesn’t blow up, no convex kinks
Deeper Reason: Unique Viscosity Solution

- basically there is a sense in which the HJB equation for the neoclassical growth model has a unique solution, independent of boundary conditions

- 3 Theorems (references two slides ahead)

  - **Theorem 1:** HJB equation has unique “nice” solution
  - **Theorem 2:** “nice” solution equals value function, i.e. solution to “sequence problem.”
  - **Theorem 3:** FD method converges to unique “nice” solution, upwind scheme picks “nice” solution

- Comments
  - “viscosity solution” = bazooka, not really needed, designed for non-differentiable PDEs
  - Important for our purposes: uniqueness property
  - typically, viscosity solution = “nice”, concave solution. Definition ⇒ doesn’t blow up, no convex kinks
References for Theorems

- **Theorems 1 and 2** originally due to Crandall and Lions (1983), “Viscosity Solutions of Hamilton-Jacobi Equations.”


- **Theorem 3** originally due to Barles and Souganidis (1991) “Convergence of approximation schemes for fully nonlinear second order equations.”
Deeper Reason: Unique Viscosity Solution

- What do I mean by unique solution, independent of boundary conditions?

- Consider HJB equation for $V(x)$ on $[x_0, x_1]$.

- MAT 101: given a boundary condition $V(x_0) = \theta$, say, HJB has a unique "classical" solution $V^{\text{classical}}(x)$

- **Theorem 1** basically says: there is a unique $\theta$ such that $V^{\text{classical}}(x)$ is "nice."

- Related: in neoclassical growth model and many other models
  - drift positive at lower end of state space
  - drift negative at upper end of state space

- Since HJB equation is forward-looking, it cannot possibly matter what happens at boundaries of state space: the HJB equation "does not see the boundary condition"
  - **exception**: models with state constraints, e.g. Huggett. But need boundary condition only if state constraint binds.
General Lesson from all this

- the MAT 101 way is **not** the right way to think about HJB equations, i.e. you **don’t** want to think of them as ODEs that require boundary conditions!
Numerical Solution of Huggett Model

http://www.princeton.edu/~moll/HACTproject.htm
Kolmogorov Forward Equation

- Given HJB equation, solving KF equation is a piece of cake!

- Recall: solving HJB equation using implicit scheme involves solving linear system

\[
((\rho + \frac{1}{\Delta})I - A_n) v^{n+1} = U(c^n) + \frac{1}{\Delta} v^n
\]

- **Key:** matrix $A_n$ encodes evolution of stochastic process

- Stationary distribution simply solves

\[
A^T g = 0.
\]

- For details see section 2 of
  [http://www.princeton.edu/~moll/HACTproject/numerical_MATLAB.pdf](http://www.princeton.edu/~moll/HACTproject/numerical_MATLAB.pdf)
Saving and Wealth Distribution given $r$

![Graph showing savings and wealth distribution](image-url)
Generalization: Continuum of \( z \)-Types

Savings \( s(a,z) \)

Wealth, \( a \)  
Productivity, \( z \)
Generalization: Continuum of $z$-Types
Transition Dynamics

movie here
Generalization: Aiyagari, Transition

(a) Capital

(b) GDP

(c) Wage

(d) Interest Rate
Background: Diffusion Processes
Diffusion Processes

• A diffusion is simply a continuous-time Markov process (with continuous sample paths, i.e. no jumps)

• Simplest possible diffusion: standard Brownian motion (sometimes also called “Wiener process”)

• Definition: a standard Brownian motion is a stochastic process $W$ which satisfies

$$W(t + \Delta t) - W(t) = \varepsilon_t \sqrt{\Delta t}, \quad \varepsilon_t \sim N(0, 1), \quad W(0) = 0$$

• Not hard to see

$$W(t) \sim N(0, t)$$

• Continuous time analogue of a discrete time random walk:

$$W_{t+1} = W_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, 1)$$
Standard Brownian Motion

- Note: mean zero, $\mathbb{E}(W(t)) = 0$...
- ... but blows up $\text{Var}(W(t)) = t$. 
Brownian Motion

• Can be generalized

\[ x(t) = x(0) + \mu t + \sigma W(t) \]

• Since \( \mathbb{E}(W(t)) = 0 \) and \( \text{Var}(W(t)) = t \)

\[ \mathbb{E}[x(t) - x(0)] = \mu t, \quad \text{Var}[x(t) - x(0)] = \sigma^2 t \]

• This is called a Brownian motion with drift \( \mu \) and variance \( \sigma^2 \)

• Can write this in differential form as

\[ dx(t) = \mu dt + \sigma dW(t) \]

where \( dW(t) \equiv \lim_{\Delta t \to 0} \varepsilon_t \sqrt{\Delta t} \), with \( \varepsilon_t \sim N(0, 1) \)

• This is called a \textbf{stochastic differential equation}

• Analogue of stochastic difference equation:

\[ x_{t+1} = \mu t + x_t + \sigma \varepsilon_t, \quad \varepsilon_t \sim N(0, 1) \]
Brownian Motion with Drift

- "x(t)" : Current3
- Forecast x : Current3
- "Forecast x + 1 SD" : Current3
- "Forecast x - 1 SD" : Current3
Further Generalizations: Diffusion Processes

- Can be generalized further (suppressing dependence of $x$ and $W$ on $t$)

$$dx = \mu(x)dt + \sigma(x)dW$$

where $\mu$ and $\sigma$ are any non-linear etc etc functions.

- This is called a “diffusion process”

- $\mu(\cdot)$ is called the drift and $\sigma(\cdot)$ the diffusion.

- All results can be extended to the case where they depend on $t$, $\mu(x, t)$, $\sigma(x, t)$ but abstract from this for now.

- The amazing thing about diffusion processes: **by choosing functions $\mu$ and $\sigma$, you can get pretty much any stochastic process you want** (except jumps)
Example 1: Ornstein-Uhlenbeck Process

- Brownian motion $dx = \mu dt + \sigma dW$ is not stationary (random walk). But the following process is

$$dx = \theta(\bar{x} - x)dt + \sigma dW$$

- Analogue of AR(1) process, autocorrelation $e^{-\theta} \approx 1 - \theta$

$$x_{t+1} = \theta \bar{x} + (1 - \theta)x_t + \sigma \varepsilon_t$$

- That is, we just choose

$$\mu(x) = \theta(\bar{x} - x)$$

and we get a nice stationary process!

- This is called an “Ornstein-Uhlenbeck process”
Ornstein-Uhlenbeck Process

Can show: stationary distribution is $N(\bar{x}, \sigma^2/(2\theta))$
Example 2: “Moll Process”

- Design a process that stays in the interval \([0, 1]\) and mean-reverts around \(1/2\)

\[
\mu(x) = \theta \left(1/2 - x\right), \quad \sigma(x) = \sigma x (1 - x)
\]

\[
dx = \theta \left(1/2 - x\right) dt + \sigma x (1 - x) dW
\]

- Note: diffusion goes to zero at boundaries \(\sigma(0) = \sigma(1) = 0\) & mean-reverts \(\Rightarrow\) always stay in \([0, 1]\)
Other Examples

- Geometric Brownian motion:
  \[ dx = \mu x dt + \sigma x dW \]
  \[ x \in [0, \infty), \text{ no stationary distribution:} \]
  \[ \log x(t) \sim N((\mu - \sigma^2/2)t, \sigma^2 t). \]

- Feller square root process (finance: “Cox-Ingersoll-Ross”)
  \[ dx = \theta (\bar{x} - x) dt + \sigma \sqrt{x} dW \]
  \[ x \in [0, \infty), \text{ stationary distribution is } \Gamma(\gamma, 1/\beta), \text{ i.e.} \]
  \[ f_\infty(x) \propto e^{-\beta x} x^{\gamma-1}, \quad \beta = 2\theta \bar{x}/\sigma^2, \quad \gamma = 2\theta \bar{x}/\sigma^2 \]

Background: HJB Equations for Diffusion Processes
Stochastic Optimal Control

• Generic problem:

\[ V(x_0) = \max_{u(t)} \mathbb{E}_0 \int_0^\infty e^{-\rho t} h(x(t), u(t)) \, dt \]

subject to the law of motion for the state

\[ dx(t) = g(x(t), u(t)) \, dt + \sigma(x(t))dW(t) \]

and \( u(t) \in U \)

for \( t \geq 0, \ x(0) = x_0 \) given.

• Deterministic problem: special case \( \sigma(x) \equiv 0 \).

• In general \( x \in \mathbb{R}^m, u \in \mathbb{R}^n \). For now do scalar case.
Stochastic HJB Equation: Scalar Case

- Claim: the HJB equation is

\[ \rho V(x) = \max_{u \in U} h(x, u) + V'(x)g(x, u) + \frac{1}{2} V''(x)\sigma^2(x) \]

- Here: on purpose no derivation ("cookbook")

- In case you care, see any textbook, e.g. chapter 2 in Stokey (2008)
Just for Completeness: Multivariate Case

• Let $x \in \mathbb{R}^m$, $u \in \mathbb{R}^n$.

• For fixed $x$, define the $m \times m$ covariance matrix

$$\sigma^2(x) = \sigma(x)\sigma(x)'$$

(this is a function $\sigma^2 : \mathbb{R}^m \rightarrow \mathbb{R}^m \times \mathbb{R}^m$)

• The HJB equation is

$$\rho V(x) = \max_{u \in U} h(x, u) + \sum_{i=1}^m \partial_i V(x)g_i(x, u) + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \sigma_{ij}^2(x)\partial_{ij}V(x)$$

• In vector notation

$$\rho V(x) = \max_{u \in U} h(x, u) + \nabla_x V(x) \cdot g(x, u) + \frac{1}{2} \text{tr} \left( \Delta_x V(x) \sigma^2(x) \right)$$

• $\nabla_x V(x)$: gradient of $V$ (dimension $m \times 1$)

• $\Delta_x V(x)$: Hessian of $V$ (dimension $m \times m$).
HJB Equation: Endogenous and Exogenous State

- Lots of problems have the form $x = (x_1, x_2)$
  - $x_1$: endogenous state
  - $x_2$: exogenous state

$$dx_1 = \tilde{g}(x_1, x_2, u)dt$$

$$dx_2 = \tilde{\mu}(x_2)dt + \tilde{\sigma}(x_2)dW$$

- Special case with

$$g(x) = \begin{bmatrix} \tilde{g}(x_1, x_2, u) \\ \tilde{\mu}(x_2) \end{bmatrix}, \quad \sigma(x) = \begin{bmatrix} 0 \\ \tilde{\sigma}(x_2) \end{bmatrix}$$

- Claim: the HJB equation is

$$\rho V(x_1, x_2) = \max_{u \in U} h(x_1, x_2, u) + \partial_1 V(x_1, x_2)\tilde{g}(x_1, x_2, u)$$

$$+ \partial_2 V(x_1, x_2)\tilde{\mu}(x_2) + \frac{1}{2} \partial_{22} V(x_1, x_2)\tilde{\sigma}^2(x_2)$$
Example 1: Real Business Cycle Model

\[ V(k_0, A_0) = \max_{c(t)_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} U(c(t)) dt \]

subject to

\[ dk = [AF(k) - \delta k - c] dt \]
\[ dA = \mu(A) dt + \sigma(A) dW \]

for \( t \geq 0, \ k(0) = k_0, \ A(0) = A_0 \) given.

- Here: \( x_1 = k, \ x_2 = A, \ u = c \)
- \( h(x, u) = U(u) \)
- \( g(x, u) = F(x) - \delta x - u \)
Example 1: Real Business Cycle Model

- HJB equation is

\[
\rho V(k, A) = \max_c U(c) + \partial_k V(k, A)[AF(k) - \delta k - c] \\
+ \partial_A V(k, A)\mu(A) + \frac{1}{2} \partial_{AA} V(k, A)\sigma^2(A)
\]
Example 2: Huggett Model

\[ V(a_0, z_0) = \max_{c(t)_{t=0}^\infty} \mathbb{E}_0 \int_{0}^{\infty} e^{-\rho t} U(c(t)) \, dt \quad \text{s.t.} \]

\[
da = [z + ra - c] \, dt
\]

\[
dz = \mu(z) \, dt + \sigma(z) \, dW
\]

\[ a \geq a \]

for \( t \geq 0 \), \( a(0) = a_0 \), \( z(0) = z_0 \) given.

- Here: \( x_1 = k \), \( x_2 = z \), \( u = c \)
- \( h(x, u) = U(u) \)
- \( g(x, u) = x_2 + rx_1 - u \)
Example 2: Huggett Model

- HJB equation is

\[
\rho V(a, z) = \max_c U(c) + \partial_a V(a, z)[z + ra - c]
\]

\[
+ \partial_z V(a, z)\mu(z) + \frac{1}{2} \partial_{zz} V(a, z)\sigma^2(z)
\]
Example 2: Huggett Model

- Special Case 1: \( z \) is a geometric Brownian motion

\[
\begin{align*}
dz &= \mu z dt + \sigma z dW \\
\rho V(a, z) &= \max_c U(c) + \partial_a V(a, z)[z + ra - c] \\
&\quad + \partial_z V(a, z)\mu z + \frac{1}{2} \partial_{zz} V(a, z)\sigma^2 z^2
\end{align*}
\]

- Special Case 2: \( z \) is a Feller square root process

\[
\begin{align*}
dz &= \theta(\bar{z} - z) dt + \sigma \sqrt{z} dW \\
\rho V(a, z) &= \max_c U(c) + \partial_a V(a, z)[z + ra - c] \\
&\quad + \partial_z V(a, z)\theta(\bar{z} - z) + \frac{1}{2} \partial_{zz} V(a, z)\sigma^2 z
\end{align*}
\]
Background: Kolmogorov Forward Equations
Kolmogorov Forward Equations

- Let $x$ be a scalar diffusion

$$dx = \mu(x)dt + \sigma(x)dW, \quad x(0) = x_0$$

- Suppose we’re interested in the evolution of the distribution of $x$, $f(x, t)$, and in particular in the limit $\lim_{t \to \infty} f(x, t)$.

- Natural thing to care about especially in heterogeneous agent models

- Example 1: $x =$ wealth
  - $\mu(x)$ determined by saving behavior and return to investments
  - $\sigma(x)$ by return risk.
  - microfound later

- Example 2: $x =$ city size (Gabaix and others)
Kolmogorov Forward Equations

- **Fact:** Given an initial distribution $f(x, 0) = f_0(x)$, $f(x, t)$ satisfies the PDE

$$\partial_t f(x, t) = -\partial_x[\mu(x)f(x, t)] + \frac{1}{2}\partial_{xx}[\sigma^2(x)f(x, t)]$$

- This PDE is called the “Kolmogorov Forward Equation”
- Note: in math this often called “Fokker-Planck Equation”
- Can be extended to case where $x$ is a vector as well.
- **Corollary:** if a stationary distribution, $\lim_{t \to \infty} f(x, t) = f(x)$ exists, it satisfies the ODE

$$0 = -\frac{d}{dx}[\mu(x)f(x)] + \frac{1}{2}\frac{d^2}{dx^2}[\sigma^2(x)f(x)]$$
Just for Completeness: Multivariate Case

• Let $x \in \mathbb{R}^m$.

• As before, define the $m \times m$ covariance matrix

$$
\sigma^2(x) = \sigma(x)\sigma(x)'
$$

• The Kolmogorov Forward Equation is

$$
\partial_t f(x, t) = -\sum_{i=1}^{m} \partial_i[\mu_i(x)f(x, t)] + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \partial_{ij}[\sigma_{ij}^2(x)f(x, t)]
$$
Application: Stationary Distribution of Huggett Model

• Recall Huggett model

\[
\rho V(a, z) = \max_c U(c) + \partial_a V(a, z)[z + ra - c] \\
+ \partial_z V(a, z)\mu(z) + \frac{1}{2} \partial_{zz} V(a, z)\sigma^2(z)
\]

• Denote the optimal saving policy function by

\[
s(a, z) = z + ra - c
\]

• Then \( g(a, z, t) \) solves

\[
\partial_t g(a, z, t) = -\partial_a [s(a, z)g(a, z, t)] \\
- \partial_z [\mu(z)g(a, z, t)] + \frac{1}{2} \partial_{zz} [\sigma^2(z)g(a, z, t)]
\]
Wealth Distribution and the Business Cycle
What We Do

Do income and wealth distribution matter for the macroeconomy?

Do macroeconomic aggregates in heterogeneous agent models with frictions behave like those in frictionless representative agent models?

Ask these questions in an economy in which

- heterogeneous producers
- face collateral constraints
- and fixed costs in production
Setup

- Continuum of entrepreneurs, heterogeneous in wealth $a$ and productivity $z$
- Continuum of productivity types, diffusion process
- For now: no workers, occupational choice etc
- Entrepreneurial preferences

$$
\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt
$$
Technologies

- Two technologies: productive and unproductive

\[ y = f_u(z, A, k) = zA\tilde{f}(k), \]

\[ y = f_p(z, A, k) = \begin{cases} 
  zAB\tilde{f}(k - \kappa), & k \geq \kappa \\
  0, & k < \kappa,
\end{cases} \]

- \( B > 1 \), but fixed cost \( \kappa > 0 \)
- \( \tilde{f} \) strictly increasing and concave, but \( f_p \) non-convex

- Sources of uncertainty:
  - \( z \): *idiosyncratic shock*, diffusion process
  - \( A \): *aggregate shock*, two-state Poisson process
Setup

• Collateral constraints

\[ k \leq \lambda a, \quad \lambda \geq 1. \]

• Profit maximization:

\[ \Pi(a, z, A; r) = \max \{ \Pi_u(a, z, A; r), \Pi_p(a, z, A; r) \} \]

\[ \Pi_j(a, z, A; r) = \max_{k \leq \lambda a} f_j(z, A, k) - (r + \delta)k, \quad j = p, u. \]

• Entrepreneurs solve

\[
\max_{\{c_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.} \quad \begin{align*}
    da_t &= [\Pi(a_t, z_t, A_t; r_t) + r_t a_t - c_t] dt
\end{align*}
\]
Plan

1. model without aggregate shocks, $A_t = 1$

2. model with aggregate shocks, $A_t \in \{A_\ell, A_h\}$, Poisson
Without Aggregate Shocks

- Capital market clearing:

\[
\int [k_u(a, z; r(t))\mathbf{1}_{\{\Pi_u > \Pi_p\}} + k_p(a, z; r(t))\mathbf{1}_{\{\Pi_u < \Pi_p\}}]g(a, z, t)dadz
\]

\[
= \int ag(a, z, t)dadz
\]  

(EQ)

\[\rho v(a, z, t) = \max_c u(c) + \partial_a v(a, z, t)[\Pi(a, z; r(t)) + r(t)a - c] \quad \text{(HJB)}\]

\[+ \partial_z v(a, z, t)\mu(z) + \frac{1}{2} \partial_{zz} v(a, z, t)\sigma^2(z) + \partial_t v(a, z, t)\]

\[
\partial_t g(a, z, t) = - \partial_a [s(a, z, t)g(a, z, t)] - \partial_z [\mu(z)g(a, z, t)] \quad \text{(KFE)}
\]

\[+ \frac{1}{2} \partial_{zz} [\sigma^2(z)g(a, z, t)],\]

\[s(a, z, t) = \Pi(a, z; r(t)) + r(t)a - c(a, z, t)\]

Given initial condition \(g_0(a, z)\), the two PDEs (HJB) and (KFE) together with (EQ) fully characterize equilibrium.
Dynamics of Wealth Distribution

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Dynamics of Wealth Distribution

Dynamics of Wealth Distribution

Density, \( g(a, z, t) \)

Wealth, \( a \)

Productivity, \( z \)
With Aggregate Shocks

- $A_t \in \{A_\ell, A_h\}$, Poisson with intensities $\phi_\ell, \phi_h$

- As in discrete time: necessary to include entire wealth distribution as state variable
  - Aggregate state: $(A_i, g), i = \ell, h$
  - Optimal saving policy function: $s_i(a, z, g), i = \ell, h$
  - Interest rate $r_i(g), i = \ell, h$
  - Capital demands: $k_{i,u}(a, z, g; r)$ and $k_{i,p}(a, z, g; r), i = \ell, h$
  - Equilibrium interest rate $r_i(g)$ solves

\[
\int \left[ k_{i,u}(a, z, g; r_i(g))1_{\Pi_u > \Pi_p} + k_{i,p}(a, z, g; r_i(g))1_{\Pi_u < \Pi_p} \right] g(a, z) \, da \, dz \\
= \int ag(a, z) \, da \, dz, \quad i = \ell, h
\]
Entrepreneurs’ Problem

• Useful to write law of motion of distribution as

\[ \partial_t g(a, z, t) = T[g(\cdot, t), s_i(\cdot, t)](a, z) \]

where \( T \) is the “Kolmogorov Forward operator”

\[
T[g, s_i](a, z) = - \partial_a [s_i(a, z, g)g(a, z)] \\
- \partial_z [\mu(z)g(a, z)] + \frac{1}{2} \partial_{zz} [\sigma^2(z)g(a, z)]
\]
• **Entrepreneurs problem: recursive formulation**

$$
\rho V_i(a, z, g) = \max_c u(c) + \partial_a V_i(a, z, g)[\Pi_i(a, z, g) + r_i(g)a - c]
$$

$$
+ \partial_z V_i(a, z, g)\mu(z) + \frac{1}{2} \partial_{zz} V_i(a, z, g)\sigma^2(z)
+ \phi_i[V_j(a, z, g) - V_i(a, z, g)]
$$

$$
+ \int \frac{\delta V_i(a, z, g)}{\delta g(a, z)} T[g, s_i](a, z)dadz.
$$

• **$\delta V_i/\delta g(a, z)$**: functional derivative of $V_i$ wrt to $g$ at $(a, z)$.

• **Saving policy function**

$$
\begin{align*}
s_i(a, z, g) &= \Pi_i(a, z, g) + r_i(g)a - c_i(a, z, g) \\
&= \Pi_i(a, z, g) + r_i(g)a - (u')^{-1}(\partial_a V_i(a, z, g)).
\end{align*}
$$

• **Recursive equilibrium**: solution to these two equations.
Approximation Method: Basic Idea

1. use **discrete time approximation** to $A$ process
2. consider only **finitely many** shocks
3. keep track of **all possible histories**
4. between shocks: same (HJB) and (KFE) as without shocks, one system for each history
5. then piece them together in correct way

- Example:
  - 5 shocks at times $t = 5, 10, 15, 20, 25$
  - $2^5 = 32$ histories
• **Obvious problem:** blows up on you, e.g. $2^8 = 256$.

• **Working on:** using this approximation as input into fancier approximation scheme

• but the economics seems robust to me, will bet anyone who thinks that things will change a lot
Approximation Method: Details

• $N$ shocks hit at times $t_n = \Delta n, n = 0, 1, 2, \ldots, N$

• Approximate Poisson process

\[
\begin{align*}
\Pr(\tilde{A}_{n+1} = A_i | \tilde{A}_n = A_i) &= 1 - e^{-\phi_i \Delta} \\
\Pr(\tilde{A}_{n+1} = A_j | \tilde{A}_n = A_i) &= e^{-\phi_i \Delta}
\end{align*}
\]

converges to Poisson as $\Delta \to 0$ and $N \to \infty$.

• Denote histories by $\tilde{A}^n = (\tilde{A}_0, \ldots, \tilde{A}_n)$ where $\tilde{A}_n \in \{A_\ell, A_h\}$

• Keep track of all

\[
v(a, z, t, \tilde{A}^n) \quad \text{and} \quad g(a, z, t, \tilde{A}^n)
\]
**Approximation Method: Details**

**Between shocks:** for all histories $\tilde{A}^n$

$$\rho v(a, z, t, \tilde{A}^n) = \max_c u(c) + \partial_a v(a, z, t, \tilde{A}^n)[\Pi(a, z, t, \tilde{A}^n) + r(t, \tilde{A}^n)a - c]$$

$$+ \partial_z v(a, z, t, \tilde{A}^n)\mu(z) + \frac{1}{2}\partial_{zz} v(a, z, t, \tilde{A}^n)\sigma^2(z) + \partial_t v(a, z, t, \tilde{A}^n)$$

$$\partial_t g(a, z, t, \tilde{A}^n) = - \partial_a[s(a, z, t, \tilde{A}^n)g(a, z, t, \tilde{A}^n)]$$

$$- \partial_z[\mu(z)g(a, z, t, \tilde{A}^n)] + \frac{1}{2}\partial_{zz}[\sigma^2(z)g(a, z, t, \tilde{A}^n)]$$

**Boundary conditions:** at all times when shocks hit $t_n$:

$$v \left( a, z, t_{n+1}, \tilde{A}^n \right) = \sum_{\tilde{A}_{n+1}=A_\ell,A_h} \Pr(\tilde{A}_{n+1}|\tilde{A}_n)v \left( a, z, t_{n+1}, \tilde{A}^{n+1} \right)$$

$$g \left( a, z, t_{n+1}, \tilde{A}^{n+1} \right) = g \left( a, z, t_{n+1}, \tilde{A}^n \right), \quad \text{all } \tilde{A}^n.$$  

**Note:** uses

1. people forward-looking, form expectations over all branches
2. continuity of $g$ with respect to $t$ (state variable)
The Experiments

- **Experiment 1**: comparison to rep. agent model
  - two economies: economy with frictions and rep. agent
  - at $t = 0$, both in steady state (corresponding to no $A$ shocks)
  - at $t = 10$, hit with $A$ shock, compare impulse responses

- **Experiment 2**: effect of wealth distribution
  - two economies with frictions: one more unequal than the other
  - at $t = 0$, both in steady state (corresponding to no $A$ shocks)
  - at $t = 10$, hit with $A$ shock, compare impulse responses
The Corresponding Representative Agent

- **RBC model** with utility function $u(c)$ and following aggregate production function

$$F(K, A) = \max_{\{k(z)\}} \int \max\{f_u(z, A, k(z)), f_p(z, A, k(z))\} \psi(z)dz \quad \text{s.t.}$$

$$\int k(z)\psi(z)dz \leq K$$

where $\psi(z)$ is the stationary distribution of the $z$-process.

- **Solution:**

$$F(K, A) = \max_{\hat{z}} AZ(\hat{z}) (K - \kappa (1 - \Psi(\hat{z})))^\alpha$$

$$Z(\hat{z}) \equiv \left( \int_0^{\hat{z}} z^{\frac{1}{1-\alpha}} \psi(z)dz + B^{\frac{1}{1-\alpha}} \int_{\hat{z}}^{\infty} z^{\frac{1}{1-\alpha}} \psi(z)dz \right)^{1-\alpha}$$
Parameterization

- **Functional forms**

\[ u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \tilde{f}(k) = k^\alpha \]

- **Productivity process: Ornstein-Uhlenbeck**

\[ d \log z_t = -\nu \log z_t + \bar{\sigma} dW_t \]

but reflected at \( \bar{z}, \bar{z} \geq 0 \).

- **Parameter values**

\[
\begin{align*}
B &= 1.8, \quad \alpha = 0.6, \quad \delta = 0.05, \quad \kappa = 6, \quad \lambda = 2, \\
\rho &= 0.05, \quad \gamma = 2, \quad \nu = 0.2, \quad \bar{\sigma}_z = 0.1 \\
A_h &= 1, \quad A_\ell = 0.95 \\
a &= 0, \quad \bar{a} = 30, \quad z = 0.6, \quad \bar{z} = 1.4, \quad T = 100
\end{align*}
\]
Frictions $\Rightarrow$ much Slower Recovery

(a) Productivity, $A_t$

(b) Capital Stock, $K_t$

(c) GDP, $Y_t$

(d) Interest Rate, $r_t$

Note: GDP and interest rate computations currently imprecise
Dynamics of Wealth Distribution

movie here
Wealth Distribution at Peak

Year = 10
Wealth Distribution at Trough

Year = 20

Wealth, $a$

Density, $g(a, z, t)$

Productivity, $z$
Wealth Distribution 10 Years after Trough

Year = 30

Density, \( g(a, z, t) \)

Wealth, \( a \)

Productivity, \( z \)
Next Steps

- Redistributive policies?
- Add workers, occupational choice
- Fancier approximation
  - computations suggest: densities live in relatively low-dimensional space
  - project densities on that space
  - use as state variable in value function
- What else?