The level of per-capita GDP currently stands at approximately $40,000 (in US $2011) in the world’s industrialized economies. What will its level be 50 years from now? 100 years from now? 200 years from now? And, how certain can we be about these values? Answers to these questions are important for evaluating long-term economic policies such as policies aimed at mitigating climate change. In this exercise you will use historical data on GDP to construct predictive distributions for per-capita values of GDP at long horizons.

The file Gerz_OECD_w3.xlsx contains annual observations on per-capita values of GDP for the OECD from 1915-1914. Call this series \( Q_t \). Let \( Y_t = \ln(Q_t/Q_{t-1}) \) denote the annual growth rate of \( Q_t \).

1. Summary statistics:

   (a) Plot \( Q_t \). Plot \( Y_t \).

   (b) Calculate the sample mean and standard deviation of \( Y_t \) over the full sample period.

   (c) Repeat (b) after dividing the sample into 25-year sub-samples.

   (d) Comment on the evolution of \( Q_t \) and \( Y_t \) over the past century.

   (e) (e.i) Compute the first 3 sample autocorrelations of \( Y_t \).
   
   (e.ii) Estimate an AR(2) model for \( Y_t \). (Be sure to include a constant.)
   
   (e.iii) Comment on the serial correlation properties of \( Y_t \).

2. Suppose \( Y_t \) is a stationary stochastic process that is well-described by an AR(1) model:
\[ Y_t = \mu + u_t, \text{ where } u_t = \phi u_{t-1} + \varepsilon_t. \] Suppose you have data \( Y_1, \ldots, Y_T \). Let \( \bar{Y}_{t:T} = \frac{1}{T} \sum_{t=1}^{T} Y_t \).

   (a) Let \( \mu_Y \) denote the mean of \( Y \). Show that \( \sqrt{T}(\bar{Y}_{t:T} - \mu_Y) \overset{d}{\to} N(0, \Omega) \), and derive an expression for \( \Omega \).

   (b) Suppose you want to forecast the average value of \( Y \) from \( T+1 \) to period \( T+h \); that is, you are interested in \( \bar{Y}_{T+1:T+h} = \frac{1}{h} \sum_{j=1}^{h} Y_{T+j} \). Suppose that \( h \) is large.
(b.i) Show that $\sqrt{h}(\bar{Y}_{T+h} - \mu_Y) \overset{d}{\to} N(0,V)$, and derive an expression for $V$.

(b.ii) Suppose you knew the value of $\mu_Y$. A researcher constructs a 95% “prediction interval” for $\bar{Y}_{T+h}$ as $\mu_Y \pm 1.96 \times \sqrt{h^{-1}V}$. Will this interval contain the realized value of $\bar{Y}_{T+h}$ with 95% probability? Explain.

(b.iii) Suppose that $h = T$ (so you are interested in forecasting the average value of $Y$ for $T$ periods in the future). Show that

$$\sqrt{T} \left[ \begin{array}{c} (\bar{Y}_{1:T} - \mu_Y) \\ (\bar{Y}_{T+1:T} - \mu_Y) \end{array} \right] \overset{d}{\to} N \left( \begin{array}{cc} 0 & \Omega \\ 0 & 0 \end{array} \right) \Omega.$$

(b.iv) Consider a 95% “prediction interval” for $\bar{Y}_{T+1:T+h}$ as $\bar{Y}_{1:T} \pm 1.96 \times \sqrt{T^{-1}(\Omega + \Omega)}$. Will this interval contain the realized value of $\bar{Y}_{T+1:T+h}$ with 95% probability? Explain.

(b.v) Discuss how the results in (b.i)-(b.iv) change if $h = \rho T$ where $\rho$ is a fixed number that is greater than zero (for example, $\rho = 0.6$ or $\rho = 2$).

(c) Using the data for $Y_t =$ growth rate of OECD per-capita GDP.

(c.i) Estimate an AR(1) model for $Y_t$ (i.e., $Y_t = \mu + u_t$ with $u_t = \phi u_{t-1} + \varepsilon_t$.

Equivalently, you can estimate $Y_t = \alpha + \phi Y_{t-1} + \varepsilon_t$.

(c.ii) Use the estimates to construct an estimate of $\Omega$.

(c.iii) Use your analysis in parts (a)-(b) to form a 95% "prediction interval" for the average value of $Y_t$ over the 2015-2054. Over 2015-2114.

(c.iv) Per capita OECD GDP in 2014 was $39,124. What is a 95% confidence interval for the value of per capita GDP in 2054? In 2114?

(d) Your answer in (c) assumed that $Y_t$ followed an AR(1) model. Suppose you were unsure of this, and instead used a Newey-West estimator for $\Omega$. 
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(d.i) Repeat part (c.ii) and (c.iv) using a Newey-West estimator with \( k = 4 \) lags.

(d.ii) How sensitive is your answer to using \( k = 4 \) lags? What about \( k = 3 \) or \( k = 5 \)? How many lags do you think you should use? Why?

3. In question 2(a)-(c), you used the model \( Y_t = \mu + u_t \), where \( u_t = \phi u_{t-1} + \epsilon_t \). In this question I want you to use the model:

\[
\begin{align*}
Y_t &= \mu_t + u_t \\
\mu_t &= \mu_{t-1} + e_t
\end{align*}
\]

where \( \begin{bmatrix} \epsilon_t \\ e_t \end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{\epsilon}^2 & 0 \\ 0 & \sigma_e^2 \end{bmatrix} \right) \)

(a) Suppose \( \sigma_{\epsilon} = 0 \). Show that this is same model as in parts 2(a)-(c).

(b) I want you to think about reasonable values for \( \sigma_{\epsilon} \).

(b.i) Show the standard deviation of \( (\mu_{t+k} - \mu_t) = k^{1/2} \sigma_{\epsilon} \).

(b.ii) Using \( \sigma_{\epsilon} = 0.002 \), show that the standard deviation of \( (\mu_{t+25} - \mu_t) = 0.01 \), and the standard deviation of \( (\mu_{t+50} - \mu_t) = 0.014 \).

(b.iii) Using \( \sigma_{\epsilon} = 0.003 \), show that the standard deviation of \( (\mu_{t+25} - \mu_t) = 0.015 \), and the standard deviation of \( (\mu_{t+50} - \mu_t) = 0.021 \).

(b.iv) Using \( \sigma_{\epsilon} = 0.001 \), show that the standard deviation of \( (\mu_{t+25} - \mu_t) = 0.005 \), and the standard deviation of \( (\mu_{t+50} - \mu_t) = 0.007 \).

(b.v) Discuss the ‘reasonableness’ of these values of \( \sigma_{\epsilon} \) for the growth rate of per-capita GDP for the OECD.

(c) Using \( \sigma_{\epsilon} = 0.002 \) and the values of \( \phi \) and \( \sigma_{\epsilon} \) you computed in (2.c.i), use the Kalman filter to compute estimates of \( \mu_t \) for \( t = 1915-2014 \). Initialize the Kalman filter using
(c.i) Why did I suggest this initialization for \(u_0\)? Is it reasonable?

(c.ii) How ‘informative’ is the initialization about the value of \(\mu_0\)? Does it restrict \(\mu_0\) in an important way?

(c.iii) After running the Kalman filter, plot the values of \(Y_t\) and \(\mu_{t|t}\). Discuss the plot.

(c.iv) Consider the interval \(\mu_{t|t} \pm 1.96 \sqrt{P_{t|t}(\mu)}\) where \(P_{t|t}(\mu)\) is “2,2” element of the Kalman filter \(P_{t|t}\) matrix. Does this interval contain \(\mu_t\) with probability 0.95. Explain.

(c.v) Plot \(\mu_{t|t}\) along with these 95% error bands. Discuss the plot.

(d) Repeat your calculations using \(\sigma_c = 0.001\) and \(\sigma_c = 0.003\). Discuss how these values of \(\sigma_c\) change the estimates of \(\mu_t\) and the error bands.

(e) Suppose you have a prior that \(\sigma_c = 0.001, 0.002\) and 0.003 each with probability 1/3. Using the \(Y_t\) data, compute the posterior distribution of \(\sigma_c\).

(f) (Somewhat harder) Using \(\sigma_c = 0.002\),

(f.i) Construct a 95% "prediction interval" for the average value of \(Y_t\) over the 2015 – 2054. Over 2015-2014.

(f.ii) Construct a 95% confidence interval for the value of per capita GDP in 2054? In 2114?

(g) (Harder still) Using the posteriors for \(\sigma_c\) in (e), repeat (f).