Tuesday Evening Exercises

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1. Suppose that $X_t$ follows the MA(1) process: $X_t = \mu + \varepsilon_t - \theta \varepsilon_{t-1}$ where $\mu$ is a constant and $\varepsilon_t$ is i.i.d. with mean 0 and variance $\sigma^2$. Let $\bar{X} = T^{-1} \sum_{t=1}^T X_t$ denote the sample mean of $X$.

(a) Show that $\bar{X} = \mu + (1 - \theta) T^{-1} \sum_{t=1}^T \varepsilon_t + R$ (where $R$ is a “remainder term”) and derive an expression for $R$.

(b) Show that $\sqrt{T} R \xrightarrow{p} 0$.

(c) Show that $T^{-1/2} \sum_{t=1}^T \varepsilon_t \Rightarrow N(0, \sigma^2)$

(d) Use the result in (a)-(c) to show that $\sqrt{T} (\bar{X} - \mu) \Rightarrow N(0, V)$, and derive $V$.

(e) In a sample of $T = 100$, $\bar{X} = 36.1$, $T^{-1} \sum_{t=1}^{100} (X_t - \bar{X})^2 = 5.8$ and $T^{-1} \sum_{t=2}^{100} (X_t - \bar{X})(X_{t-1} - \bar{X}) = 1.9$. Construct a 95% confidence interval for $\mu$.

2. Consider the AR(2) model:

$$X_t = 5.0 + 1.3 X_{t-1} - 0.5 X_{t-2} + \varepsilon_t$$

where $\varepsilon_t \sim iid(0,4)$

(a) Verify that the roots of the autoregressive polynomial are outside the unit circle.

(b) Compute the eigenvalues of the model’s companion matrix and verify that they are the inverses of the roots you computed in (a).

(c) Assume that the initial conditions are chosen so that the process is covariance stationary.
   i. What is the mean of $X$?
   ii. What is the variance of $X$?
   iii. What is the correlation between $X_t$ and $X_{t+4}$.

3. Suppose that $X_t$ follows the stationary AR($p$) process

$$\phi(L)X_t = \beta + \varepsilon_t$$

where $\varepsilon_t \sim iid(0, \sigma^2)$, and $\phi(L) = 1 - \phi_1 L - ... - \phi_p L^p$.

(a) Derive the mean of $X$. Show that the mean can be written as $\beta / \phi(1)$ where $\phi(1) = 1 - \phi_2 - ... - \phi_p$ is the AR polynomial $\phi(z)$ evaluated at $z = 1$.

(b) Let $Y_t = (1 + L + L^2)X_t$. Show that $Y_t$ is covariance stationary.

4. Consider the MA(1) process

$$Y_t = \varepsilon_t - 3\varepsilon_{t-1}$$

where $\varepsilon_t$ is white noise with $\sigma^2_e = 1$. 
(a) Show that the autocovariances of $Y_t$ can also be represented by the MA(1) model

$$Y_t = \varepsilon_t - \frac{1}{3} \varepsilon_{t-1}$$

where $\varepsilon_t$ is white noise with variance $\sigma^2$. Derive the value of $\sigma^2$.

(b) Show that the representation using $\varepsilon_t$ is the Wold representation for $Y$.

(c) Derive a formula that expresses $\varepsilon_t$ as a function of current and lagged values of $e_t$, that is, determine the values of $c_i$ that yield

$$\varepsilon_t = \sum_{i=0}^{\infty} c_i e_{t-i}.$$

(d) Suppose $e_t \sim iid(0, \sigma^2)$. Is $\varepsilon_t \sim iid(0, \sigma^2)$? Explain.

5. Suppose $y_t$ follows the AR(1) model

$$y_t = \phi y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim iidN(0, 1)$ and $y_0 = 0$. You have data for $t = 1, 2, ..., 50, 52, ..., 100$. That is, you are missing data for $t = 51$. Denote the observations as $(Y_{1:50}, Y_{52:100})$. Write an explicit formula for the likelihood function, that is the density of $(Y_{1:50}, Y_{52:100})$ as a function of $\phi$. Note that I want an explicit formula – do NOT use the Kalman filter here.