Wednesday Evening Exercises

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1. Suppose that $Y_t$ follows the AR(1) process with heteroskedastic errors: $Y_t = \phi Y_{t-1} + \varepsilon_t$, where $\varepsilon_t|Y_{1:t-1} \sim N(0, \sigma^2_{t-1})$ with $\sigma^2_{t-1} = \omega + \alpha Y^2_{t-1}$ for $t = 2, \ldots, T$. Suppose $Y_0 = 0$.

(a) Write the explicit joint density/likelihood function for $Y_{1:T}$ (conditional on $Y_0 = 0$).

(b) The file 518_EX1_5.csv contains a realization of $T = 100$ observations for $Y_t$.

i. Suppose $\phi = 0.8$, $\omega = 1$ and $0.0 \leq \alpha \leq 0.9$. Compute the MLE of $\alpha$. (Hint: Have your computer compute the value of the likelihood for a grid of values of $\alpha$ and numerically choose the maximum.)

ii. You are Bayesian. Before looking at the data you know that $\phi = 0.8$, and $\omega = 1$, but are unsure about the value of $\alpha$. Your prior is $P(\alpha = 0.4) = 0.6$ and $P(\alpha = 0.7) = 0.4$. Use the data to find the posterior for $\alpha$.

2. Consider the model used in the notes to derive the Kalman filter. Assume that $E(w_t v_t') = G$. (In the notes $G = 0$). Derive the Kalman filter.

3. Suppose

$$ y_t = \xi_t + w_t $$

$$ \xi_t = 0.8 \xi_{t-1} + v_t $$

where $w_t \sim iidN(0,2)$, $v_t \sim iidN(0,3)$, $\{w_t\}$ and $\{v_t\}$ are independent, and $\xi_0 = 0$.

(a) Write an explicit (non-recursive) formula that expresses $(y_{1:t}, \xi_{1:t})$ as a function of $(v_{1:t}, w_{1:t})$.

(b) Suppose $t$ is large.

i. Show that the distribution of $(\xi_t, y_t|\xi_{t-1} = 2.5)$ is normal, and derive the mean vector and covariance matrix.

ii. You are now told that $y_t = 4.1$. Find $E(\xi_t|\xi_{t-1} = 2.5, y_t = 4.1)$ and $\text{var}(\xi_t|\xi_{t-1} = 2.5, y_t = 4.1)$.

4. Suppose that $y_1$ and $y_2$ are scalar random variables with

$$ y_1 = \xi + \varepsilon_1 $$

$$ y_2 = \xi + \varepsilon_2 $$

where $\xi$, $\varepsilon_1$ and $\varepsilon_2$ are mutually independent random variables with $\xi \sim N(0, \sigma^2)$, $\varepsilon_1 \sim N(0,1)$ and $\varepsilon_2 \sim N(0,1)$. A researcher has data on $y_1$ and $y_2$ and wants to estimate the value of $\xi$. They propose the estimator $\hat{\xi} = \frac{1}{2}(y_1 + y_2)$.

(a) Suppose $\sigma^2 = 1$. Compute the mean squared error (MSE) of $\hat{\xi}$.

(b) A more general estimator is $\hat{\xi} = \lambda_1 y_1 + \lambda_2 y_2$, where $\lambda_1$ and $\lambda_2$ are constants.

i. Suppose $\sigma^2 = 1$. What values of $\lambda_1$ and $\lambda_2$ yield the estimator with the lowest MSE?
ii. Is there a value of $\sigma^2$ that rationalizes the estimator with $\lambda_1 = \lambda_2 = \frac{1}{2}$?

(c) An even more general estimator is $\tilde{\xi} = g(y_1, y_2)$. What function $g$ yields the estimator with the lowest MSE? What is the value of the MSE?

5. Consider the model $y_t = s_t + \varepsilon_t$ where $\varepsilon_t \sim iidN(0, 1)$ and $s_t$ is a 0-1 binary variable with $P(s_t = 1|s_{t-1} = 0) = 0.3$ and $P(s_t = 1|s_{t-1} = 1) = 0.8$. Assume $s_t$ and $\varepsilon_t$ are independent for all $t$ and $k$.

(a) Suppose the history of information on $y$ tells you that $P(s_{t-1} = 1|y_{1:t-1}) = 0.6$. You observe $y_t = 1.5$. Compute $P(s_t = 1|y_{1:t})$.

(b) Generalize your calculations and derive a recursive algorithm for computing $P(s_t = 1|y_{1:t})$ as a function of $y_t$ and $P(s_{t-1} = 1|y_{1:t-1})$.

(c) Explain how the algorithm in (b) can be used the compute the likelihood function/joint density of $y_{1:T}$. 