Exercises for Thursday Evening

1. Presented by: Flavia Cifarelli

Suppose that $y_t = x_t\beta + u_t$, with $u_t = 0.8u_{t-1} + \varepsilon_t$, and $x_t = \varepsilon_{t+1}$, where ε_t is iid(0,1). Let $\hat{\beta}$ denote the OLS estimator of β.

(a) Show that $\sqrt{T}(\hat{\beta}-\beta)\xrightarrow{d} N(0,V)$, and derive an expression for V.

(b) Noting that $(1-0.8\mathrm{L})u_t = \varepsilon_t$, the GLS estimator of β can be formed by regressing $(1-0.8\mathrm{L})y_t$ onto $(1-0.8\mathrm{L})x_t$. Derive the probability limit of the GLS estimator.

2. Presented by: Beatrice Retali

Suppose that $y_t = x_t\beta + u_t$, where $u_t = \phi u_{t-1} + \varepsilon_t$, $x_t = e_t + \theta e_{t-1}$, where ε_t and e_t are both i.i.d. with mean zero and variance σ_ε^2 and σ_e^2, and ε_t and e_τ are independent for all t and τ. Let $\hat{\beta}$ denote the OLS estimator of β based on a sample of size T, and let $\hat{u}_t$ denote the OLS residual.

(a) Show that $\sqrt{T}(\widehat{\beta}-\beta)\xrightarrow{d} N(0,V)$ $\sqrt{T}(\widehat{\beta}-\beta)\xrightarrow{d} N(0,V)$ and derive an expression for V.

(b) Suppose that $T = 100$, $\widehat{\beta} = 2.1$, $\frac{1}{100}\sum_{t=1}^{100} x_t^2 = 5$, $\frac{1}{99}\sum_{t=2}^{100} x_t x_{t-1} = 2.5$, $\frac{1}{98}\sum_{t=3}^{100} x_t x_{t-2} = 1.0$, $\frac{1}{100}\sum_{t=1}^{100} \hat{u}_t^2 = 4$, $\frac{1}{99}\sum_{t=2}^{100} \hat{u}_t\hat{u}_{t-1} = 3.6$, $\frac{1}{98}\sum_{t=3}^{100} \hat{u}_t\hat{u}_{t-2} = 3.1$, $\frac{1}{99}\sum_{t=2}^{100} x_t\hat{u}_{t-1} = 0.8$, $\frac{1}{99}\sum_{t=2}^{100} \hat{u}_t x_{t-1} = 0.2$. Use your result in (a) to construct an approximate 95% confidence interval for β.

3. Presented by: Laurant Ott

Suppose $y_t = \mu + u_t$ where u_t follows the stationary ARMA(1,1) process $u_t = \phi u_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$ where $\varepsilon_t \sim$ iid $(0, \sigma^2)$.

(a) Show that $\sqrt{T}(\bar{y}-\mu)\xrightarrow{d} N(0,V)$ and derive an expression for V. (Hint: use the formula for the ACGF for an ARMA(1,1) model to derive V.)

(b) In a sample with $T = 400$, $\overline{y} = 5.4$. A researcher uses the data $y_t - \overline{y}$ to estimate the AR parameter and finds $\hat{\phi} = 0.66$, $\hat{\theta} = -0.81$ and $\hat{\sigma}_\varepsilon = 1.2$. Construct a 95% confidence interval for μ.

4. Presented by: Evert Reins

Suppose Y_t follows the AR(1) model $Y_t = \phi Y_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim i.i.d.,(0,\sigma^2)$. You are interested in forecasting Y_{T+2} given $Y_{1:T} = (Y_1, Y_2, \ldots, Y_T)$.

(a) Show that $\mathrm{E}(Y_{T+2} \mid Y_{1:T}) = \gamma Y_T$, with $\gamma = \phi^2$.

(b) Let $\hat{\phi}$ denote the OLS estimator of ϕ, obtained from the regression of Y_{t+1} onto Y_t for t = 1,…, T-1. Let $\hat{\gamma} = \hat{\phi}^2$ denote an estimator of γ. Show that $\sqrt{T}\left(\hat{\gamma} - \gamma\right) \xrightarrow{d} N(0, V_{\hat{\gamma}})$ and derive an expression for $V_{\hat{\gamma}}$.

(c) An alternative estimator of γ is obtained by regressing Y_{t+2} onto Y_t for t = 1, … , T-2 using OLS. Call this estimator $\tilde{\gamma}$. Show that $\sqrt{T}\left(\tilde{\gamma} - \gamma\right) \xrightarrow{d} N(0, V_{\tilde{\gamma}})$ and derive an expression for $V_{\tilde{\gamma}}$.

(d) Is $\hat{\gamma}$ preferred to $\tilde{\gamma}$? Explain.

Some Additional Exercises

1. y_t follows the stationary AR(1) model $y_t = \phi y_{t-1} + \varepsilon_t$. A researcher wants to estimate ϕ. He cannot find data on y_t, but can find estimates of y_t based on survey. Let x_t denote the survey estimate of y_t, and suppose that $x_t = y_t + u_t$, where u_t is iid$(0,\sigma_u^2)$ and u_t is independent of ε_j for all t and j. The researcher estimates ϕ by regressing x_t onto x_{t-1} using x_{t-2} as an instrument.

(a) Suppose that $\phi \neq 0$. Derive the asymptotic distribution of the IV estimator

(b) Suppose $\phi = 0$, show that the IV estimator is not consistent. Derive the limiting distribution of the estimator.

(c) How would you test $\phi = 0$?

2. Suppose that y_t follows the AR(1) process

$$y_t = \mu + u_t$$

$$u_t = \rho u_{t-1} + \varepsilon_t$$

where ε_t is $iid(0,\sigma^2)$, and $|\rho| < 1$. Let

$$\bar{y} = T^{-1}\sum y_t$$

(a) Show that $\bar{y} \xrightarrow{p} \mu$.

(b) Show that $\sqrt{T}(\bar{y} - \mu) \xrightarrow{d} N(0,V)$ and derive an expression for V.

(c) Let $\hat{\lambda}_0 = \frac{1}{T}\sum_{i=1}^{n}(y_t - \bar{y})^2$. Show that $\hat{\lambda}_0 \xrightarrow{p} \lambda_0$.

(d) Show that $\sqrt{T}\left(\hat{\lambda}_0 - \lambda_0\right) \xrightarrow{d} N(0,U)$, and derive an expression for U. (Feel free to make any additional assumptions necessary to show this result.)

3. Suppose that $y_t = x_t\beta + u_t$, where $u_t = \phi u_{t-1} + \varepsilon_t$, $x_t = e_t + \theta e_{t-1}$, where ε_t and e_t are both i.i.d. with mean zero and variance σ_ε^2 and σ_e^2, and ε_t and e_τ are independent for all t and τ. Let $\hat{\beta}$ denote the OLS estimator of β based on a sample of size T, and let $\hat{u}_t$ denote the OLS residual.

(a) Show that $\sqrt{T}(\hat{\beta} - \beta) \xrightarrow{d} N(0,V)$ and derive an expression for V.

(b) Suppose that $T=100$, $\widehat{\beta}=2.1$, $\frac{1}{100}\sum_{t=1}^{100} x_t^2=5$, $\frac{1}{99}\sum_{t=2}^{100} x_t x_{t-1}=2.5$, $\frac{1}{98}\sum_{t=3}^{100} x_t x_{t-2}=1.0$, $\frac{1}{100}\sum_{t=1}^{100} \hat{u}_t^{\,2}=4$, $\frac{1}{99}\sum_{t=2}^{100} \hat{u}_t\hat{u}_{t-1}=3.6$, $\frac{1}{98}\sum_{t=3}^{100} \hat{u}_t\hat{u}_{t-2}=3.1$, $\frac{1}{99}\sum_{t=2}^{100} x_t\hat{u}_{t-1}=0.8$, $\frac{1}{99}\sum_{t=2}^{100} \hat{u}_t x_{t-1}=0.2$. Construct a 95% confidence interval for β.

(c) An alternative to OLS in this problem is GLS. Explain how you would construct the GLS estimator. Is GLS preferred to OLS in this situation? Explain.

4. Suppose that a sequence of random variables y_t is generated by the model:

$$y_t = \mu + u_t$$
$$u_t = \varepsilon_t + \theta\varepsilon_{t-1}$$

where $\varepsilon_t \sim Niid(0,\sigma^2)$. From a realization of size $T=100$, I calculate

$$\bar{y} = \frac{1}{T}\sum_{t=1}^{T} y_t = 4$$

$$\sum_{t=2}^{T}(y_t-\bar{y})(y_{t-1}-\bar{y}) = 30$$

$$\sum_{t=1}^{T}(y_t-\bar{y})^2 = 100$$

(a) Show that $\bar{y}$ is the ordinary least squares estimate of μ.

(b) Calculate an estimate of the variance of the OLS estimator of μ.

(c) Construct a 95% confidence interval for μ.

5. Suppose that y_t follows the AR(1) process

$$y_t = \mu + u_t$$

$$u_t = \rho u_{t-1} + \varepsilon_t$$

where ε_t is $iid(0,\sigma^2)$, and $|\rho|<1$. Let

$$\hat{\mu}^{ols} = T^{-1}\sum y_t$$

(e) Show that $\sqrt{T}(\hat{\mu}^{ols}-\mu)\xrightarrow{d} N(0,V_{ols})$ and derive an expression for V_{ols}.

(b) Suppose that $u_0 = 0$. Explain how you would construct the GLS estimator of μ.

(c) Let $\hat{\mu}^{gls}$ denote the GLS from part c. Show that $\sqrt{T}(\hat{\mu}^{gls} - \mu) \xrightarrow{d} N(0, V_{gls})$ and derive an expression for V_{gls}.

(d) Is the GLS estimator better that the OLS estimator? Explain.

(e) Suppose that $T = 100$, and $\hat{\mu}^{ols} = 2$. Let $\hat{u}_t = y_t - \widehat{\mu}^{ols}$. The regression of $\hat{u}_t$ onto $\hat{u}_{t-1}$ yields a regression coefficient of 0.4 and the standard error of the regression is 1.1. Construct a 95% confidence interval for μ.

6. Suppose that y and x follow the process

$$y_t = ax_{t-1} + \varepsilon_t$$
$$x_t = \phi x_{t-1} + v_t$$

where $v_t = \varepsilon_t + e_t$, where ε and e are mutually independent iid $(0,1)$ processes. Suppose that $|\phi| < 1$. Let $\hat{\alpha}$ and $\hat{\phi}$ denote the OLS estimators of α and ϕ.

(a) Derive asymptotic distribution of the 2×1 vector $(\hat{\alpha}, \hat{\phi})$.

(b) In a sample of 100 observations $\hat{\alpha} = 1.3$ and $\hat{\phi} = 0.72$. Derive a 95% confidence interval for α.

7. Suppose that Y_t follows a stationary AR(1), and you have data Y_t, $t = 1, \ldots, 100$, and find $\bar{Y} = T^{-1}\sum_{t=1}^{T} Y_t = 18.1$, $\hat{\lambda}_0 = T^{-1}\sum_{t=1}^{T}(Y_t - \bar{Y})^2 = 2.4$, and $\hat{\lambda}_1 = (T-1)^{-1}\sum_{t=1}^{T}(Y_t - \bar{Y})(Y_{t-1} - \bar{Y}) = 1.0$. Construct a 95% confidence interval for $\mu = E(Y)$. (Hint: Proceed in three steps. Step 1: Think about the appropriate asymptotic covariance matrix for $\sqrt{T}(\bar{Y} - \mu)$. Step 2: Think about how you would estimate the covariance matrix from the data given in the problem. Step 3: Construct the confidence interval.)

9. Suppose that y_t is iid$(0,\sigma^2)$ with $E(y_t^4) = \kappa$. Let $\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} y_t^2$.

(a) Show that $\sqrt{T}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} N(0, V_{\hat{\sigma}^2})$ and derive an expression for $V_{\hat{\sigma}^2}$.

(b) Suppose that $T = 100$, $\hat{\sigma}^2 = 2.1$ and $\frac{1}{T}\sum_{t=1}^{T} y_t^4 = 16$. Construct a 95% confidence interval for σ, where σ is the standard deviation of y.

10. Suppose that y_t follows the AR(1) process $y_t = \phi y_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim$ iidN$(0,\sigma_\varepsilon^2)$. Let $\hat{\phi}$ denote the OLS estimator, $\hat{\varepsilon}_t = y_t - \hat{\phi} y_{t-1}$ denote the OLS residual, and $\hat{\sigma}_\varepsilon^2 = \frac{1}{T}\sum_{t=1}^{T} \hat{\varepsilon}_t^2$ denote the estimator of σ_ε^2.

(a) Suppose that $|\phi| < 1$ and $\{y_t\}$ is stationary. Show that $\hat{\sigma}_\varepsilon^2 \xrightarrow{p} \sigma_\varepsilon^2$

(b) Suppose that $\phi = 1$ and $y_0 = 0$. Show that $\hat{\sigma}_\varepsilon^2 \xrightarrow{p} \sigma_\varepsilon^2$. (Hint: Answer this after studying the unit root autoregression.)

11. (a) Suppose that y_t, for $t = 1, \ldots, T$ is generated by the model: $y_t = \mu + u_t$, where $u_t = \varepsilon_t - \theta\varepsilon_{t-1}$, where $\varepsilon_t \sim$ i.i.d. with mean zero and variance σ^2.

(a.1) Show that $\sqrt{T}(\bar{Y} - \mu) \xrightarrow{d} N(0, V)$ and derive an expression for V.

(a.2) Propose an estimator for V and sketch a proof showing that the estimator is consistent.

(b) Now suppose that y_t follows the same model as in (a) but the parameter μ undergoes a change during the sample. That is for $1 \le t \le \tau$, the parameter takes on the value μ_1 and for $\tau+1 \le t \le T$ the parameter takes on the value μ_2. (The parameters θ and σ^2 do not change.) Let $\bar{Y}_1 = \frac{1}{\tau}\sum_{t=1}^{\tau} y_t$ and $\bar{Y}_2 = \frac{1}{T-\tau}\sum_{t=\tau+1}^{T} y_t$. Let $\rho = \tau/T$.

(b.1) Suppose that ρ is fixed as $T \to \infty$. Show that $\sqrt{T}\left[\left(\bar{Y}_1 - \bar{Y}_2\right) - \left(\mu_1 - \mu_2\right)\right] \xrightarrow{d} N(0, \Omega)$ and derive an expression for Ω.

(b.2) Discuss how you would construct an approximate 95% confidence interval for $\mu_1 - \mu_2$ using the data y_t, $t = 1, \ldots, T$.

(b.3) Describe how you would test the null hypothesis that $\mu_1 = \mu_2$ if you know the break date τ.

12. Suppose that y_t follows a stationary Gaussian zero-mean ARMA(1,1) process, $(1-\rho L)y_t = (1-\theta L)\varepsilon_t$. Let $\hat{\rho}$ denote the IV estimator of ρ obtained by "regressing" y_t onto y_{t-1} using y_{t-2} as an instrument. Let T denote the sample size.

(a) Suppose that the true value of ρ is $\rho = 0.5$ and the true value of θ is $\theta = 0.1$.

(i) Show that $\hat{\rho}$ is consistent.

(ii) Derive the asymptotic distribution of $\hat{\rho}$.

(b) Suppose that the true value of ρ is $\rho = 0.5$ and the true value of θ is $\theta = 0.5$.

(i) Is $\hat{\rho}$ is consistent? Explain.

(ii) Derive the asymptotic distribution of $\hat{\rho}$.

13. Consider the IV model with $L = K = 1$, and conditionally homoskedastic errors. (That is $y_t = z_t\delta + \varepsilon_t$, x_t is a instrument, and $\text{var}(\varepsilon_t x_t) = \sigma_\varepsilon^2 \sigma_X^2$.)

In a sample of size 200, the following sample moment matrices are calculated:

$S_{yy} = 4.33$
$S_{zy} = 1.87$
$S_{zz} = 1.12$
$S_{xy} = 0.56$
$S_{xz} = 0.58$
$S_{xx} = 1.06$

(a) Compute $\hat{\delta}$ and its standard error (based on the usual large sample approximation).

(b) Compute a 95% confidence for δ using the usual large sample approximation to the distribution of $\hat{\delta}$.

(c) Compute the F-statistic from the regression of z onto x. What is the p-value for this statistic? Is there evidence that x is a weak instrument?

(d) Use the Anderson-Rubin method to compute a 95% confidence interval for δ. How does this confidence interval compare to the confidence interval that you computed in (a). Is this what you have expected given the F-statistic in (c)? Explain.

14. y_t follows the stationary AR(1) model $y_t = \phi y_{t-1} + \varepsilon_t$. A researcher wants to estimate ϕ. He cannot find data on y_t, but can find estimates of y_t based on survey. Let x_t denote the survey estimates of y_t, and suppose that $x_t = y_t + u_t$, where u_t is iid(0, σ_u^2) and u_t is independent of ε_j for all t and j. The researcher estimates ϕ by regressing x_t onto x_{t-1} using x_{t-2} as an instrument.

(a) Suppose that $\phi \neq 0$. Derive the asymptotic distribution of the IV estimator

(b) Suppose $\phi = 0$, show that the IV estimator is not consistent. Derive limiting distribution.

(c) How would you test $\phi = 0$?

15. Suppose that y_t follows the MA(1) process $y_t = (1-\theta L)\varepsilon_t$, where $\varepsilon_t \sim$ iid(0, σ_ε^2) and has as many higher-order moments as needed to answer the questions below.

(a) Show that the variance of y is given by $\sigma_y^2 = \sigma_\varepsilon^2(1+\theta^2)$.

(b) Let $\hat{\sigma}_y^2 = T^{-1}\sum_{t=1}^{T} y_t^2$.

(b.i) Show that $\hat{\sigma}_y^2 \xrightarrow{p} \sigma_y^2$

(b.ii) Show that $\sqrt{T}(\hat{\sigma}_y^2 - \sigma_y^2) \xrightarrow{d} N(0,V)$ and derive an expression for V in terms of moments of y.

(c) Using a sample of size $T = 100$, I find $\hat{\sigma}_y^2 = 1.52$, $T^{-1}\sum_{t=1}^{T}(y_t^2 - \hat{\sigma}_y^2)^2 = 4.8$, $T^{-1}\sum_{t=1}^{T}(y_t^2 - \hat{\sigma}_y^2)(y_{t-1}^2 - \hat{\sigma}_y^2) = 1.1$, $T^{-1}\sum_{t=1}^{T}(y_t^2 - \hat{\sigma}_y^2)(y_{t-2}^2 - \hat{\sigma}_y^2) = -0.4$, $T^{-1}\sum_{t=1}^{T} y_t y_{t-1}^3 = 3.6$, and $T^{-1}\sum_{t=1}^{T} y_t^3 y_{t-1} = 3.1$. Construct a 95% confidence interval for σ_y^2.

16.. y_t follows the stationary AR(1) model $y_t = \phi y_{t-1} + \varepsilon_t$. A researcher wants to estimate ϕ. He cannot find data on y_t, but can find estimates of y_t based on survey. Let x_t denote the survey estimates of y_t, and suppose that $x_t = y_t + u_t$, where u_t is iid$(0, \sigma_u^2)$ and u_t is independent of ε_j for all t and j. The researcher estimates ϕ by regressing x_t onto x_{t-1} using x_{t-2} as an instrument.

(a) Suppose that $\phi \neq 0$. Derive the asymptotic distribution of the IV estimator

(b) Suppose $\phi = 0$, show that the IV estimator is not consistent. Derive limiting distribution.

(c) How would you test $\phi = 0$?