Exercises For Wednesday Evening

1. Presented by: Vera Zabrodina (Basel)

Suppose that y_{1t} and y_{2t} are scalar random variables with

$$y_{1t} = x_t + \varepsilon_{1t}$$

$$y_{2t} = x_t + \varepsilon_{2t}$$

where x_t, ε_{1t}, and ε_{2t} are mutually independent i.i.d. sequences of $N(0,1)$ random variables. A researcher has data on y_{1t} and y_{2t} and would like to use these data to estimate the value of x_t. She proposes the estimator $\hat{x}_t = \frac{1}{2}(y_{1t} + y_{2t})$.

(a) Compute the mean squared error (MSE) of $\hat{x}_t$.

(b) A more general estimator is $\tilde{x}_t = \lambda_1 y_{1t} + \lambda_2 y_{2t}$, where λ_1 and λ_2 are two constants. What values of λ_1 and λ_2 yield the estimator with the smallest MSE?

(c) An even more general estimator is $\breve{x}_t = g(y_{1t}, y_{2t})$. What function g yields the estimator with the smallest MSE?

2. Presented by: Pascal Meighty (Lausanne)

Consider the state-space model

$$y_t = \beta x_t + v_t$$

$$x_t = \phi x_{t-1} + \varepsilon_t$$

where x and y are scalars,

$$\begin{pmatrix} v_t \\ \varepsilon_t \end{pmatrix} \sim i.i.d.\ N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_v^2 & \sigma_{v\varepsilon} \\ \sigma_{v\varepsilon} & \sigma_\varepsilon^2 \end{bmatrix} \right)$$

with $\sigma_{v\varepsilon} \neq 0$, and $\{x\}$ is not observed. Using the usual Kalman filter notation, let $x_{t/k} = E(x_t \mid \{y_i\}_{i=1}^{k})$ and $P_{t/k} = Var(x_t \mid \{y_i\}_{i=1}^{k})$. Derive an algorithm that computes $x_{t/t}$ and $P_{t/t}$ as a function of $x_{t-1/t-1}$, $P_{t-1/t-1}$ and y_t.

3. Presented by: Tobias Lehmann (Lausanne)

Consider the model $y_t = s_t + \varepsilon_t$ where $\varepsilon_t \sim$ i.i.d. N(0,1) and s_t is a 0-1 binary random variable with $P(s_t = 1 | s_{t-1} = 0) = 0.3$ and $P(s_t = 1 | s_{t-1} = 1) = 0.8$.

(a) Suppose that the history of information on y tells you that $P(s_{t-1} = 1 \mid y_{1:t-1}) = 0.6$. You observe $y_t = 1.5$. Compute $P(s_t = 1 \mid y_{1:t})$.

(b) Generalize your calculations and derive a recursive algorithm for computing $P(s_t = 1 \mid y_{1:t})$ as a function of y_t and $P(s_{t-1} = 1 \mid y_{1:t-1})$. Explain how this result can be used the compute the likelihood function/joint density of $y_{1:T}$.

4. Presented by: Seoni Han (Graduate)

Suppose that $y_t = x_t + \varepsilon_t$, where $x_t = 0.8x_{t-1} + e_t$, and were $\begin{bmatrix} \varepsilon_t \\ e_t \end{bmatrix} \sim i.i.d.\, N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\right)$.

Suppose you know that $x_0 = 2$ and $y_1 = 4.6$.

(a) Derive the minimum mean square error estimate of x_1.

(b) What is the mean squared error of the estimate in (a)?

Now suppose now that $\{\varepsilon_t\}$ and $\{e_t\}$ are mutually independent iid processes with (i) $\varepsilon_t = -2$ with probability 0.5 and $\varepsilon_t = 2$ with probability 0.5, and (ii) $e_t = -1$ with probability 0.5 and $e_t = 1$ with probability 0.5. Suppose you know that $x_0 = 2$ and $y_1 = 4.6$

(c) Derive the linear minimum mean square error estimate of x_1.

(e) What is the mean squared error of this estimate?

(f) Is the estimate in (e) the minimum mean squared estimate? Explain.

5. Presented by: Severin Lenhard (Bern)

Suppose that $y_t = x_t\beta + u_t$, where $u_t = \phi u_{t-1} + \varepsilon_t$, where ε_t and x_t are both i.i.d. with mean zero and variance σ_ε^2 and σ_x^2, and ε_t and x_τ are independent for all t and τ. Let $\hat{\beta}$ denote the OLS estimator of β based on a sample of size T, and let $\hat{u}_t$ denote the OLS residual.

(a) Show that $\sqrt{T}(\hat{\beta} - \beta) \xrightarrow{d} N(0,V)$ $\sqrt{T}(\hat{\beta} - \beta) \xrightarrow{d} N(0,V)$ and derive an expression for V.

(b) Suppose that $T = 100$, $\widehat{\beta} = 2.1$, $\frac{1}{100}\sum_{t=1}^{100} x_t^2 = 5$, $\frac{1}{99}\sum_{t=2}^{100} x_t x_{t-1} = 0.5$, $\frac{1}{98}\sum_{t=3}^{100} x_t x_{t-2} = 0.03$, $\frac{1}{100}\sum_{t=1}^{100} \hat{u}_t^{\,2} = 4$, $\frac{1}{99}\sum_{t=2}^{100} \hat{u}_t \hat{u}_{t-1} = 3.6$, $\frac{1}{98}\sum_{t=3}^{100} \hat{u}_t \hat{u}_{t-2} = 3.1$, $\frac{1}{99}\sum_{t=2}^{100} x_t \hat{u}_{t-1} = 0.2$, $\frac{1}{99}\sum_{t=2}^{100} \hat{u}_t x_{t-1} = -0.05$. Construct a 95% confidence interval for β.