Exercises For Wednesday Evening

1. Suppose that \( y_{1t} \) and \( y_{2t} \) are scalar random variables with

\[
y_{1t} = x_t + \varepsilon_{1t} \\
y_{2t} = x_t + \varepsilon_{2t}
\]

where \( x_t, \varepsilon_{1t}, \) and \( \varepsilon_{2t} \) are mutually independent i.i.d. sequences of \( N(0,1) \) random variables. A researcher has data on \( y_{1t} \) and \( y_{2t} \) and would like to use these data to estimate the value of \( x_t \). She proposes the estimator \( \hat{x}_t = \frac{1}{2}(y_{1t} + y_{2t}) \).

(a) Compute the mean squared error (MSE) of \( \hat{x}_t \).

(b) A more general estimator is \( \tilde{x}_t = \lambda_1 y_{1t} + \lambda_2 y_{2t} \), where \( \lambda_1 \) and \( \lambda_2 \) are two constants. What values of \( \lambda_1 \) and \( \lambda_2 \) yield the estimator with the smallest MSE?

(c) An even more general estimator is \( \tilde{x}_t = g(y_{1t}, y_{2t}) \). What function \( g \) yields the estimator with the smallest MSE?

2. Consider the state-space model

\[
y_t = \beta x_t + \nu_t \\
x_{t+1} = \phi x_t + \varepsilon_t
\]

where \( x \) and \( y \) are scalars,

\[
\begin{pmatrix} \nu_t \\ \varepsilon_t \end{pmatrix} \sim i.i.d. \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_\nu & \sigma_{\nu \varepsilon} \\ \sigma_{\nu \varepsilon} & \sigma^2_\varepsilon \end{bmatrix} \right)
\]

with \( \sigma_{\nu \varepsilon} \neq 0 \), and \( \{x\} \) is not observed. Using the usual Kalman filter notation, let \( x_{i|k} = E(x_t \mid \{y_{1t}\}_{i=1}^k) \) and \( P_{i|k} = Var(x_t \mid \{y_{1t}\}_{i=1}^k) \). Derive an algorithm that computes \( x_{i|t} \) and \( P_{i|t} \) as a function of \( x_{i-1|t-1} \), \( P_{i-1|t-1} \) and \( y_t \).

3. Consider the model \( y_t = s_t + \varepsilon_t \) where \( \varepsilon_t \sim i.i.d. \mathcal{N}(0,1) \) and \( s_t \) is a 0-1 binary random variable with \( P(s_t = 1 \mid s_{t-1} = 0) = 0.3 \) and \( P(s_t = 1 \mid s_{t-1} = 1) = 0.8 \).

(a) Suppose that the history of information on \( y \) tells you that \( P(s_{t-1} = 1 \mid y_{1:t-1}) = 0.6 \). You observe \( y_t = 1.5 \). Compute \( P(s_t = 1 \mid y_{1:t}) \).
(b) Generalize your calculations and derive a recursive algorithm for computing $P(s_t = 1 \mid y_{1:t})$ as a function of $y_t$ and $P(s_{t-1} = 1 \mid y_{1:t-1})$. Explain how this result can be used to compute the likelihood function/joint density of $y_{1:T}$.

4. Suppose that $y_t = x_t + \varepsilon_t$, where $x_t = 0.8x_{t-1} + e_t$, and where

$$\begin{bmatrix}
\varepsilon_t \\
e_t
\end{bmatrix} \sim \text{i.i.d. } N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Suppose you know that $x_0 = 2$ and $y_1 = 4.6$.

(a) Derive the minimum mean square error estimate of $x_1$.

(b) What is the mean squared error of the estimate in (a)?

Now suppose now that $\{\varepsilon_t\}$ and $\{e_t\}$ are mutually independent iid processes with (i) $\varepsilon_t = -2$ with probability 0.5 and $\varepsilon_t = 2$ with probability 0.5, and (ii) $e_t = -1$ with probability 0.5 and $e_t = 1$ with probability 0.5. Suppose you know that $x_0 = 2$ and $y_1 = 4.6$

(c) Derive the linear minimum mean square error estimate of $x_1$.

(e) What is the mean squared error of this estimate?

(f) Is the estimate in (e) the minimum mean squared estimate? Explain.

5. Suppose that $Y_t$ follows the AR(1) model with heteroskedastic errors: $Y_t = \phi Y_{t-1} + \varepsilon_t$, where $\varepsilon_t | Y_{0:t-1} \sim N(0, \sigma_{t-1}^2)$ where $\sigma_{t-1}^2 = \omega + \alpha Y_{t-1}^2$ for $t = 2, \ldots, T$. Suppose that $Y_0 = 0$.

(a) Write the explicit joint density/likelihood function for $Y_{1:T}$ (conditional on $Y_0 = 0$).

(b) (If you have time and a computer). The file 518_EX1_5.csv contains a realization of $T=100$ observations for $Y_t$.

(i) Suppose $\phi = 0.8$, $\omega = 1$ and $0.0 \leq \alpha \leq 0.9$. Compute the MLE of $\alpha$.

(ii) You are Bayesian. Before looking at the data you know that $\phi = 0.8$ and $\omega = 1$, but are unsure about the value of $\alpha$. Your prior is $P(\alpha = 0.4) = 0.6$ and $P(\alpha = 0.7) = 0.4$. Use the data to find the posterior for $\alpha$. 

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