Exercises For Tuesday Evening

1. Presented by: Anja Garbely

Consider the AR(2) model:

$$X_t = 2.0 + 1.3X_{t-1} - .5X_{t-2} + \varepsilon_t$$

where $\varepsilon_t \sim iid(0,4)$.

(a) Verify that the roots of the autoregressive polynomial are outside the unit circle.

(b) Compute the eigenvalues of model's companion matrix are verify that they are the inverses of the roots you computed in (a).

(c) Assume that the initial conditions are chosen so that the process is covariance stationary.

(i) What is the mean of X?

(ii) What is the variance of X?

(iii) What it the correlation between X_t and X_{t-5}?

2. Presented by: Felix Schonenberger

Consider the MA(2) process:

$$X_t = \varepsilon_t + .5\varepsilon_{t-1} - .36\varepsilon_{t-2},$$

where $\varepsilon_t \sim iid(0,4)$.

(a) Verify that the process is invertible.

(b) Construct three other MA processes that have the same autocovariances as the process above. (That is, construct three MA processes with MA coefficients and/or innovation variances different from the one above and from one another.)

3. Presented by: Elio Bolliger

(a) Suppose that X_t follows the MA(1) process: $X_t = e_t - 0.5e_{t-1}$ where e_t is *i.i.d.* with mean 0 and unit variance. Derive the Wold representation for X_t. Denote the Wold shocks as ε_t. Derive the variance of ε_t. Derive a formula that expresses ε_t as a function of current and lagged values of e_t.

(b) Suppose that X_t follows the MA(1) process: $X_t = e_t - 2.0\, e_{t-1}$ where e_t is *i.i.d.* with mean 0 and unit variance. Derive the Wold representation for X_t. Denote the Wold shocks as ε_t.

Derive the variance of ε_t. Derive a formula that expresses ε_t as a function of current and lagged values of e_t.

4. Presented by: Matthias Van Den Heuvel

Suppose that X_t follows the MA(1) process: $X_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1}$ where μ is a constant and ε_t is $i.i.d.$ with mean 0 and variance σ^2.

(a) Show that $\bar{X} = \mu + (1-\theta)T^{-1}\sum_{t=1}^{T}\varepsilon_t + R$ (where R is a "remainder term") and derive an expression for R.

(b) Show that $\sqrt{T}R \xrightarrow{p} 0$.

(c) Show that $T^{-1/2}\sum_{t=1}^{T}\varepsilon_t \xrightarrow{d} N(0,\sigma^2)$

(d) Use the result in (a)-(c) to show that $\sqrt{T}(\bar{X}-\mu) \xrightarrow{d} N(0,V)$, and derive V.

(e) (b) In a sample of $T = 100$, $\bar{X} = 31.2$, $T^{-1}\sum_{t=1}^{100}(X_t - \bar{X})^2 = 5$ and $T^{-1}\sum_{t=2}^{100}(X_t - \bar{X})(X_{t-1} - \bar{X}) = 1.5$. Construct a 95% confidence interval for μ.

Some Additional Exercices

1. For the general AR(p) model, prove that the roots of the AR polynomial are the reciprocals of the eigenvalues of the companion matrix.

2. Consider the MA(2) process:

$$X_t = \varepsilon_t - \varepsilon_{t-1} - 6.0\varepsilon_{t-2},$$

where $\varepsilon_t \sim iid(0,1)$. Suppose that you have data on X_t for $t \le T$. How would you use these data to forecast X_{T+1}? Be specific, and provide an explicit formula.

3. Suppose u_t follows the stationary ARMA(1,1) process $u_t = \phi u_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$, and let

$\lambda_k = E(u_t u_{t+k}) = E(u_t u_{t-k})$.

(a) Derive the moving average representation for u_t. (That is, find the values of c_i in the representation $u_t = c_0\varepsilon_t + c_1\varepsilon_{t-1} + c_2\varepsilon_{t-2} + c_3\varepsilon_{t-3} + \ldots$)

(b) Show that $\lambda_k = \phi\lambda_{k-1}$ for $k \ge 2$.

4. Let $\phi(L) = (1 - 1.2L + .6L^2)$ and $\phi(L)^{-1} = c(L) = c_0 + c_1 L + c_2 L^2 + \ldots$. Calculate c_i, for $i = 1, 2, \ldots, 10$.

5. Suppose that a random variable y_t is generated by one two possible stochastic processes: (i) $y_t = 0.4y_{t-1} + \varepsilon_t$ or (ii) $y_t = \varepsilon_t + 0.4\varepsilon_{t-1}$, where ε_t is iid with mean 0 and variance σ^2. Suppose that you had a partial realization of the process (a sample) of length T, $(y_1, y_2, \ldots, y_T)$. How would you use these data to choose between process (i) and process (ii)? Explain.

6. Suppose that y_t follows the stationary AR(p) process $\phi(\mathrm{L})y_t = \varepsilon_t$, where ε_t is $iid(0,\sigma_\varepsilon^2)$. Let $x_t = (1+\mathrm{L})y_t$. Prove that x_t is covariance stationary.

7. Suppose that Y_t follows a MA(2) model:

$$Y_t = \varepsilon_t - .2\varepsilon_{t-1} + .4\varepsilon_{t-2}$$

where $\varepsilon_t \sim iidN(0,1)$. You need to make a forecast of Y_{T+1}, but the only piece of information that you have is $Y_T = 2.0$.

(a) What is your forecast value of Y_{T+1}?

(b) What is the variance of the forecast error associated with this forecast?

(c) Suppose that you used data $\{Y_t\}_{t=1}^{T}$ to construct your forecast of Y_{T+1}, and suppose that T was large. What would be the variance of the forecast error? (Does your answer depend on the invertibility of the MA process?)

8. Suppose that Y_t follows the AR(1) model with heteroskedastic errors: $Y_t = \phi_1 Y_{t-1} + \varepsilon_t$, where $\varepsilon_t | Y_{0:t-1} \sim N(0, \sigma_{t-1}^2)$ where $\sigma_{t-1}^2 = \omega + \alpha Y_{t-1}^2$ for $t = 2, \ldots, T$. Suppose that $Y_0 = 0$.

(a) Write the explicit joint density/likelihood function for $Y_{1:T}$ (conditional on $Y_0 = 0$).

(b) The file **518_EX1_5.csv** contains a realization of T =100 observations for Y_t.

(i) Suppose $\phi = 0.8$, $\omega = 1$ and $0.0 \leq \alpha \leq 0.9$. Compute the MLE of α.

(ii) You are Bayesian. Before looking at the data you know that $\phi = 0.8$ and $\omega = 1$, but are unsure about the value of α. Your prior is $P(\alpha = 0.4) = 0.6$ and $P(\alpha = 0.7) = 0.4$. Use the data to find the posterior for α.