Exercises For Tuesday Evening

1. Consider the AR(2) model: \( X_t = 1.3X_{t-1} - .5X_{t-2} + \varepsilon_t \), where \( \varepsilon_t \sim iid(0, 9) \).
   
   (a) Compute the roots of the AR polynomial
   
   (b) Compute the eigenvalues of the companion matrix.
   
   (c) Verify that the roots of the AR polynomial are the reciprocals of the eigenvalues of the companion matrix.
   
   (d) For a general AR(2) model, prove that the roots of the AR polynomial are the reciprocals of the eigenvalues of the companion matrix. (Hint: write out the characteristic polynomial of the companion matrix.)
   
   (e) Assume that the initial conditions are chosen so that the process is covariance stationary. Derive the autocovariances of the \( X \) process.

2. Suppose that \( Y_t \) follows the covariance stationary AR(\( p \)) process: \( \phi(L)Y_t = \varepsilon_t \), where \( \varepsilon_t \sim iid(0, \sigma^2) \). Let \( X_t = (1+L)Y_t \). Show that \( X_t \) is covariance stationary.

3. (a) Consider the MA(1) process: \( X_t = \varepsilon_t - 0.8\varepsilon_{t-1} \) where \( \varepsilon_t \sim iid(0, 1) \).
   
   (i) Verify that the process is invertible.
   
   (ii) Construct another MA process with the same autocovariances as the process above.
   
   (b) Consider the MA(2) process: \( X_t = \varepsilon_t - 2.1\varepsilon_{t-1} - 2.7\varepsilon_{t-2} \), where \( \varepsilon_t \sim iid(0, 4) \).
   
   (i) Is the process invertible?
   
   (ii) Construct three other MA processes that have the same autocovariances as the process above. (That is, construct three MA processes with MA coefficients and/or innovation variances different from the one above and from one another.)
   
   (iii) Which of the processes that you constructed in (b) are invertible?
   
   (iv) What is the variance of the 'Wold' shocks for \( X_t \)?

4. Suppose that \( X_t \) follows the MA(1) process: \( X_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1} \) where \( \mu \) is a constant and \( \varepsilon_t \) is i.i.d. with mean 0 and variance \( \sigma^2 \).
   
   (a) Show that \( \bar{X} = \mu + (1-\theta)T^{-1}\sum_{t=1}^{T} \varepsilon_t + R \) (where \( R \) is a “remainder term”) and derive an
expression for $R$.

(b) Show that $\sqrt{TR} \xrightarrow{p} 0$.

(c) Show that $T^{-1/2} \sum_{t=1}^{T} \epsilon_i \xrightarrow{d} N(0, \sigma^2)$

(d) Use the result in (a)-(c) to show that $\sqrt{T} (\bar{X} - \mu) \xrightarrow{d} N(0, V)$, and derive $V$.

(e) (b) In a sample of $T = 100$, $\bar{X} = 31.2$, $\sum_{t=1}^{100} (X_i - \bar{X})^2 = 5$ and $\sum_{t=2}^{100} (X_i - \bar{X})(X_{t-1} - \bar{X}) = 1.5$. Construct a 95% confidence interval for $\mu$. 
3. Presented by: Valentino DeSilvestro (Bern)

(a) Consider the MA(1) process: \( X_t = \varepsilon_t - 0.8\varepsilon_{t-1} \) where \( \varepsilon_t \sim i.i.d.(0,1) \).

(i) Verify that the process is invertible.

(ii) Construct another MA process with the same autocovariances as the process above.

(b) Suppose that \( X_t \) follows the MA(1) process: \( X_t = v_t - 5.0v_{t-1} \) where \( v_t \) is i.i.d. with mean 0 and variance 1. Write the Wold representation as \( X_t = c(L)\varepsilon_t = \varepsilon_t + c_1\varepsilon_{t-1} + c_2\varepsilon_{t-2} + ... \)

(i) Derive the values of \( c_i \) for \( i = 1, 2, ... \) and the variance of \( \varepsilon_t \).

(ii) Derive the function that relates the Wold errors \( \varepsilon_t \) to current and lagged values of \( v_t \).

(c) (i) Using the same process for \( X_t \) as in (b), you learn that \( X_t = 3 \). Predict the value of \( X_{t+1} \).

(ii) Suppose \( t \) is large and you know the values of \( X_1, X_2, ..., X_t \). How would you predict \( X_{t+1} \)?

4. Presented by: Laura Nowzohour

Suppose \( Y_t = 0.5Y_{t-1} + \varepsilon_t \), where \( \varepsilon_t = \sigma e_t, e_t \sim i.i.d. N(0,1), ln(\sigma) = \rho ln(\sigma_{t-1}) + 0.10\varepsilon_{t-1} \) and \( Y_0 = 0, \sigma_0 = 1, \varepsilon_0 = 0 \). You have data \( Y_t \) for \( t = 1, ..., T \) collected in a vector \( Y_{1:T} \).

(a) Derive the likelihood function \( f(Y_{1:T} | \rho) \).

(b) In the Excel file TUE_Q4.XLSX you will find data for \( Y_{1:T} \) with \( T = 100 \).

(b.i) Find the value of \( f(Y_{1:T} | \rho) \) for \( \rho = 0.8 \).

(b.ii) Find the value of \( f(Y_{1:T} | \rho) \) for \( \rho = 0.7 \).

(b.iii) You have a prior for \( \rho \) that has the form: \( P(\rho = 0.7) = 0.4 \) and \( P(\rho = 0.8) = 0.6 \). Derive the posterior for \( \rho \).

1. Presented by: Eleonora Brandimarti (Geneva)

(Hint: Review Week 1: Thursday night, exercise 4).

Suppose that \( X_t \) follows the MA(1) process: \( X_t = \mu + \varepsilon_t - \theta \varepsilon_{t-1} \) where \( \mu \) is a constant and \( \varepsilon_t \) is i.i.d. with mean 0 and variance \( \sigma^2 \).
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(a) Show that $\bar{X} = \mu + (1 - \theta)T^{-1} \sum_{t=1}^{T} \epsilon_t + R$ (where $R$ is a “remainder term”) and derive an expression for $R$.

(b) Show that $\sqrt{T}R \xrightarrow{p} 0$.

(c) Show that $T^{-1/2} \sum_{t=1}^{T} \epsilon_t \xrightarrow{d} N(0,\sigma^2)$

(d) Use the result in (a)-(c) to show that $\sqrt{T}(\bar{X} - \mu) \xrightarrow{d} N(0,V)$, and derive $V$.

(e) (b) In a sample of $T = 100$, $\bar{X} = 31.2$, $T^{-1} \sum_{t=1}^{100} (X_t - \bar{X})^2 = 5$ and $T^{-1} \sum_{t=2}^{100} (X_t - \bar{X})(X_{t-1} - \bar{X}) = 1.5$. Construct a 95% confidence interval for $\mu$. 