Tuesday Evening

1. Gema Lax-Martinez (Lausanne)

(a) For an AR(2) model, prove that the roots of the AR polynomial are the reciprocals of the eigenvalues of the companion matrix. (Hint: write out the characteristic polynomial of the companion matrix.)

(b) Extend the result in (a) to the general AR(p) model.

2. David Gallusser (Basel)

(a) Consider the MA(1) process: $X_t = \varepsilon_t - 0.9\varepsilon_{t-1}$ where $\varepsilon_t \sim$ iid(0,1).

(i) Verify that the process is invertible.

(ii) Construct another MA process with the same autocovariances as the process above.

(b) Consider the MA(2) process: $X_t = \varepsilon_t - 1.1\varepsilon_{t-1} + 0.18\varepsilon_{t-2}$ where $\varepsilon_t \sim$ iid(0,1).

(i) Verify that the process is invertible.

(ii) Construct three other MA processes that have the same autocovariances as the process above. (That is, construct three MA processes with MA coefficients and/or innovation variances different from the one above and from one another.)

(c) Suppose that X_t follows the $MA(1)$ process: $X_t = \varepsilon_t - 4.0\varepsilon_{t-1}$ where ε_t is $i.i.d.$ with mean 0 and variance 1. Derive the Wold representation for X_t.

3. Yu Wu (EPFI)

$Y_t = X_t + V_t$. Let $\{\varepsilon_t\}$ and $\{e_t\}$ be mutually uncorrelated white noise processes with unit variances.

(a) Suppose $X_t = \varepsilon_t$ and $V_t = e_t$.

(i) Show that Y_t has the representation $Y_t = a_t$, where a_t is white noise. Derive the value of σ_a.

(ii) Write an expression for a_t as a function of current and lagged values of ε_t and e_t.

(b) Suppose $X_t = (1 - 0.5\text{L})\varepsilon_t$ and $V_t = (1 + 0.9\text{L})e_t$.

(i) Show that Y_t has an invertible MA(1) representation, say $Y_t = (1 - \theta\text{L})a_t$, where a_t is white noise. Derive the value of θ and the σ_a.

(ii) Write an expression for a_t as a function of current and lagged values of ε_t and e_t.

(c) Suppose $(1 - 0.5\text{L})X_t = \varepsilon_t$ and $(1 + 0.9\text{L})V_t = e_t$.

(i) Show that Y_t has an ARMA(2,1) representation $(1 - 0.5\text{L})(1 + 0.9\text{L})Y_t = (1 - \theta\text{L})a_t$, where a_t is white noise. Derive the value of θ and the σ_a.

(ii) Write an expression for a_t as a function of current and lagged values of ε_t and e_t.

(d) (Not For Presentation in Class. Everyone should do this after working (a)-(c)) Suppose X_t follows the ARMA(p_X,q_X) process $\phi_X(\text{L})X_t = \theta_X(\text{L})\varepsilon_t$ and V_t follows the ARMA(p_V,q_V) process $\phi_V(\text{L})X_t = \theta_V(\text{L})e_t$. Show that Y_t follows the ARMA(p_Y, q_Y) process $\phi_Y(\text{L})Y_t = \theta_Y(\text{L})a_t$, where a_t is white nose, $p_Y = p_X + p_V$ and $q_Y = \max(p_X + q_V, p_V + q_X)$.

4. Robert Oleschak (SNB)

Suppose $Y_t = \phi Y_{t-1} + \varepsilon_t$, where $\varepsilon_t = \sigma_t e_t$, $e_t \sim i.i.d.\ N(0,1)$, $ln(\sigma_t) = \rho\, ln(\sigma_{t-1}) + \varepsilon_{t-1}$ and $Y_0 = 0$, $\sigma_0 = 1$, $\varepsilon_0 = 0$. Let $\theta = (\phi, \rho)$. You have data Y_t, for $t = 1, \ldots, T$ collected in a vector $Y_{1:T}$.

(a) Find the likelihood function $f(Y_{1:T} \mid \theta)$.

(b) In the Excel file Tuesday.XLSX you will find data for $Y_{1:T}$ with T =100.

(i) Assume that $\rho = 0.9$. Plot the log-likelihood as a function of ϕ. Find the MLE of ϕ.

(ii) Assume that $\phi = 0.8$. Plot the log-likelihood as a function of ρ. Find the MLE of ρ.

(iii) Find the MLE of $\theta = (\phi, \rho)$.