Take-Home Exercise

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In this exercise you will study the evolution of inflation in the U.S. over the last 35 years. Of particular interest is the nature of the recent sharp increases in inflation and what these portend for future inflation.

Part 1: Background calculations

1. Suppose that $a_t \sim iid(0, \sigma_a^2)$ for $t = 1, ..., T$. Let $\hat{\sigma}_a^2 = T^{-1} \sum_{t=1}^T a_t^2$.
   (a) Show that $\sqrt{T}(\hat{\sigma}_a^2 - \sigma_a^2) \Rightarrow N(0, V_{\hat{\sigma}_a})$ and derive an expression for $V_{\hat{\sigma}_a}$. Note 1: The problem does not assume that $a$ is normally distributed. Note 2: Make sure to state any additional assumptions you need to make on the existence of moments, etc.
   (b) Use the $\delta$-method to show that $\sqrt{T}(\hat{\sigma}_a - \sigma_a) \Rightarrow N(0, V_{\hat{\sigma}_a})$ and derive an expression for $V_{\hat{\sigma}_a}$.

2. Suppose $y_t = \tau_t + \varepsilon_t$ where $\tau_t = \tau_{t-1} + \eta_t$ and $\{\eta_t\}$ and $\{\varepsilon_t\}$ are mutually independent $iid$ sequences of zero-mean normally distributed random variables with standard deviations $\sigma_\eta$ and $\sigma_\varepsilon$. The initial value of $\tau_0 \sim N(0, \kappa^2)$ and is independent of $(\varepsilon_1, \eta_1)$ for $t \geq 1$.
   (a) Show that $x_t = (1 - L)y_t$ is covariance stationary with mean zero, and autocovariances $\lambda_k$ with $\lambda_k = 0$ for $|k| > 1$. Show how the values of $\lambda_0$ and $\lambda_1$ are related to $\sigma_\varepsilon$ and $\sigma_\eta$.
   (b) Show that $x_t$ has an MA(1) representation, say $\Delta y_t = (1 - \theta L)a_t$.
      i. Show how the parameters $\theta$ and $\sigma_a$ are related to $\lambda_0$ and $\lambda_1$.
      ii. Show how the parameters $\theta$ and $\sigma_a$ are related to $\sigma_\varepsilon$ and $\sigma_\eta$.
      iii. Suppose $\sigma_a = 1$ and $\theta = 0.8$.
         A. What are the values of $\lambda_0$ and $\lambda_1$?
         B. What are the values of $\sigma_\varepsilon$ and $\sigma_\eta$?

3. Let $\hat{\theta}$ and $\hat{\sigma}_a^2$ denote estimators of $\theta$ and $\sigma_a^2$ using $y_{1:T}$. Suppose
   \[
   \sqrt{T} \begin{bmatrix} (\hat{\theta} - \theta) \\ (\hat{\sigma}_a - \sigma_a) \end{bmatrix} \Rightarrow N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \begin{bmatrix} V_{\hat{\theta}} & 0 \\ 0 & V_{\hat{\sigma}_a} \end{bmatrix} \right) \]
   Let $\hat{\sigma}_\varepsilon$ and $\hat{\sigma}_\eta$ denote estimators of $\sigma_\varepsilon$ and $\sigma_\eta$ constructed from $(\hat{\theta}, \hat{\sigma}_a)$ using the formula you derived in Q2. Use the $\delta$-method to show that
   \[
   \sqrt{T} \begin{bmatrix} (\hat{\sigma}_\varepsilon - \sigma_\varepsilon) \\ (\hat{\sigma}_\eta - \sigma_\eta) \end{bmatrix} \Rightarrow N \left( 0, V_{(\hat{\sigma}_\varepsilon, \hat{\sigma}_\eta)} \right) \]
   and derive an expression for the $2 \times 2$ matrix $V_{(\hat{\sigma}_\varepsilon, \hat{\sigma}_\eta)}$ as a function of $(V_{\hat{\theta}}, V_{\hat{\sigma}_a})$.

4. What are the Kalman filter recursive formulae for computing $\tau_{t|t} = \mathbb{E}(\tau_t|y_{1:t})$ and $P_{t|t} = \text{var}(\tau_t|y_{1:t})$.
   What are the appropriate initial values for the recursion, i.e., the values of $\tau_{0|0}$ and $P_{0|0}$?
5. What is the the Kalman smoothing recursive formulae for computing $\tau_{t|T} = E(\tau_t|y_{1:T})$ and $P_{t|T} = \text{var}(\tau_t|y_{1:T})$. (Hint: For this, you should consult Hamilton, or another source. We did not cover this in lecture.)

6. Consider out-of-sample forecasts of $\tau_t$ and $y_t$.
   (a) Show that $\tau_{T+k|T} = \tau_{T|T}$ for all $k \geq 0$.
   (b) Show that $P_{T+k|T} = P_{T|T} + k \times \sigma_n^2$ for $k \geq 0$.
   (c) Show that $E(y_{T+k}|y_{1:T}) = \tau_{T+k|T}$ for $k \geq 0$.
   (d) Show that $\text{var}(y_{T+k}|y_{1:T}) = P_{T+k|T} + \sigma_n^2$ for $k \geq 0$.

7. Let $\overline{x} = T^{-1} \sum_{t=1}^{T} x_t$.
   (a) Show that $\mu_x = E(x_t) = 0$.
   (b) Show that $\sqrt{T}(\overline{x} - \mu_x) \Rightarrow N(0, V)$ and derive an expression for $V$.

**Part 2: Empirical Analysis**

The file `pgdp_2022.xlsx` contains values of the GDP price deflator for the United States, quarterly, from 1960:Q1-2021:Q4. Denote these values by $P_{t}^{GDP}$. Compute the values of inflation $y_t = 400 \times \ln(P_{t}^{GDP}/P_{t-1}^{GDP})$ (measured in percentage points at an annual rate) and $x_t = (1 - L)y_t$, the change in the inflation rate.

To begin, I want you to analyse data from the pre-Covid period and after the inflationary periods of the 1970s and early 1980s. So, use data from the sample period 1985:Q1-2019:Q4. Use the framework outlined above to answer the following questions.

1. Plot $y_t$ and $x_t$. Comment on the evolution of U.S. inflation over the sample period.

2. Test the null hypothesis that $x_t$ has a mean of $\mu_x = 0$ using data from the 1985:Q1-2019:Q4. How did you compute the standard error for $\overline{x}$ used in the test? (See your answer to Q7 in Part 1.)

3. Using data for $x_t$ to estimate an MA(1) model. Do not include a constant in the model. (When $\mu_x = 0$, the value of the constant term in zero.)
   (a) What are the values of $\hat{\theta}$ and $\hat{\sigma}_a$.
   (b) The standard error of $\hat{\theta}$ is undoubtedly computed by the software you are using (Eviews, STATA, R, etc.). What is the standard error for $\hat{\theta}$. How is the value of this standard error related to $V_{\hat{\theta}}$ in Q3 of Part 1. (You may assume that your software computed the standard error correctly.)
   (c) Most software does not report a standard error for $\hat{\sigma}_a$. You will need to compute this yourself.
      i. Construct the residuals, $\hat{a}_t = (1 - \hat{\theta}L)\overline{x}_t$.
      ii. Use these residuals in place of $a_t$ and compute the value $V_{\hat{\sigma}_a}$ using the formula you derived in Q1 in Part 1.
      iii. Use the estimate of $V_{\hat{\sigma}_a}$ to compute the standard error for $\hat{\sigma}_a$. Based on this, how precise is the estimate of $\hat{\sigma}_a$ constructed from the sample data?
4. Using the estimates from the MA(1) model:
   (a) Estimate the values of \( \hat{\sigma}_\varepsilon \) and \( \hat{\sigma}_\eta \). (Use the analysis from Q3 in Part 1.)
   (b) Estimate the value of the covariance matrix \( V(\hat{\sigma}_\varepsilon, \hat{\sigma}_\eta) \) defined Q3 of Part 1.
   (c) Use your answers to (a) and (b) to construct approximate 90% confidence intervals for \( \sigma_\varepsilon \) and \( \sigma_\eta \).

5. Using the estimated values of \( \sigma_\varepsilon \) and \( \sigma_\eta \):
   (a) Use the Kalman filter that you derived in Q4 of Part 1 to compute \( \tau_{t|t} \) and \( P_{t|t} \). (In that question you showed that \( \tau_{0|0} = 0 \) and \( P_{0|0} = \kappa^2 \). For this application, what is a reasonable for \( \kappa \)? Hint: What is a reasonable range of values for \( \tau_{1984:Q4} \)?)
      i. Plot \( y_t \) and \( \tau_{t|t} \). Say something interesting about the plot.
      ii. Add the 67% error bands \( \tau_{t|t} \pm \sqrt{P_{t|t}} \) to the plot. In what sense do these (approximately) contain the true value of \( \tau_t \) with probability 0.67?
   (b) Use the Kalman smoother that you derived in Q5 of Part 1 to compute \( \tau_{t|T} \) and \( P_{t|T} \).
      i. Plot \( y_t, \tau_{t|T} \) and the 67% error bands \( \tau_{t|T} \pm \sqrt{P_{t|T}} \). Discuss how this plot differs from the corresponding plot with the filtered values.

6. When I plotted \( x_t \) over 1985:Q1-2019:Q4, it looked like the process became more variable around 2005:Q1.
   (a) Estimate the MA(1) model for \( x_t \) over the two different sample periods 1985:Q1-2004:Q4 and 2005:Q1-2019:Q4.
      i. Discuss how the estimated values of \( \theta \) and \( \sigma_a \) differ across the two periods.
      ii. Are the differences statistically significant? (Can you derive a procedure to test the null of no change in these parameters?)
   (b) Using the results from (a), estimate that values of \( \sigma_\varepsilon \) and \( \sigma_\eta \) over the two sample periods. Based on these estimates, what is responsible for the increase in variance of \( x_t \) in the second sample period?

7. Using the estimated values of \( \sigma_\varepsilon \) and \( \sigma_\eta \) from the 2005:Q1-2019:Q4 sample:
   (a) Construct the Kalman filter values of \( \tau_{t|t} \) and \( P_{t|t} \) over 2020:Q1-2021:Q4. (Use data beginning in 2005:Q1 to start the Kalman filter.)
   (b) Plot inflation, \( \tau_{t|T} \) and the error bands \( \tau_{t|T} \pm \sqrt{P_{t|T}} \) over this Covid period. Say something interesting.
   (c) Using your results in Q6 of Part 1:
      i. Forecast the values of \( \tau \) over 2022:Q1-2023:Q3. Construct the 90% forecast intervals for \( \tau_t \). (Hint: These are \( \tau_{T+k|T} \pm 1.64\sqrt{P_{T+k|T}} \).
      ii. Forecast the values of inflation, \( y_t \), over 2022:Q1-2023:Q3. Construct the 90% forecast intervals for \( y_{T+k} \).
      iii. Do these forecasts and forecast intervals seem reasonable?
8. Some have argued that inflation has been buffeted by large *transitory* shocks during the Covid period. Suppose the value of $\sigma_\varepsilon$ increased by a factor of 2 in 2020:Q1. Repeat the forecasting exercise in question 7. Describe how this changes the forecasts and forecast intervals.

9. Some have argued that inflation has been buffeted by large *permanent* shocks during the Covid period. Suppose the value of $\sigma_\eta$ increased by a factor of 2 in 2020:Q1. Repeat the forecasting exercise in question 7. Describe how this changes the forecasts and forecast intervals.