Studienzentrum Gerzensee Doctoral Program in Economics
Econometrics Week 4 Miscellaneous Exercises

Basics

1. For the general AR(p) model, prove that the roots of the AR polynomial are the reciprocals of the eigenvalues of the companion matrix.

2. Consider the AR(2) model:

$$X_t = 1.3X_{t-1} - .5X_{t-2} + \varepsilon_t$$

where $\varepsilon_t \sim iid(0,4)$.

(a) Verify that the roots of the autoregressive polynomial are outside the unit circle, or equivalently, that the eigenvalues of the model's companion matrix have modulus less than 1.

(b) Assume that the initial conditions are chosen so that the process is covariance stationary. Derive the autocovariances for the X_t process.

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(c) What must be assumed about the initial conditions so that the process is covariance stationary?

3. Consider the MA(2) process:

$$X_t = \varepsilon_t + .4\varepsilon_{t-1} - .32\varepsilon_{t-2},$$

where $\varepsilon_t \sim iid(0,4)$.

(a) Verify that the process is invertible.

(b) Construct three other MA processes that have the same autocovariances as the process above. (That is, construct three MA processes with MA coefficients and/or innovation variances different from the one above and from one another.)

4. Consider the MA(2) process:

$$X_t = \varepsilon_t - \varepsilon_{t-1} - 6.0\varepsilon_{t-2},$$

where $\varepsilon_t \sim iid(0,1)$. Suppose that you have data on X_t for $t \le T$. How would you use these data to forecast X_{T+1}? Be specific, and provide an explicit formula.

5. Suppose u_t follows the stationary ARMA(1,1) process $u_t = \phi u_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$, and let $\lambda_k = E(u_t u_{t+k}) = E(u_t u_{t-k})$.

(a) Derive the moving average representation for u_t. (That is, find the values of c_i in the representation $u_t = c_0\varepsilon_t + c_1\varepsilon_{t-1} + c_2\varepsilon_{t-2} + c_3\varepsilon_{t-3} + \ldots$)

(b) Show that $\lambda_k = \phi\lambda_{k-1}$ for $k \geq 2$.

6. Suppose that X_t is generated by:

$$X_t = Y_t + u_t$$

where

$$Y_t = \phi Y_{t-1} + \varepsilon_t$$

and ε_t and u_t are mutually independent iid mean zero processes.

(a) Show that X_t has an ARMA(1,1) representation: $(1-\phi L)X_t = (1-\theta L)e_t$. (Hint: What are the autocovariances of $\varepsilon_t + u_t - \phi u_{t-1}$? What are the autocovariances of $e_t - \theta e_{t-1}$?)

(b) How are the sets of parameters $(\sigma_u^2, \sigma_\varepsilon^2, \phi)$ and $(\theta, \sigma_e^2, \phi)$ related?

(c) Can any ARMA(1,1) model be written as an AR(1) plus independent white noise?

(d) Suppose that you have data on X_t for $t \leq T$. You do not have data for Y or u. How would you forecast X_{T+4}?

(e) How would your answer to the last question change if you had data on Y_t and u_t for $t \leq T$?

7. Suppose that Y_t has an ARIMA(0,1,1) representation:

$$(1-L)Y_t = (1+0.4L)w_t$$

where w_t is white noise.

(a) Can Y_t be represented as the sum of a random walk (say $\tau_t = \tau_{t-1} + \varepsilon_t$) and white noise (say a_t), where ε_t and a_t are independent white noise?

(b) Can Y_t be represented as the sum of a random walk (say $\tau_t = \tau_{t-1} + \varepsilon_t$) and white noise (say a_t), where ε_t and a_t are correlated white noise? (i.e. ε_t and a_t are correlated but $(a_t \varepsilon_t)'$ is vector white noise)

(c) Can you deduce the value the value of $\rho = cor(a_t \varepsilon_t)$ from the autocovariances of Y_t? If you cannot deduce the value of ρ, can you bound its values?

8. Let $\phi(L) = (1 - 1.2L + .6L^2)$ and $\phi(L)^{-1} = c(L) = c_0 + c_1 L + c_2 L^2 + \dots$. Calculate c_i, for $i = 1, 2, \dots, 10$.

9. Suppose that a random variable y_t is generated by one two possible stochastic processes: (i) $y_t = 0.4 y_{t-1} + \varepsilon_t$ or (ii) $y_t = \varepsilon_t + 0.4\varepsilon_{t-1}$, where ε_t is iid with mean 0 and variance σ^2. Suppose that you had a partial realization of the process (a sample) of length T, $(y_1, y_2, \dots, y_T)$. How would you use these data to choose between process (i) and process (ii)? Explain.

10. Suppose that y_t follows the stationary AR(p) process $\phi(\mathrm{L}) y_t = \varepsilon_t$, where ε_t is $iid(0, \sigma_\varepsilon^2)$. Let $x_t = (1+\mathrm{L}) y_t$. Prove that x_t is covariance stationary.

11. Suppose that Y_t follows a MA(2) model:

$$Y_t = \varepsilon_t - .2\varepsilon_{t-1} + .4\varepsilon_{t-2}$$

where $\varepsilon_t \sim iidN(0,1)$. You need to make a forecast of Y_{T+1}, but the only piece of information that you have is $Y_T = 2.0$.

(a) What is your forecast value of Y_{T+1}?

(b) What is the variance of the forecast error associated with this forecast?

(c) Suppose that you used data $\{Y_t\}_{t=1}^T$ to construct your forecast of Y_{T+1}, and suppose that T was large. What would be the variance of the forecast error? (Does your answer depend on the invertibility of the MA process?)

12. Suppose that X_t follows the $MA(1)$ process: $X_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1}$ where μ is a constant and ε_t is $i.i.d.$ with mean 0 and variance σ^2. Show that $\overline{X} \xrightarrow{p} \mu$.

13. (a) Suppose that X_t follows the $MA(1)$ process: $X_t = \varepsilon_t - 0.33\varepsilon_{t-1}$ where ε_t is *i.i.d.* with mean 0 and variance 1. Derive the Wold representation for X_t.

(b) Suppose that X_t follows the $MA(1)$ process: $X_t = \varepsilon_t - 3.0\varepsilon_{t-1}$ where ε_t is *i.i.d.* with mean 0 and variance 1. Derive the Wold representation for X_t.

14. (a) Consider the MA(1) process: $X_t = \varepsilon_t - 0.9\varepsilon_{t-1}$ where $\varepsilon_t \sim$ iid(0,1).

(i) Verify that the process is invertible.

(ii) Construct another MA process with the same autocovariances as the process above.

(b) Consider the MA(2) process: $X_t = \varepsilon_t - 1.1\varepsilon_{t-1} + 0.18\varepsilon_{t-2}$ where $\varepsilon_t \sim$ iid(0,1).

(i) Verify that the process is invertible.

(ii) Construct three other MA processes that have the same autocovariances as the process above. (That is, construct three MA processes with MA coefficients and/or innovation variances different from the one above and from one another.)

(c) Suppose that X_t follows the $MA(1)$ process: $X_t = \varepsilon_t - 4.0\varepsilon_{t-1}$ where ε_t is *i.i.d.* with mean 0 and variance 1. Derive the Wold representation for X_t.

15. $Y_t = X_t + V_t$. Let $\{\varepsilon_t\}$ and $\{e_t\}$ be mutually uncorrelated white noise processes with unit variances.

(a) Suppose $X_t = \varepsilon_t$ and $V_t = e_t$.

(i) Show that Y_t has the representation $Y_t = a_t$, where a_t is white noise. Derive the value of σ_a.

(ii) Write an expression for a_t as a function of current and lagged values of ε_t and e_t.

(b) Suppose $X_t = (1 - 0.5\mathrm{L})\varepsilon_t$ and $V_t = (1 + 0.9\mathrm{L})e_t$.

(i) Show that Y_t has an invertible MA(1) representation, say $Y_t = (1 - \theta\mathrm{L})a_t$, where a_t is white noise. Derive the value of θ and the σ_a.

(ii) Write an expression for a_t as a function of current and lagged values of ε_t and e_t.

(c) Suppose $(1 - 0.5\text{L})X_t = \varepsilon_t$ and $(1 + 0.9\text{L})V_t = e_t$.

(i) Show that Y_t has an ARMA(2,1) representation $(1 - 0.5\text{L})(1 + 0.9\text{L})Y_t = (1 - \theta\text{L})a_t$, where a_t is white noise. Derive the value of θ and the σ_a.

(ii) Write an expression for a_t as a function of current and lagged values of ε_t and e_t.

(d) (Not For Presentation in Class. Everyone should do this after working (a)-(c)) Suppose X_t follows the ARMA(p_X,q_X) process $\phi_X(\text{L})X_t = \theta_X(\text{L})\varepsilon_t$ and V_t follows the ARMA(p_V,q_V) process $\phi_V(\text{L})X_t = \theta_V(\text{L})e_t$. Show that Y_t follows the ARMA(p_Y, q_Y) process $\phi_Y(\text{L})Y_t = \theta_Y(\text{L})a_t$, where a_t is white nose, $p_Y = p_X + p_V$ and $q_Y = \max(p_X + q_V, p_V + q_X)$.

16. Suppose $Y_t = \phi Y_{t-1} + \varepsilon_t$, where $\varepsilon_t = \sigma_t e_t$, $e_t \sim i.i.d.\ N(0,1)$, $ln(\sigma_t) = \rho\ ln(\sigma_{t-1}) + \varepsilon_{t-1}$ and $Y_0 = 0$, $\sigma_0 = 1$, $\varepsilon_0 = 0$. Let $\theta = (\phi, \rho)$. You have data Y_t, for $t = 1, \dots, T$ collected in a vector $Y_{1:T}$.

(a) Find the likelihood function $f(Y_{1:T} \mid \theta)$.

Signal Extraction, Kalman Filtering, and so forth.

1. Suppose

$$y_t = \xi_t + w_t$$
$$\xi_t = 0.8\xi_{t-1} + v_t$$

where $w_t \sim iidN(0,2)$ and $v_t \sim iidN(0,3)$ and $\{w_t\}$ and $\{v_t\}$ are independent. Suppose that you know $\xi_{t-1} = 3.4$, and $y_t = 4.1$. Find $E(\xi_t \mid \xi_{t-1} = 3.4,\ y_t = 4.1)$ and $\text{var}(\xi_t \mid \xi_{t-1} = 3.4,\ y_t = 4.1)$.

2. Suppose that y_t follows the AR(1) model $y_t = \phi y_{t-1} + \varepsilon_t$, for $t = 1,2,...,100$, with $y_0 = 0$ and $\varepsilon_t \sim$ Niid(0, σ^2). Suppose data on y_{50} is missing.

Suppose that you know the values of ϕ and σ^2.

(a) Find an expression for $E(y_{50} \mid y_{49})$

(b) Write down an expression that would allow you to calculate $E(y_{50} \mid \{y_{49}, y_{51}\})$

(c) How would you construct $E(y_{50} \mid \{\{y_1, y_{2,} ... y_{49}, y_{51}, y_{52}, ..., y_{100}\})$?

3. Consider the state-space model

$$y_t = \beta x_t + v_t$$
$$x_t = \phi x_{t-1} + \varepsilon_t$$

where x and y are scalars,

$$\begin{pmatrix} v_t \\ \varepsilon_t \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_v^2 & \sigma_{v\varepsilon} \\ \sigma_{v\varepsilon} & \sigma_\varepsilon^2 \end{bmatrix} \right)$$

with $\sigma_{v\varepsilon} \neq 0$, and $\{x\}$ is not observed. Using the usual Kalman filter notation, let $x_{t/k} = E(x_t \mid \{y_i\}_{i=1}^k)$ and $P_{t/k} = Var(x_t \mid \{y_i\}_{i=1}^k)$. Derive an algorithm that computes $x_{t/t}$ and $P_{t/t}$ as a function of $x_{t-1/t-1}$, $P_{t-1/t-1}$ and y_t.

4. Suppose that y_{1t} and y_{2t} are scalar random variables with

$$y_{1t} = x_t + \varepsilon_{1t}$$

$$y_{2t} = x_t + \varepsilon_{2t}$$

where x_t, ε_{1t}, and ε_{2t} are mutually independent i.i.d. sequences of $N(0,1)$ random variables. A researcher has data on y_{1t} and y_{2t} and would like to use these data to estimate the value of x_t. He proposes the estimator $\hat{x}_t = \frac{1}{2}(y_{1t} + y_{2t})$.

(a) Compute the mean squared error of $\hat{x}_t$.

(b) A more general estimator is $\tilde{x}_t = \lambda_1 y_{1t} + \lambda_{2t} y_{2t}$, where λ_1 and λ_2 are two constants. What values of λ_1 and λ_2 yield the estimator with the smallest mean squared error?

5. Suppose that $y_t = x_t + \varepsilon_t$, where $x_t = 0.8x_{t-1} + e_t$, and were
$\begin{bmatrix} \varepsilon_t \\ e_t \end{bmatrix} \sim iidN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\right)$. Suppose you know that $x_0 = 2$ and $y_1 = 6$.

(a) Derive the minimum mean square error estimate of x_1.

(b) What is the mean squared error of the estimate in (a)?

Now suppose now that $\{\varepsilon_t\}$ and $\{e_t\}$ are mutually independent iid processes with (i) $\varepsilon_t = -2$ with probability 0.5 and $\varepsilon_t = 2$ with probability 0.5, and (ii) $e_t = -1$ with probability 0.5 and $e_t = 1$ with probability 0.5. Suppose you know that $x_0 = 2$ and $y_1 = 6$

(c) Derive the linear minimum mean square error estimate of x_1.

(e) What is the mean squared error of this estimate?

(f) Is the estimate in (e) the minimum mean squared estimate? Explain.

6. Suppose that $y_t = x_t + u_t$, where $x_t = \varepsilon_t + 0.8\varepsilon_{t-1}$, and

$$\begin{bmatrix} \varepsilon_t \\ u_t \end{bmatrix} \sim iidN\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 & 3 \\ 3 & 4 \end{bmatrix} \right).$$ You are told that $y_{100} = 6$.

(a) Compute the best (minimum mean square error) estimate of x_{100}.

(b) Compute the best (minimum mean square error) estimate of x_{101}.

7. Suppose that $y_{it} = x_t + \varepsilon_{it}$, for $i = 1, \ldots, n$, $(x_t, \{\varepsilon_{it}\}_{i=1}^{n})$ are *i.i.d.* through time, and normally distributed with $x_t \sim N(0, \sigma_X^2)$, $\varepsilon_{it} \sim N(0, \sigma_\varepsilon^2)$, and ε_{it} is independent of ε_{jt} (for $j \neq i$) and x_t.

(a) Show that $x_{t/t} = \lambda\, \bar{Y}_t$, where $\bar{Y}_t = \frac{1}{n}\sum_{i=1}^{n} y_{it}$, and derive an expression for λ.

(b) Show that $\lim_{n\to\infty}\lambda = 1$

(c) Show that $plim_{n\to\infty} x_{t/t} = x_t$.

(d) Show that $x_{t/t}$ converges in mean square to x_t as $n \to \infty$.

8. Consider the model $y_t = s_t + \varepsilon_t$ where $\varepsilon_t \sim$ iidN(0,1) and s_t is a 0-10 binary random variable with $P(s_t = 1 | s_{t-1} = 0) = 0.3$ and $P(s_t = 1 | s_{t-1} = 1) = 0.8$. Suppose that the history of information on y tells you that $P(s_{t-1} = 1 \mid y_{1:t-1}) = 0.6$. You observe $y_t = 1.5$. Compute $P(s_t = 1 \mid y_{1:t})$.

9. Suppose x_t evolves as $x_t = 0.9x_{t-1} + u_t$ and $y_t = x_t + v_t$ where $(u_t, v_t) \sim$ i.i.d. $N(0, I_2)$. You learn that
$x_{t-1} = 0.0$ and $y_{t-1} = 2.0$

(a) Derive the probability density of $y_t \mid (x_{t-1} = 0.0$ and $y_{t-1} = 2.0)$.

(b) You learn that $y_t = 1.0$. Derive the probability density of
$x_t \mid (x_{t-1} = 0.0, y_{t-1} = 2.0$, and $y_t = 1.0)$.

10. Suppose x_t is a binary random variable with $P(x_t = 1 \mid x_{t-1} = 0) = 0.2$ and $P(x_t = 1 \mid x_{t-1} = 1) = 0.9$. The random variable y_t is related to x_t by the equation $y_t = x_t + v_t$ where
$v_t \sim$ i.i.d. N(0,1) and is independent of x_j for all t and j. You learn that $x_{t-1} = 0.0$ and $y_{t-1} =$ 2.0

(a) Derive the probability density of $y_t \mid (x_{t-1} = 0.0$ and $y_{t-1} = 2.0)$.

(b) You learn that $y_t = 1.0$. Derive the probability density of $x_t \mid (x_{t-1} = 0.0, y_{t-1} = 2.0, \text{ and } y_t = 1.0))$.

Frequency Domain Descriptive Statistics

1. Suppose that y_t is a series that is available semiannually (that is, twice per year), once in the winter and once in the summer. Suppose that the semiannual seasonal process for the series is $y_t = 0.9\, y_{t-2} + \varepsilon_t$, where ε_t is iid(0, σ^2).

(a) Derive and plot the spectrum of y. Discuss how the seasonality in the process is evident in spectrum.

(b) A researcher proposes to use $x_t = 0.5(1+\text{L})$, as a "seasonally adjusted" version of y. Compute the gain of the filter 0.5(1+L). Does this filter attenuate and/or eliminate the seasonality in y? Explain.

2. x_t follows the AR(1) process $x_t = 0.9x_{t-1} + \varepsilon_t$, and $y_t = (1-\text{L})x_t$.

(a) Draw time series plots of hypothetical realizations of x and y.

(b) Use the gain of the filter (1-L) to explain why the plot of y is "choppier" than the realization of x.

3. Suppose that x_t is generated by $x_t = y_t + u_t$, where $y_t = 0.6y_{t-1} + \varepsilon_t$, and ε_t and u_t are mutually independent iid N(0,1) processes.

(a) Plot the spectrum of y.
(b) Plot the spectrum of u.
(c) Plot the spectrum of x.

4. Compute and plot the gain of the "Kuznets" filter given in the lecture notes. Where is the gain maximized?

5. For each of the stationary stochastic processes given below:

(a) Generate a realization of length $T = 500$. (Use the stationary distribution for any initial conditions, so the realization is a draw from the stationary process.) Plot the time series.

(b) Compute the spectrum of the stochastic process.

(c) Discuss what the spectrum in (b) tells you about the characteristics of the realization plotted in (a).

Processes to use: Let $\varepsilon_t \sim i.i.d.\ N(0,1)$

(i) (White noise) $y_t = \varepsilon_t$

(ii) (AR(1)) $y_t = 0.95y_{t-1} + \varepsilon_t$

(iii) (MA(1)) $y_t = \varepsilon_t - 0.95\varepsilon_{t-1}$

(iv) (MA(4) $y_t = \varepsilon_t + 0.8\varepsilon_{t-4}$

Likelihood functions for time series models

1. Y_t follows the stationary AR(2) model $Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$, $\varepsilon_t \sim Niid(0,\sigma^2)$. Write the explicit joint density/likelihood function for $Y_{1:T}$.

2. Y_t follows the MA(1) model $Y_t = \varepsilon_t - \theta\varepsilon_{t-1}$, where $\varepsilon_t \sim Niid(0,\sigma^2)$ for $t = 1, \ldots, T$, and $\varepsilon_t = 0$ for $t \leq 0$.

(a) Write the explicit joint density/likelihood function for $Y_{1:T}$. Discuss how you would compute the MLE of θ and σ^2.

(b) Does the result in (a) require that $|\theta| < 1$? Explain.

(c) Suppose $\varepsilon_t \sim Niid(0,\sigma^2)$ for $t \leq 0$. How would you modify your answer to (a) and (b)?

3. Suppose that Y_t follows the AR(1) model with heteroskedastic errors: $Y_t = \phi_1 Y_{t-1} + \varepsilon_t$, where $\varepsilon_t | \varepsilon_{1:t-1} \sim N(0, \sigma_{t-1}^2)$ where $\sigma_{t-1}^2 = \omega + \alpha Y_{t-1}^2$ for $t = 2, \ldots, T$. Suppose that $Y_0 = 0$.

(a) Write the explicit joint density/likelihood function for $Y_{1:T}$ (conditional on $\varepsilon_0 = Y_0 = 0$).

(b) Suppose ω and α were known. What is the MLE of ϕ?

4. Y_t follows the stationary AR(1) model $Y_t = \phi_1 Y_{t-1} + \varepsilon_t$, $\varepsilon_t \sim Niid(0,\sigma^2)$. You have data for $Y_{1:100}$ and $Y_{102:T}$ (so that data are missing at time $t = 101$). Write the joint density/likelihood for $(Y_{1:100}, Y_{102:T})$.

Asymptotics

1. Suppose that y_t follows the AR(1) process

$$y_t = \mu + u_t$$

$$u_t = \rho u_{t-1} + \varepsilon_t$$

where ε_t is $iid(0,\sigma^2)$, and $|\rho|<1$. Let

$$\overline{y} = T^{-1}\sum y_t$$

(a) Show that $\overline{y} \xrightarrow{p} \mu$.

(b) Show that $\sqrt{T}(\overline{y}-\mu) \xrightarrow{d} N(0,V)$ and derive an expression for *V*.

(c) Let $\hat{\lambda}_0 = \frac{1}{T}\sum_{i=1}^{n}(y_t - \overline{y})^2$. Show that $\hat{\lambda}_0 \xrightarrow{p} \lambda_0$.

(d) Show that $\sqrt{T}\left(\hat{\lambda}_0 - \lambda_0\right) \xrightarrow{d} N(0,U)$, and derive an expression for *U*. (Feel free to make any additional assumptions necessary to show this result.)

2. Suppose that X_t follows the MA(1) process: $X_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1}$ where μ is a constant and ε_t is $i.i.d.$ with mean 0 and variance σ^2.

(a) Show that $\overline{X} = \mu + (1-\theta)T^{-1}\sum_{t=1}^{T}\varepsilon_t + R$ (where *R* is a "remainder term") and derive an expression for *R*.

(b) Show that $\sqrt{T}R \xrightarrow{p} 0$.

(c) Show that $T^{-1/2}\sum_{t=1}^{T}\varepsilon_t \xrightarrow{d} N(0,\sigma^2)$

(d) Use the result in (a)-(c) to show that $\sqrt{T}(\overline{X}-\mu) \xrightarrow{d} N(0,V)$, and derive *V*.

(e) (b) In a sample of $T = 100$, $\overline{X} = 31.2$, $T^{-1}\sum_{t=1}^{100}(X_t - \overline{X})^2 = 5$ and $T^{-1}\sum_{t=2}^{100}(X_t - \overline{X})(X_{t-1} - \overline{X}) = 1.5$. Construct a 95% confidence interval for μ.

(Questions 3-7 require the FCLT.)

3. (a) Assume that y_t is generated by

$$y_t = u_t + \varepsilon_t$$

where

$$u_t = u_{t-1} + a_t$$

$$\begin{bmatrix} a_t \\ \varepsilon_t \end{bmatrix} \sim Niid\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), t = 1, \ldots, T$$

and $u_0 = 0$.

Show that

$$T^{-3/2} \sum_{t=1}^{T} y_t \Rightarrow \int_0^1 W_1(s) ds$$

where $W_1(s)$ is a standard Weiner process.

(b) Suppose x_t is generated by

$$x_t = \frac{1}{T} u_t + \varepsilon_t$$

where u and ε follow the same processes as in part (a). Show that

$$T^{-1/2} \sum_{t=1}^{T} x_t \Rightarrow \int_0^1 W_1(s) ds + W_2(1)$$

where $W_1(s)$ and $W_2(s)$ are independent standard Weiner processes.

4. Let $y_t = \mu + \varepsilon_t$, where $\varepsilon_t \sim$ iidN(0,1), and let $\bar{y} = T^{-1} \sum_{t=1}^{T} y_t$

(a) Show that $T^{-1/2} \sum_{t=1}^{[sT]} (y_t - \bar{y}) \Rightarrow W(s) - sW(1)$, where $W(s)$ is a Wiener process.

(b) Find the limiting distribution of $A_T = \sum_{t=1}^{T} \left[\sum_{i=1}^{t} (y_t - \bar{y}) \right]^2$ (in terms of $W(s)$), after dividing A_T by the appropriate power of T, that is, derive the asymptotic distribution of $T^{-q} A_T$ for the appropriate value of q.

5. Suppose that y_t follows the model

$$y_t = \mu + u_t$$
$$u_t = u_{t-1} + \varepsilon_t$$

where ε_t is i.i.d. $N(0,\sigma^2)$, and $u_0 = 0$.

(a) Show that $T^{-3/2}\sum_{t=1}^{T}(y_t/\sigma) \Rightarrow \int_0^1 W(s)ds$.

(b) Is $\bar{y}$ a consistent estimator of μ?

(c) Show that $\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=2}^{T}(y_t - y_{t-1})^2 \xrightarrow{p} \sigma^2$.

(d) Show that y_1 is the MLE of μ.

(e) Based on a sample of $T = 1000$ observations, I compute $\bar{y} = 4.3, \hat{\sigma}^2 = .64$ and find that $y_1 = 1.4$. Construct a 95% confidence interval for μ.

6. Suppose that $y_t = y_{t-1} + u_t$, where $y_0 = 0$, $u_t = \theta(L)\varepsilon_t$, $\theta(L) = (1 - \theta_1 L - \theta_2 L^2)$ and $\varepsilon_t \sim$ iidN$(0,\sigma^2)$. Show that

$$\frac{1}{T^2}\sum_{t=1}^{T} y_t^2 \Rightarrow \theta(1)^2\sigma^2 \int_0^1 W(s)^2\, ds,$$ where $W(s)$ is standard Wiener process.

(Hint: Show that $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} y_t \Rightarrow \theta(1)\sigma W(s)$.)

7. Suppose that $y_t = \frac{1}{T}u_t + \varepsilon_t$, where $u_t = u_{t-1} + e_t$, $u_0 = 0$, and where $\{\varepsilon_t\}$ and $\{e_t\}$ are mutually independent sequences of standard normal random variables.

(a) Show that $\frac{1}{T}\sum_{t=1}^{T} y_t^2 \xrightarrow{p} 1$.

(b) Let $x_t = x_{t-1} + y_t$, with $x_0 = 0$. Derive a limiting representation for $\frac{1}{T}x_T^2$.

Linear Models

1. Suppose that $y_t = x_t\beta + u_t$, where $u_t = \phi u_{t-1} + \varepsilon_t$, $x_t = e_t + \theta e_{t-1}$, where ε_t and e_t are both i.i.d. with mean zero and variance σ_ε^2 and σ_e^2, and ε_t and e_τ are independent for all t and τ. Let $\hat{\beta}$ denote the OLS estimator of β based on a sample of size T, and let $\hat{u}_t$ denote the OLS residual.

(a) Show that $\sqrt{T}(\hat{\beta}-\beta)\xrightarrow{d} N(0,V)$ and derive an expression for V.

(b) Suppose that $T = 100$, $\hat{\beta} = 2.1$, $\frac{1}{100}\sum_{t=1}^{100} x_t^2 = 5$, $\frac{1}{99}\sum_{t=2}^{100} x_t x_{t-1} = 2.5$, $\frac{1}{98}\sum_{t=3}^{100} x_t x_{t-2} = 1.0$, $\frac{1}{100}\sum_{t=1}^{100} \hat{u}_t^2 = 4$, $\frac{1}{99}\sum_{t=2}^{100} \hat{u}_t\hat{u}_{t-1} = 3.6$, $\frac{1}{98}\sum_{t=3}^{100} \hat{u}_t\hat{u}_{t-2} = 3.1$, $\frac{1}{99}\sum_{t=2}^{100} x_t\hat{u}_{t-1} = 0.8$, $\frac{1}{99}\sum_{t=2}^{100} \hat{u}_t x_{t-1} = 0.2$. Construct a 95% confidence interval for β.

(c) An alternative to OLS in this problem is GLS. Explain how you would construct the GLS estimator. Is GLS preferred to OLS in this situation? Explain.

2. Suppose that a sequence of random variables y_t is generated by the model:

$$y_t = \mu + u_t$$
$$u_t = \varepsilon_t + \theta\varepsilon_{t-1}$$

where $\varepsilon_t \sim Niid(0,\sigma^2)$. From a realization of size $T = 100$, I calculate

$$\bar{y} = \frac{1}{T}\sum_{t=1}^{T} y_t = 4$$

$$\sum_{t=2}^{T} (y_t - \bar{y})(y_{t-1} - \bar{y}) = 30$$

$$\sum_{t=1}^{T} (y_t - \bar{y})^2 = 100$$

(a) Show that $\bar{y}$ is the ordinary least squares estimate of μ.

(b) Calculate an estimate of the variance of the OLS estimator of μ.

(c) Construct a 95% confidence interval for μ.

3. Suppose that y_t follows the AR(1) process

$$y_t = \mu + u_t$$

$$u_t = \rho u_{t-1} + \varepsilon_t$$

where ε_t is $iid(0,\sigma^2)$, and $|\rho|<1$. Let

$$\hat{\mu}^{ols} = T^{-1}\sum y_t$$

(a) Show that $\sqrt{T}(\hat{\mu}^{ols} - \mu)\xrightarrow{d} N(0,V_{ols})$ and derive an expression for V_{ols}.

(b) Suppose that $u_0 = 0$. Explain how you would construct the GLS estimator of μ.

(c) Let $\hat{\mu}^{gls}$ denote the GLS from part c. Show that $\sqrt{T}(\hat{\mu}^{gls} - \mu)\xrightarrow{d} N(0,V_{gls})$ and derive an expression for V_{gls}.

(d) Is the GLS estimator better that the OLS estimator? Explain.

(e) Suppose that $T = 100$, and $\hat{\mu}^{ols} = 2$. Let $\hat{u}_t = y_t - \hat{\mu}^{ols}$. The regression of $\hat{u}_t$ onto $\hat{u}_{t-1}$ yields a regression coefficient of 0.4 and the standard error of the regression is 1.1. Construct a 95% confidence interval for μ.

4. Suppose $y_t = \mu + u_t$ where u_t follows the stationary ARMA(1,1) process $u_t = \phi u_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$, where $\varepsilon_t \sim$ iid $(0, \sigma^2)$.

(a). Show that $\sqrt{T}(\bar{y} - \mu)\xrightarrow{d} N(0,V)$ and derive an expression for V.

(b) In a sample with $T = 400$, $\bar{y} = 5.4$. A researcher uses the data $y_t - \bar{y}$ to estimate the ARMA parameters and finds $\hat{\phi} = 0.66$, $\hat{\theta} = 0.85$, and $\hat{\sigma}_\varepsilon = 1.2$. Construct a 95% confidence interval for μ.

5. Suppose that y and x follow the process

$$y_t = ax_{t-1} + \varepsilon_t$$
$$x_t = \phi x_{t-1} + v_t$$

where $v_t = \varepsilon_t + e_t$, where ε and e are mutually independent iid$(0,1)$ processes. Suppose that $|\phi| < 1$. Let $\hat{\alpha}$ and $\hat{\phi}$ denote the OLS estimators of α and ϕ.

(a) Derive asymptotic distribution of the 2×1 vector $(\hat{\alpha}, \hat{\phi})$.

(b) In a sample of 100 observations $\hat{\alpha} = 1.3$ and $\hat{\phi} = 0.72$. Derive a 95% confidence interval for α.

6. Suppose that Y_t follows a stationary AR(1), and you have data Y_t, $t = 1, \ldots, 100$, and find $\bar{Y} = T^{-1}\sum_{t=1}^{T} Y_t = 18.1$, $\hat{\lambda}_0 = T^{-1}\sum_{t=1}^{T}(Y_t - \bar{Y})^2$ =2.4, and $\hat{\lambda}_1 = (T-1)^{-1}\sum_{t=1}^{T}(Y_t - \bar{Y})(Y_{t-1} - \bar{Y})$ = 1.0. Construct a 95% confidence interval for $\mu = E(Y)$. (Hint: Proceed in three steps. Step 1: Think about the appropriate asymptotic covariance matrix for $\sqrt{T}(\bar{Y} - \mu)$. Step 2: Think about how you would estimate the covariance matrix from the data given in the problem. Step 3: Construct the confidence interval.)

8. Suppose that $y_t = x_t\beta + u_t$, with $u_t = \varepsilon_t - \theta\varepsilon_{t-1}$, and $x_t = \varepsilon_{t+1}$, where ε_t is iid(0,1). Let $\hat{\beta}$ denote the OLS estimator of β and $\hat{u}_t$ denote the OLS residual

(a) Show that $\sqrt{T}(\hat{\beta} - \beta) \xrightarrow{d} N(0, V)$, and derive an expression for V.

(b) Noting that $(1-\theta L)^{-1}u_t = \varepsilon_t$, the GLS estimator of β can be formed by regressing $(1-\theta L)^{-1}y_t$ onto $(1-\theta L)^{-1}x_t$. Derive the probability limit of the GLS estimator.

9. Suppose that y_t is iid$(0,\sigma^2)$ with $E(y_t^4) = \kappa$. Let $\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} y_t^2$.

(a) Show that $\sqrt{T}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} N(0, V_{\hat{\sigma}^2})$ and derive an expression for $V_{\hat{\sigma}^2}$.

(b) Suppose that $T = 100$, $\hat{\sigma}^2 = 2.1$ and $\frac{1}{T}\sum_{t=1}^{T} y_t^4 = 16$. Construct a 95% confidence interval for σ, where σ is the standard deviation of y.

10. Suppose that y_t follows the AR(1) process $y_t = \phi y_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim$ iidN$(0,\sigma_\varepsilon^2)$. Let $\hat{\phi}$ denote the OLS estimator, $\hat{\varepsilon}_t = y_t - \hat{\phi} y_{t-1}$ denote the OLS residual, and $\hat{\sigma}_\varepsilon^2 = \frac{1}{T}\sum_{t=1}^{T} \hat{\varepsilon}_t^2$ denote the estimator of σ_ε^2.

(a) Suppose that $|\phi| < 1$ and $\{y_t\}$ is stationary. Show that $\hat{\sigma}_\varepsilon^2 \xrightarrow{p} \sigma_\varepsilon^2$

(b) Suppose that $\phi = 1$ and $y_0 = 0$. Show that $\hat{\sigma}_\varepsilon^2 \xrightarrow{p} \sigma_\varepsilon^2$. (Hint: Answer this after studying the unit root autoregression.)

11. (a) Suppose that y_t, for $t = 1, \ldots, T$ is generated by the model: $y_t = \mu + u_t$, where $u_t = \varepsilon_t - \theta\varepsilon_{t-1}$, where $\varepsilon_t \sim$ i.i.d. with mean zero and variance σ^2.

(a.1) Show that $\sqrt{T}(\bar{Y} - \mu) \xrightarrow{d} N(0, V)$ and derive an expression for V.

(a.2) Propose an estimator for V and sketch a proof showing that the estimator is consistent.

(b) Now suppose that y_t follows the same model as in (a) but the parameter μ undergoes a change during the sample. That is for $1 \le t \le \tau$, the parameter takes on the value μ_1 and for $\tau+1 \le t \le T$ the parameter takes on the value μ_2. (The parameters θ and σ^2 do not change.) Let $\bar{Y}_1 = \frac{1}{\tau}\sum_{t=1}^{\tau} y_t$ and $\bar{Y}_2 = \frac{1}{T-\tau}\sum_{t=\tau+1}^{T} y_t$. Let $\rho = \tau/T$.

(b.1) Suppose that ρ is fixed as $T \to \infty$. Show that

$\sqrt{T}\left[\left(\bar{Y}_1 - \bar{Y}_2\right) - \left(\mu_1 - \mu_2\right)\right] \xrightarrow{d} N(0, \Omega)$ and derive an expression for Ω.

(b.2) Discuss how you would construct an approximate 95% confidence interval for
$\mu_1 - \mu_2$ using the data y_t, $t = 1, \ldots, T$.

(b.3) Describe how you would test the null hypothesis that $\mu_1 = \mu_2$ if you know the break date τ.

12. Suppose that y_t follows a stationary Gaussian zero-mean ARMA(1,1) process, $(1-\rho L)y_t = (1-\theta L)\varepsilon_t$. Let $\hat{\rho}$ denote the IV estimator of ρ obtained by "regressing" y_t onto y_{t-1} using y_{t-2} as an instrument. Let T denote the sample size.

(a) Suppose that the true value of ρ is $\rho = 0.5$ and the true value of θ is $\theta = 0.1$.

(i) Show that $\hat{\rho}$ is consistent.

(ii) Derive the asymptotic distribution of $\hat{\rho}$.

(b) Suppose that the true value of ρ is $\rho = 0.5$ and the true value of θ is $\theta = 0.5$.

(i) Is $\hat{\rho}$ is consistent? Explain.

(ii) Derive the asymptotic distribution of $\hat{\rho}$.

13. Consider the IV model with $L = K = 1$, and conditionally homoskedastic errors. (That is $y_t = z_t\delta + \varepsilon_t$, x_t is a instrument, and $\text{var}(\varepsilon_t x_t) = \sigma_\varepsilon^2\sigma_X^2$.)

In a sample of size 200, the following sample moment matrices are calculated:

$S_{yy} = 4.33$
$S_{zy} = 1.87$
$S_{zz} = 1.12$
$S_{xy} = 0.56$
$S_{xz} = 0.58$
$S_{xx} = 1.06$

(a) Compute $\hat{\delta}$ and its standard error (based on the usual large sample approximation).

(b) Compute a 95% confidence for δ using the usual large sample approximation to the distribution of $\hat{\delta}$.

(c) Compute the F-statistic from the regression of z onto x. What is the p-value for this statistic? Is there evidence that x is a weak instrument?

(d) Use the Anderson-Rubin method to compute a 95% confidence interval for δ. How does this confidence interval compare to the confidence interval that you computed in (a). Is this what you have expected given the F-statistic in (c)? Explain.

14. y_t follows the stationary AR(1) model $y_t = \phi y_{t-1} + \varepsilon_t$. A researcher wants to estimate ϕ. He cannot find data on y_t, but can find estimates of y_t based on survey. Let x_t denote the survey estimates of y_t, and suppose that $x_t = y_t + u_t$, where u_t is iid$(0, \sigma_u^2)$ and u_t is independent of ε_j for all t and j. The researcher estimates ϕ by regressing x_t onto x_{t-1} using x_{t-2} as an instrument.

(a) Suppose that $\phi \neq 0$. Derive the asymptotic distribution of the IV estimator

(b) Suppose $\phi = 0$, show that the IV estimator is not consistent. Derive limiting distribution.

(c) How would you test $\phi = 0$?

15. Suppose that y_t follows the MA(1) process $y_t = (1-\theta L)\varepsilon_t$, where $\varepsilon_t \sim$ iid(0, σ_ε^2) and has as many higher-order moments as needed to answer the questions below.

(a) Show that the variance of y is given by $\sigma_y^2 = \sigma_\varepsilon^2(1+\theta^2)$.

(b) Let $\hat{\sigma}_y^2 = T^{-1}\sum_{t=1}^T y_t^2$.

(b.i) Show that $\hat{\sigma}_y^2 \xrightarrow{p} \sigma_y^2$

(b.ii) Show that $\sqrt{T}(\hat{\sigma}_y^2 - \sigma_y^2) \xrightarrow{d} N(0,V)$ and derive an expression for V in terms of moments of y.

(c) Using a sample of size $T = 100$, I find $\hat{\sigma}_y^2 = 1.52$, $T^{-1}\sum_{t=1}^T (y_t^2 - \hat{\sigma}_y^2)^2 = 4.8$, $T^{-1}\sum_{t=1}^T (y_t^2 - \hat{\sigma}_y^2)(y_{t-1}^2 - \hat{\sigma}_y^2) = 1.1$, $T^{-1}\sum_{t=1}^T (y_t^2 - \hat{\sigma}_y^2)(y_{t-2}^2 - \hat{\sigma}_y^2) = -0.4$, $T^{-1}\sum_{t=1}^T y_t y_{t-1}^3 =$ 3.6, and $T^{-1}\sum_{t=1}^T y_t^3 y_{t-1} = 3.1$. Construct a 95% confidence interval for σ_y^2.

16.. y_t follows the stationary AR(1) model $y_t = \phi y_{t-1} + \varepsilon_t$. A researcher wants to estimate ϕ. He cannot find data on y_t, but can find estimates of y_t based on survey. Let x_t denote the survey estimates of y_t, and suppose that $x_t = y_t + u_t$, where u_t is iid(0, σ_u^2) and u_t is independent of ε_j for all t and j. The researcher estimates ϕ by regressing x_t onto x_{t-1} using x_{t-2} as an instrument.

(a) Suppose that $\phi \neq 0$. Derive the asymptotic distribution of the IV estimator

(b) Suppose $\phi = 0$, show that the IV estimator is not consistent. Derive limiting distribution.

(c) How would you test $\phi = 0$?

17. A researcher carries out a QLR test using **25%** trimming and there are $q = 5$ restrictions. Answer the following questions using the values of the QLR Statistic with **15%** Trimming that can be found in the Stock-Watson undergraduate textbook, and the table of critical values for the $F_{m,\infty}$ Distribution.

(a) The QLR F-statistic is 4.2. Should she reject the null at the 5% level?

(b) The QLR F-statistic is 2.1. Should she reject the null at the 5% level?

(c) The QLR F-statistic is 3.5. Should she reject the null at the 5% level?

VAR models

1. Suppose that y_t is generated by the $ARIMA(0,1,2)$ model:

$$\Delta y_t = \varepsilon_t + 0.4\varepsilon_{t-1} + 0.2\varepsilon_{t-2}$$

(a) Derive the impulse response function for Δy_t.
(b) Derive the impulse response function for y_t.

2. Suppose that a sequence of random variables y_t is generated by the stationary AR(1) model:

$$y_t = \phi y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim Niid(0,\sigma^2)$. From a realization of size $T = 100$, I calculate
$\sum_{t=2}^{T} y_t y_{t-1} = 48$, $\sum_{t=2}^{T} y_t^2 = 100$, and $\sum_{t=2}^{T} y_{t-1}^2 = 99$

(a) Calculate the least squares estimator of ϕ.

(b) Let $\{\delta_k\}_{k=0}^{\infty}$ denote the sequence of "impulse responses" model. Construct an approximate 95% confidene interval for δ_4.

3. Consider the bivariate structural VAR model

$$x_t = \alpha y_t + \sum_{i=1}^{p} (\phi_{xx,i} x_{t-i} + \phi_{xy,i} y_{t-i}) + \varepsilon_{zt}$$

$$y_t = \lambda x_t + \sum_{i=1}^{p} (\phi_{yx,i} x_{t-i} + \phi_{yy,i} y_{t-i}) + \varepsilon_{yt}$$

where ε_{xt} and ε_{yt} are mutually independent iid sequences.

(a) Suppose that you know that $\alpha = 0.3$. Explain how you would construct an 95% confidence interval for λ.

(b) Suppose that you know that $0 \le \alpha \le 1$. Explain how you would construct an 95% confidence interval for λ.

4. Let $x_t = \Delta y_t$. Consider the structural VAR model:

$$x_t = \gamma_0 z_t + \phi x_{t-1} + \gamma_1 z_{t-1} + \varepsilon_{xt}$$

$$z_t = \rho z_{t-1} + \varepsilon_{zt}$$

show that $\lim_{k\to\infty} \frac{\partial y_{t+k}}{\partial \varepsilon_{zt}} = \frac{(\gamma_0 - \gamma_1)}{(1-\phi)(1-\rho)}$

5. Consider the bivariate structural VAR model

$$x_t = \alpha y_t + \sum_{i=1}^{p} (\phi_{xx,i} x_{t-i} + \phi_{xy,i} y_{t-i}) + \varepsilon_{zt}$$

$$y_t = \lambda x_t + \sum_{i=1}^{p} (\phi_{yx,i} x_{t-i} + \phi_{yy,i} y_{t-i}) + \varepsilon_{yt}$$

where ε_{xt} and ε_{yt} are mutually independent sequences. Let $\varepsilon_t = (\varepsilon_{xt}\ \varepsilon_{yt})'$. Suppose that

$$\operatorname{var}(\varepsilon_t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{for } t < T/2 \\ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{for } t \geq T/2 \end{cases}$$

(a) Show how you can use this information to identify the structural VAR parameters and the shocks.

(b) Is the model identified with the variance of ε_{xt} is equal to 2 in the second half of the sample also? Explain.