Supplement to Lectures 5 and 6

Spectral Approach to Distributional Dynamics
(based on EF&G discussion of Alvarez-Lippi in March 2019)

Distributional Macroeconomics
Part II of ECON 2149

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Purpose of these Notes

• Explain “eigenvalue-eigenfunction approach” or “spectral approach” to distributional dynamics

• Used in two recent papers


These notes are based on a discussion of Alvarez-Lippi at the NBER EF&G Meeting in San Francisco on March 1, 2019
Overview

Key message: even though GLLM, Alvarez-Lippi study environment with

- continuous time \( t \)
- continuous state \( x \)

conceptually everything is the same as with discrete time & states

Approach has two steps:

1. express transition dynamics in terms of eigenvalues & eigenvectors/functions
2. analytic solution for these
Plan for Explanation

1. *Discrete* time, *discrete* states

2. *Continuous* time, *discrete* states

3. *Continuous* time, *continuous* states

Again, point is: it’s all the same!
1. Discrete time, discrete states

- \( x_{it} \in \{x_1, \ldots, x_N\} \Rightarrow \text{distribution} = \text{vector} \ p_t \in \mathbb{R}^N \) (histogram)

- Dynamics of distribution

\[
\mathbf{p}_{t+1} = \mathbf{A}^\top \mathbf{p}_t,
\]

where \( \mathbf{A} = N \times N \) transition matrix

- Example: symmetric two-state process, \( \mathbf{A} = \begin{bmatrix} 1 - \phi & \phi \\ \phi & 1 - \phi \end{bmatrix} \)

- Stationary distribution solves

\[
\mathbf{p}_\infty = \mathbf{A}^\top \mathbf{p}_\infty
\]

i.e. eigenvector corresponding to unit eigenvalue: \( \lambda \mathbf{v} = \mathbf{A}^\top \mathbf{v}, \lambda = 1 \)

- But what about transition dynamics? Spectral approach
1. Discrete time, discrete states

- Spectral approach to distributional dynamics
  - assume $\mathbf{A}^T$ is diagonalizable
  - denote eigenvalues by $\lambda_1 > \lambda_2 > \ldots > \lambda_N$
  - corresponding eigenvectors by $\mathbf{v}_1, \ldots, \mathbf{v}_N$

Result: can write $p_0 = \sum_{j=1}^{N} b_j \mathbf{v}_j$ and hence $p_t = \sum_{j=1}^{N} \lambda_j^t b_j \mathbf{v}_j$

- Two-state example from previous slide: $\lambda_1 = 1$ and $\lambda_2 = 1 - 2\phi$

  \[ \Rightarrow \quad p_t = b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1 - 2\phi)^t b_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

- Similarly, IRFs for moments of distribution $H_t := \sum_{i=1}^{N} f(x_i) p_{it} = \mathbf{f}^T p_t = \sum_{j=1}^{N} \lambda_j^t b_j (\mathbf{f}^T \mathbf{v}_j)$
1. Discrete time, discrete states

- Spectral approach to distributional dynamics
  - assume $A^T$ is diagonalizable
  - denote eigenvalues by $\lambda_1 > \lambda_2 > \ldots > \lambda_N$
  - corresponding eigenvectors by $v_1, \ldots, v_N$

Result: can write $p_0 = \sum_{j=1}^{N} b_j v_j$ and hence $p_t = \sum_{j=1}^{N} \lambda_j^t b_j v_j$

- Two-state example from previous slide: $\lambda_1 = 1$ and $\lambda_2 = 1 - 2\phi$

  \[ p_t = b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1 - 2\phi)^t b_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

- Similarly, IRFs for moments of distribution

  \[ H_t = \sum_{j=1}^{N} \lambda_j^t b_j[p_0] b_j[f], \quad b_j[f] := f^T v_j \]

  Discrete analogue of Alvarez-Lippi’s main formula
Quick summary so far

Approach has two steps:

1. express transition dynamics in terms of eigenvalues & eigenvectors
   • very general

2. analytic solution for these
   • only works in particular cases, e.g. two-state example

Will come back this...
2. Continuous time, discrete states

Assume process for $x_{it} = \text{finite-state Poisson process}$

Everything the same except

$$\dot{p}(t) = A^T p(t)$$

$$\lambda_j \leq 0$$

$$p(t) = \sum_{j=1}^{N} e^{\lambda_j t} b_j v_j$$

$$H(t) = \sum_{j=1}^{N} e^{\lambda_j t} b_j [p_0] b_j [f]$$
3. **Continuous time, continuous states**

Now suppose $x$ is continuous rather than discrete.

Everything still the same but need a bit of **new vocabulary**:

- vector $\mathbf{p} \leftrightarrow$ function $p$
- matrix $\mathbf{A} \leftrightarrow$ (linear) operator $\mathcal{A}$
- transpose $\mathbf{A}^T \leftrightarrow$ adjoint $\mathcal{A}^*$

For example: distribution is now a function

$$p(x, t)$$

rather than a vector $\mathbf{p}(t)$
3. Continuous time, continuous states

- Particular example: Brownian motion \( dx_{it} = \mu dt + \sigma dW_{it} \)

- Question: how characterize \( p(x, t) \)?

- Useful fact: \( p \) satisfies Kolmogorov Forward equation

\[
\frac{\partial p(x, t)}{\partial t} = -\mu \frac{\partial p(x, t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 p(x, t)}{\partial x^2}
\]

- Now comes the key: write this in terms of differential operator

\[
\frac{\partial p}{\partial t} = A^* p, \quad A^* := -\mu \frac{\partial}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}
\]

which is exact analogue of

\[
\dot{p}(t) = A^T p(t)
\]
3. Continuous time, continuous states

- This goes further: just like $A^T$, $A^*$ has eigenvalues & eigenvectors
- The eigenvalues $\lambda_j$ and eigenfunctions $\varphi_j(x)$ of $A^*$ solve
  \[
  \lambda \varphi = A^* \varphi \iff \lambda \mathbf{v} = A^T \mathbf{v}
  \]
- Also everything else is “the same”
  \[
  \frac{\partial p}{\partial t} = A^* p \iff \dot{p}(t) = A^T p(t)
  \]
  \[
  p(x, t) = \sum_{j=1}^{\infty} e^{\lambda_j t} b_j \varphi_j(x) \iff p(t) = \sum_{j=1}^{N} e^{\lambda_j t} b_j \mathbf{v}_j
  \]
  and similarly for IRFs ...
- Aside “for the interested nerd”: comes from quantum mechanics
3. Continuous time, continuous states

- This goes further: just like $A^T$, $A^*$ has eigenvalues & eigenfunctions

- The eigenvalues $\lambda_j$ and eigenfunctions $\varphi_j(x)$ of $A^*$ solve

  $$\lambda \varphi = A^* \varphi \iff \lambda v = A^T v$$

- Also everything else is “the same”

  $$\frac{\partial p}{\partial t} = A^* p \iff \dot{p}(t) = A^T p(t)$$

  $$p(x, t) = \sum_{j=1}^{\infty} e^{\lambda_j^t} b_j \varphi_j(x) \iff p(t) = \sum_{j=1}^{N} e^{\lambda_j^t} b_j v_j$$

  and similarly for IRFs ...

- Aside “for the interested nerd”: comes from quantum mechanics

3. **Continuous time, continuous states**

- Finally: these eigenvalue problems are differential equations, can be solved analytically in special cases

- For example: eigenvalue problem from previous slide

\[ \lambda \varphi = \mathcal{A}^* \varphi, \quad \mathcal{A}^* := -\mu \frac{\partial}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} \quad \& \text{boundary conditions} \]

is simply an ODE

\[ \lambda \varphi(x) = -\mu \varphi'(x) + \frac{\sigma^2}{2} \varphi''(x) \quad \& \text{boundary conditions} \]

- Analytic solutions with \( \sigma^2/2 = 1 \), reflected on \([0, 1]\)

\[ \lambda_0 = 0, \quad \lambda_j = \frac{\mu^2}{2} + \frac{\pi^2 j^2}{2}, \quad j = 1, 2, \ldots \]

\[ \varphi_j(x) = \pm \frac{e^{-\mu x}}{\sqrt{1 + \mu^2/(\pi^2 j^2)}} \left\{ \cos(x \pi j) + \frac{\mu}{\pi j} \sin(x \pi j) \right\} \]

which is similar to Alvarez Lippi’s formulas – see Linetsky (2005) “On the Transition Densities for Reflected Diffusions”
Summary of Approach

Conceptually, everything is the same as with discrete time & states!

Two steps:

1. express transition dynamics in terms of eigenvalues & eigenvectors/functions

2. analytic solution for these

Analogies:

- vector \( p \) ↔ function \( p \)
- matrix \( A \) ↔ operator \( A \)
- transpose \( A^T \) ↔ adjoint \( A^* \)
Applications
Main message: standard theories of top inequality $\Rightarrow$ very slow transition dynamics, too slow relative to data.

GLLM main theorem in a nutshell:

1. spectral approach: $p(t) = \sum_{j=1}^{N} e^{-|\lambda_j|t} b_j v_j \approx b_1 v_1 + e^{-|\lambda_2|t} b_2 v_2$

2. analytic formula for $|\lambda_2| = \frac{1}{2} \frac{\mu^2}{\sigma^2} + \zeta$ (but not higher eigenvalues)

3. $|\lambda_2|$ very small for any reasonable parameterization
Main results:

1. “selection effect” decouples price adj frequency & output response (active in menu cost models but not in Calvo)
2. “price plans” may yield hump-shaped IRFs
3. monetary policy less effective when volatility is high (but only if monetary expansion lags volatility shock sufficiently)

Relative to GLLM: all eigenvalues rather than just spectral gap!

Analytic characterization of whole profile of IRF. Example:
Alvarez-Lippi’s Main Theorem

Impulse response after $t$ periods:

$$H(t; f, \hat{\rho}) = \sum_{j=1}^{\infty} e^{\lambda_j t} b_j[f] b_j[\hat{\rho}]$$

and analytic solutions for $\lambda_j, b_j[f], b_j[\hat{\rho}]$, e.g.

$$\lambda_j = - \left[ \zeta + \frac{\sigma^2}{8\bar{x}} (j\pi)^2 \right], \quad j = 1, 2, \ldots$$

Exact analogue of

$$H_t = \sum_{j=1}^{N} \lambda_j^t b_j[p_0] b_j[f], \quad b_j[f] := f^T v_j$$
Other References


• Neither of these papers ever saw the light of the day, but you can find a draft of the former via google