Sectoral vs. Aggregate Shocks: A Structural Factor Analysis of Industrial Production

Supplementary Material

Andrew T. Foerster
Department of Economics, Duke University
Pierre-Daniel G. Sarte
Research Department, Federal Reserve Bank of Richmond
Mark W. Watson
Department of Economics, Princeton University

Contents

1 The Model

The general model takes the following form:

$$\max \ E_t \sum_{t=0}^{\infty} \beta^t \sum_{j=1}^{N} \left(\frac{C_{jt}^{1-\sigma} - 1}{1-\sigma} - \psi L_{jt} \right)$$

subject to

$$C_{jt} + \sum_{i=1}^{N} M_{jit} + \sum_{i=1}^{N} X_{jit} = Y_{jt},$$

$$K_{jt+1} = Z_{jt} + (1-\delta) K_{jt}, \tag{1}$$

$$Z_{jt} = \Pi_{i=1}^{N} X_{ijt}^{\theta_{ij}}, \ \sum_{i=1}^{N} \theta_{ij} = 1 \tag{2}$$

$$Y_{jt} = A_{jt} K_{jt}^{\alpha_j} \Pi_{i=1}^{N} M_{ijt}^{\gamma_{ij}} L_{jt}^{1-\alpha_j - \Sigma_{i=1}^{N} \gamma_{ij}}.$$

In each sector j, the production of final goods takes place using as materials the amount M_{ijt} of output produced in sector i. In addition, investment goods in sector j, Z_{jt}, are produced using the amount X_{ijt} of output produced in sector i.

An input-output matrix for this economy is an $N \times N$ matrix Γ with typical element, γ_{ij}, the share of sector i in the output of sector j. A sectoral investment matrix is an $N \times N$ matrix Θ with typical element, θ_{ij}, the share of sector i in total investment made by sector j.

The first-order necessary conditions are:

$$\begin{aligned} C_{jt} &: \ C_{jt}^{-\sigma} = \lambda_{jt}, \\ L_{jt} &: \ \psi = \lambda_{jt} \frac{Y_{jt}}{L_{jt}} \left(1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij} \right) \end{aligned}$$

Combining these 2 equations gives

$$L_{jt} : \psi L_{jt} = C_{jt}^{-\sigma} Y_{jt} \left(1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij} \right). \tag{3}$$

$$M_{ijt} : \lambda_{it} = \lambda_{jt} \gamma_{ij} \frac{Y_{jt}}{M_{ijt}},$$

or

$$M_{ijt} : C_{it}^{-\sigma} = C_{jt}^{-\sigma} \gamma_{ij} \frac{Y_{jt}}{M_{ijt}}. \tag{4}$$

$$X_{ijt} : \lambda_{it} = \mu_{jt} \theta_{ij} \frac{Z_{jt}}{X_{ijt}}, \tag{5}$$

where μ_{jt} is the Lagrange multiplier associated with the capital accumulation equation in sector j, equation (1).

$$K_{jt+1} : \mu_{jt} = \beta\lambda_{jt+1}\alpha_j \left(\frac{Y_{jt+1}}{K_{jt+1}}\right) + \beta\mu_{jt+1}(1-\delta). \tag{6}$$

Thus, the basic equations of the model, equations (3) through (6) along with the capital accumulation equation and the 2 production functions, represent a set of $6N+2N^2$ equations, where $6N + 2N^2 = 28080$ when $N = 117$. For this reason, it is helpful to reduce the system analytically if possible.

2 Finding the Steady State Analytically

Some key steady state equations are:

$$\psi L_j = \lambda_j \left(1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij}\right) Y_j, \tag{7}$$

$$M_{ij} = \frac{\lambda_j}{\lambda_i}\gamma_{ij} Y_j, \tag{8}$$

$$X_{ij} = \frac{\mu_j}{\lambda_i}\theta_{ij} Z_j, \tag{9}$$

$$Z_j = \delta K_j, \tag{10}$$

$$\mu_j \left[\frac{1-\beta(1-\delta)}{\beta\alpha_j}\right] = \lambda_j \left(\frac{Y_j}{K_j}\right), \tag{11}$$

$$Z_j = \Pi_{i=1}^{N} X_{ij}^{\theta_{ij}}, \tag{12}$$

$$Y_j = A_j K_j^{\alpha_j} \Pi_{i=1}^{N} M_{ij}^{\gamma_{ij}} L_j^{1-\alpha_j-\Sigma_{i=1}^{N}\gamma_{ij}} \tag{13}$$

Let y denote the log of variable Y. Further, let $z = (z_1, ..., z_N)^T$, $l = (l_1, ..., l_N)^T$, etc., $m = (m_{11}, m_{12}, ..., m_{1N}, m_{21}, m_{22}, ..., m_{NN})^T$ and $x = (x_{11}, x_{12}, ..., x_{1N}, x_{21}, x_{22}, ..., x_{NN})^T$. Using this notation, we can write equation (9) as follows,

$$x_{ij} = \ln \mu_j - \ln \lambda_i + \ln \theta_{ij} + z_j,$$

or, in vector form,

$$x = M_\mu \ln \mu - M_{x\lambda} \ln \lambda + M_z z + \operatorname{vec}\left(\ln \Theta^T\right), \tag{14}$$

where $M_\mu = M_z = 1_{N\times 1} \otimes I_N$ and $M_{x\lambda} = I_N \otimes 1_{N\times 1}$. Equation (12) implies that

$$z = \widetilde{\Theta} x, \tag{15}$$

where

$$\widetilde{\Theta}_{N\times N^2} = \begin{bmatrix} \theta_{11} & 0 & ... & \theta_{21} & 0 & ... & \theta_{N1} & 0 & ... & & \\ 0 & \theta_{12} & 0 & ... & \theta_{22} & 0 & ... & \theta_{N2} & 0 & & \\ & & \theta_{13} & & & \theta_{23} & & & \theta_{N3} & & \\ & & & ... & & & ... & & & ... & \\ & & & & \theta_{1N} & 0 & ... & \theta_{2N} & & \theta_{NN} & \end{bmatrix}.$$

Substituting equation (14) into (15) gives

$$z = \widetilde{\Theta} M_\mu \ln \mu - \widetilde{\Theta} M_{x\lambda} \ln \lambda + \widetilde{\Theta} M_z z + \widetilde{\Theta} \mathrm{vec}\left(\ln \Theta^T\right),$$

or

$$\underbrace{(I_N - \widetilde{\Theta} M_z)}_{\varnothing} z = \widetilde{\Theta} M_\mu \ln \mu - \widetilde{\Theta} M_{x\lambda} \ln \lambda + \widetilde{\Theta} \mathrm{vec}\left(\ln \Theta^T\right),$$

where

$$\widetilde{\Theta} M_\mu = \begin{bmatrix} \sum_i \theta_{i1} & & & \\ & \sum_i \theta_{i2} & & \\ & & ... & \\ & & & \sum_i \theta_{iN} \end{bmatrix} \equiv I_N,$$

and

$$\widetilde{\Theta} M_{x\lambda} = \begin{bmatrix} \theta_{11} & \theta_{21} & ... & \theta_{N1} \\ \theta_{12} & \theta_{22} & ... & \theta_{N2} \\ & & ... & \\ \theta_{1N} & \theta_{2N} & ... & \theta_{NN} \end{bmatrix} = \Theta^T,$$

so that

$$\ln \mu = \Theta^T \ln \lambda - \widetilde{\Theta} \mathrm{vec}\left(\ln \Theta^T\right). \tag{16}$$

From the Euler equation in the steady state (11), we have

$$\ln \mu + \ln \left[\frac{1-\beta(1-\delta)}{\beta\alpha}\right] = \ln \lambda + y - k$$

or

$$k = \ln \lambda - \ln \mu + y - \ln \left[\frac{1-\beta(1-\delta)}{\beta\alpha}\right],$$

where α is a vector with values α_j and the expression in square brackets is understood to be an element by element operation. Substituting equation (16) into this last equation gives

$$k = \ln \lambda - \Theta^T \ln \lambda + \widetilde{\Theta} \mathrm{vec}\left(\ln \Theta^T\right) + y - \ln \left[\frac{1-\beta(1-\delta)}{\beta\alpha}\right]$$

or

$$k = (I_N - \Theta^T) \ln \lambda + y + C_k, \tag{17}$$

where

$$C_k = \widetilde{\Theta}\text{vec}\left(\ln \Theta^T\right) - \ln\left[\frac{1-\beta(1-\delta)}{\beta\alpha}\right].$$

From the labor supply equation (7), we have

$$l = -\ln\psi + \ln\lambda + y + \ln C_l, \tag{18}$$

where

$$C_l = \Phi \times 1_{N\times 1},$$

$$\Phi = (I_N - \alpha_d - \Sigma_\gamma),$$

where α_d is a diagonal matrix with the values α_j on the diagonal, and

$$\Sigma_\gamma \equiv \begin{bmatrix} \sum_i \gamma_{i1} & & & \\ & \sum_i \gamma_{i2} & & \\ & & ... & \\ & & & \sum_i \gamma_{iN} \end{bmatrix}.$$

From the materials equation (8), we have

$$m_{ij} = \ln\lambda_j - \ln\lambda_i + \ln\gamma_{ij} + y_j,$$

or, in vector form,

$$m = M_{m\lambda}\ln\lambda + M_y y + \text{vec}(\ln\Gamma^T), \tag{19}$$

where $M_{m\lambda} = 1_{N\times 1}\otimes I_N - I_N \otimes 1_{N\times 1}$ and $M_y = 1_{N\times 1}\otimes I_N$.

We are now ready to solve for the multipliers, λ, in closed form as functions the structural parameters of the model only. From the definition of sectoral production (13), we have

$$y = a + \alpha_d k + \widetilde{\Gamma} m + \Phi l,$$

where

$$\widetilde{\Gamma}_{N\times N^2} = \begin{bmatrix} \gamma_{11} & 0 & ... & \gamma_{21} & 0 & ... & \gamma_{N1} & 0 & ... & \\ 0 & \gamma_{12} & 0 & ... & \gamma_{22} & 0 & ... & \gamma_{N2} & 0 & \\ & & \gamma_{13} & & & \gamma_{23} & & & \gamma_{N3} & \\ & & & ... & & & ... & & & ... \\ & & & & \gamma_{1N} & 0 & ... & \gamma_{2N} & & \gamma_{NN} \end{bmatrix}.$$

Substituting equations (17), (18), and (19) in this last expression gives

$$\begin{aligned} y \;=\; & a + \alpha_d[(I_N - \Theta^T)\ln\lambda + y + C_k] \\ & +\widetilde{\Gamma}[M_{m\lambda}\ln\lambda + M_y y + \text{vec}(\ln\Gamma^T)] \\ & +\Phi[-\ln\psi + \ln\lambda + y + \ln C_l] \end{aligned}$$

or

$$\underbrace{[I - \alpha_d - \widetilde{\Gamma} M_y - \Phi]}_{\varnothing} y = a + \alpha_d C_k + \widetilde{\Gamma}\mathrm{vec}(\ln \Gamma^T) - \Phi \ln \psi + \Phi \ln C_l$$
$$+ (\alpha_d (I_N - \Theta^T) + \widetilde{\Gamma} M_{m\lambda} + \Phi) \ln \lambda$$

so that

$$\ln \lambda = -(\alpha_d (I_N - \Theta^T) + \widetilde{\Gamma} M_{m\lambda} + \Phi)^{-1} [a + \alpha_d C_k + \widetilde{\Gamma}\mathrm{vec}(\ln \Gamma^T) - \Phi \ln \psi + \Phi \ln C_l], \quad (20)$$

and $\lambda = e^{\ln \lambda}$. This allows us to directly solve for consumption, $C = \lambda^{-\frac{1}{\sigma}}$. We can also solve for μ using equation (16),

$$\ln \mu = \Theta^T \ln \lambda - \widetilde{\Theta}\mathrm{vec}\left(\ln \Theta^T\right).$$

To solve for output in the steady state, we first use the resource constraint,

$$\lambda^{-\frac{1}{\sigma}} + M_{my} Y + M_{xz} Z = Y, \quad (21)$$

where

$$M_{my} = \begin{bmatrix} \gamma_{11} & \gamma_{12}\frac{\lambda_2}{\lambda_1} & \cdots & \gamma_{1N}\frac{\lambda_N}{\lambda_1} \\ \gamma_{21}\frac{\lambda_1}{\lambda_2} & \gamma_{22} & \cdots & \gamma_{2N}\frac{\lambda_N}{\lambda_2} \\ \cdots & & \cdots & \\ \gamma_{N1}\frac{\lambda_1}{\lambda_N} & \gamma_{N2}\frac{\lambda_2}{\lambda_N} & \cdots & \gamma_{NN} \end{bmatrix} \text{ and } M_{xz} = \begin{bmatrix} \theta_{11}\frac{\mu_1}{\lambda_1} & \theta_{12}\frac{\mu_2}{\lambda_1} & \cdots & \theta_{1N}\frac{\mu_N}{\lambda_1} \\ \theta_{21}\frac{\mu_1}{\lambda_2} & \theta_{22}\frac{\mu_2}{\lambda_2} & \cdots & \theta_{2N}\frac{\mu_N}{\lambda_2} \\ \cdots & & \cdots & \\ \theta_{N1}\frac{\mu_1}{\lambda_N} & \theta_{N2}\frac{\mu_2}{\lambda_N} & \cdots & \theta_{NN}\frac{\mu_N}{\lambda_N} \end{bmatrix}.$$

Now, $Z = \delta K$ and from (17),

$$K = e^{(I_N - \Theta^T)\ln \lambda + y + C_k} = \underbrace{\begin{bmatrix} C_{\exp_1} & 0 & \cdots & 0 \\ 0 & C_{\exp_2} & \cdots & 0 \\ \cdots & & \cdots & \\ 0 & 0 & \cdots & C_{\exp_N} \end{bmatrix}}_{C\exp} Y,$$

where $C_{\exp_i}$ is the i^{th} element of the vector $e^{(I_N - \Theta^T)\ln \lambda + C_k}$ and $\ln \lambda$ is given by (20). Substituting these expressions into (21) gives

$$\lambda^{-\frac{1}{\sigma}} + M_{my} Y + \delta M_{xz} C_{\exp} Y = Y,$$

so that

$$Y = (I - M_{my} - \delta M_{xz} C_{\exp})^{-1} \lambda^{-\frac{1}{\sigma}}. \quad (22)$$

From here, solving for the remaining variables in the steady state is straightforward.

3 Dynamics of the System

The model solution is described by a set of $6N + 2N^2$ equations. The following sections describe these equations and outlines an analytical system reduction to $2N$ equations. The basic equations of the system are:

$$\psi L_{jt} = C_{jt}^{-\sigma} Y_{jt} \left(1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij}\right),$$

$$C_{it}^{-\sigma} = C_{jt}^{-\sigma} \gamma_{ij} \frac{Y_{jt}}{M_{ijt}},$$

$$\lambda_{it} = \mu_{jt} \theta_{ij} \frac{Z_{jt}}{X_{ijt}},$$

where $\lambda_{it} = C_{it}^{-\sigma}$.

$$\mu_{jt} = \beta \lambda_{jt+1} \alpha \left(\frac{Y_{jt+1}}{K_{jt+1}}\right) + \beta \mu_{jt+1}(1 - \delta),$$

$$C_{jt} + \sum_{i=1}^{N} M_{jit} + \sum_{i=1}^{N} X_{jit} = Y_{jt},$$

$$K_{jt+1} = Z_{jt} + (1 - \delta) K_{jt},$$

$$Z_{jt} = \Pi_{i=1}^{N} X_{ijt}^{\theta_{ij}}, \;\; \sum_{i=1}^{N} \theta_{ij} = 1,$$

$$Y_{jt} = A_{jt} K_{jt}^{\alpha_j} \Pi_{i=1}^{N} M_{ijt}^{\gamma_{ij}} L_{jt}^{1-\alpha-\Sigma_{i=1}^{N}\gamma_{ij}}.$$

4 Log-linearized Equations

The "hat" notation stands for percent deviation from steady state.

$$\widehat{L}_{jt} = -\sigma \widehat{C}_{jt} + \widehat{Y}_{jt},$$

$$-\sigma \widehat{C}_{it} = -\sigma \widehat{C}_{jt} + \widehat{Y}_{jt} - \widehat{M}_{ijt}$$

$$\widehat{\lambda}_{it} = \widehat{\mu}_{jt} + \widehat{Z}_{jt} - \widehat{X}_{ijt}$$

$$\widehat{\mu}_{jt} = \widetilde{\beta} \widehat{\lambda}_{jt+1} + \widetilde{\beta} \widehat{Y}_{jt+1} - \widetilde{\beta} \widehat{K}_{jt+1} + \beta(1 - \delta) \widehat{\mu}_{jt+1},$$

where $\widetilde{\beta} = 1 - \beta(1 - \delta)$,

$$S_{C_j} \widehat{C}_{jt} + \sum_{i=1}^{N} S_{M_{ji}} \widehat{M}_{jit} + \sum_{i=1}^{N} S_{X_{ji}} \widehat{X}_{jit} = \widehat{Y}_{jt},$$

$$\widehat{K}_{jt+1} = \delta\widehat{Z}_{jt} + (1-\delta)\widehat{K}_{jt},$$

$$\widehat{Z}_{jt} = \sum_{i=1}^{N} \theta_{ij}\widehat{X}_{ijt},$$

$$\widehat{Y}_{jt} = \widehat{A}_{jt} + \alpha_j\widehat{K}_{jt} + \sum_{i=1}^{N} \gamma_{ij}\widehat{M}_{ijt} + \left(1-\alpha_j - \sum_{i=1}^{N}\gamma_{ij}\right)\widehat{L}_{jt}.$$

Let $z_t = (\widehat{Z}_{1t}, ..., \widehat{Z}_{Nt})^T$, $l_t = (\widehat{L}_{1t}, ..., \widehat{L}_{Nt})^T$, etc., $m_t = (\widehat{M}_{11t}, \widehat{M}_{12t}, ..., \widehat{M}_{1Nt}, \widehat{M}_{21t}, \widehat{M}_{22t}, ..., \widehat{M}_{NNt})^T$ and $x_t = (\widehat{X}_{11t}, \widehat{X}_{12t}, ..., \widehat{X}_{1Nt}, \widehat{X}_{21t}, \widehat{X}_{22t}, ..., \widehat{X}_{NNt})^T$. Given this notation, we can express the log-linearized equations as follows:

$$l_t = -\sigma c_t + y_t, \tag{23}$$

$$m_t = M_y y_t + M_c c_t, \tag{24}$$

where $M_y = 1_{N\times 1} \otimes I_N$ and $M_c = \sigma(I_N \otimes 1_{N\times 1}) - \sigma(1_{N\times 1} \otimes I_N)$.

$$x_t = M_\mu \mu_t - M_{x\lambda}\lambda_t + M_z z_t, \tag{25}$$

where $M_\mu = M_z = 1_{N\times 1} \otimes I_N$ and $M_{x\lambda} = I_N \otimes 1_{N\times 1}$.

$$\mu_t = \widetilde{\beta}\lambda_{t+1} + \widetilde{\beta}y_{t+1} - \widetilde{\beta}k_{t+1} + \beta(1-\delta)\mu_{t+1}, \tag{26}$$

where the $E_t(.)$ operator is understood to apply to the forward variables.

$$S_c c_t + S_m m_t + S_x x_t = y_t, \tag{27}$$

where

$$S_c = \begin{bmatrix} S_{C_1} & & \\ & ... & \\ & & S_{C_N} \end{bmatrix},\ S_m = \begin{bmatrix} S_{M_{11}} & S_{M_{12}} & ... & 0 & ... & 0 \\ & & ... & & & \\ 0 & ... & 0 & ... & S_{M_{N,N-1}} & S_{M_{N,N}} \end{bmatrix},$$

$$\text{and } S_x = \begin{bmatrix} S_{X_{11}} & S_{X_{12}} & ... & 0 & ... & 0 \\ & & ... & & & \\ 0 & ... & 0 & ... & S_{X_{N,N-1}} & S_{X_{N,N}} \end{bmatrix}.$$

$$k_{t+1} = \delta z_t + (1-\delta)k_t, \tag{28}$$

$$z_t = \widetilde{\Theta}x_t, \tag{29}$$

where $\widetilde{\Theta}$ is defined as in section 2.

$$y_t = a_t + \alpha_d k_t + \widetilde{\Gamma}m_t + \Phi l_t, \tag{30}$$

where $\widetilde{\Gamma}$ and Φ are defined as in section 2.

5 System Reduction

The objective of this section is to reduce the system of $6N + 2N^2$ log-linear equations described in the previous section to one with a set of N flows (consumption), N states, (capital), and N driving processes (productivity).

Substitute equation (25) into (29) to get

$$z_t = \widetilde{\Theta} M_\mu \mu_t - \widetilde{\Theta} M_{x\lambda} \lambda_t + \widetilde{\Theta} M_z z_t,$$

or

$$\widetilde{\Theta} M_\mu \mu_t = \widetilde{\Theta} M_{x\lambda} \lambda_t,$$

(since $I - \widetilde{\Theta} M_z = 0$) which gives alternatively

$$\mu_t = \Theta^T \lambda_t, \tag{31}$$

(since $\widetilde{\Theta} M_\mu = I_N$ and $\widetilde{\Theta} M_{x\lambda} = \Theta^T$).

Now, substitute equation (31) into the linearized Euler equation, (26),

$$\Theta^T \lambda_t = \widetilde{\beta} \lambda_{t+1} + \widetilde{\beta} y_{t+1} - \widetilde{\beta} k_{t+1} + \beta(1-\delta)\Theta^T \lambda_{t+1},$$

or

$$\Theta^T \lambda_t = \left(\widetilde{\beta} + \beta(1-\delta)\Theta^T\right) \lambda_{t+1} + \widetilde{\beta} y_{t+1} - \widetilde{\beta} k_{t+1}.$$

Since $\lambda_t = -\sigma c_t$, this last expression becomes

$$-\sigma \Theta^T c_t = -\sigma \left(\widetilde{\beta} + \beta(1-\delta)\Theta^T\right) c_{t+1} + \widetilde{\beta} y_{t+1} - \widetilde{\beta} k_{t+1}. \tag{32}$$

From equations (23), (24), and (30), we have

$$\underbrace{(I - \widetilde{\Gamma} M_y - \Phi)}_{\alpha_d} y_t = a_t + \alpha_d k_t + \underbrace{(\widetilde{\Gamma} M_c - \sigma \Phi)}_{Q_c} c_t$$

or, equivalently,

$$y_t = \alpha_d^{-1} a_t + k_t + \alpha_d^{-1} Q_c c_t. \tag{33}$$

Substitute this last expression into the Euler equation (32) to get

$$-\sigma \Theta^T c_t = -\sigma \left(\widetilde{\beta} + \beta(1-\delta)\Theta^T\right) c_{t+1} + \widetilde{\beta}[\alpha_d^{-1} a_{t+1} + k_{t+1} + \alpha_d^{-1} Q_c c_{t+1}] - \widetilde{\beta} k_{t+1},$$

or

$$-\sigma \Theta^T c_t = \left[-\sigma \left(\widetilde{\beta} + \beta(1-\delta)\Theta^T\right) + \widetilde{\beta} \alpha_d^{-1} Q_c\right] c_{t+1} + \widetilde{\beta} \alpha_d^{-1} a_{t+1}, \tag{34}$$

which is our first equation in a system that involves only consumption, capital, and the exogenous shocks.

To obtain the second equation, we re-write the resource constraint,

$$S_c c_t + S_m \left(M_y y_t + M_c c_t\right) + S_x \left(M_\mu \mu_t - M_{x\lambda}\lambda_t + M_z z_t\right) = y_t. \tag{35}$$

Recall that $\mu_t = \Theta^T \lambda_t$ in (31) so that $M_\mu \mu_t - M_{x\lambda}\lambda_t = (M_\mu \Theta^T - M_{x\lambda})\lambda_t = -\sigma(M_\mu \Theta^T - M_{x\lambda})c_t$. Moreover, from (28), we have $z_t = \frac{k_{t+1}}{\delta} - \frac{(1-\delta)}{\delta}k_t$. It follows that (35) becomes

$$S_c c_t + S_m \left(M_y y_t + M_c c_t\right) - \sigma S_x (M_\mu \Theta^T - M_{x\lambda})c_t + \frac{S_x M_z}{\delta}k_{t+1} - \frac{S_x M_z (1-\delta)}{\delta}k_t = y_t,$$

or

$$(I - S_m M_y)y_t = \left[S_c + S_m M_c - \sigma S_x (M_\mu \Theta^T - M_{x\lambda})\right] c_t + \frac{S_x M_z}{\delta}k_{t+1} - \frac{S_x M_z (1-\delta)}{\delta}k_t.$$

Recall that $y_t = \alpha_d^{-1} a_t + k_t + \alpha_d^{-1} Q_c c_t$ from (33). Therefore, we have

$$\begin{aligned} & (I - S_m M_y)\left[\alpha_d^{-1} a_t + k_t + \alpha_d^{-1} Q_c c_t\right] \\ = \; & \left[S_c + S_m M_c - \sigma S_x (M_\mu \Theta^T - M_{x\lambda})\right] c_t + \frac{S_x M_z}{\delta}k_{t+1} - \frac{S_x M_z (1-\delta)}{\delta}k_t \end{aligned}$$

or

$$\begin{aligned} & \frac{S_x M_z}{\delta}k_{t+1} \\ = \; & \left[(I - S_m M_y)\alpha_d^{-1} Q_c - S_c - S_m M_c + \sigma S_x (M_\mu \Theta^T - M_{x\lambda})\right] c_t \\ & + \left[I - S_m M_y + \frac{S_x M_z (1-\delta)}{\delta}\right] k_t \\ & + (I - S_m M_y)\alpha_d^{-1} a_t, \end{aligned} \tag{36}$$

which is the second equation of our system.

We summarize equations (34) and (36) as follows:

$$\begin{aligned} & \begin{bmatrix} -\sigma\left(\widetilde{\beta} + \beta(1-\delta)\Theta^T\right) + \widetilde{\beta}\alpha_d^{-1} Q_c, & 0 \\ 0, & \frac{S_x M_z}{\delta} \end{bmatrix} E_t \begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix} \\ = & \begin{bmatrix} -\sigma \Theta^T, & 0 \\ (I - S_m M_y)\alpha_d^{-1} Q_c - S_c - S_m M_c + \sigma S_x (M_\mu \Theta^T - M_{x\lambda}), & I - S_m M_y + \frac{S_x M_z (1-\delta)}{\delta} \end{bmatrix} \begin{bmatrix} c_t \\ k_t \end{bmatrix} \\ & + \begin{bmatrix} 0 \\ (I - S_m M_y)\alpha_d^{-1} \end{bmatrix} a_t + \begin{bmatrix} -\widetilde{\beta}\alpha_d^{-1} \\ 0 \end{bmatrix} E_t(a_{t+1}). \end{aligned} \tag{37}$$

At this stage, the dynamics of the system can be solved using standard linear rational expectations toolkits as described in Blanchard and Khan (1980), King, Plosser, Rebelo (1998), and Klein (2000). Our calculations are based on the algorithms described in King and Watson (2002).

6 Solution and Policy Functions

The policy functions associated with (37) take the form:

$$\begin{bmatrix} c_{1t} \\ \ldots \\ c_{Nt} \\ k_{1t} \\ \ldots \\ k_{Nt} \end{bmatrix} = \begin{bmatrix} \ldots & & & & & \\ & & & & & \\ & & & & & \\ 1 & 0 & \ldots & 0 & \ldots & 0 \\ \ldots & & & & & \\ 0 & \ldots & 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} k_{1t} \\ \ldots \\ k_{Nt} \\ \delta_{1t} \\ \ldots \\ \delta_{Nt} \end{bmatrix},$$

$$\underbrace{\begin{bmatrix} a_{1t} \\ \ldots \\ a_{Nt} \end{bmatrix}}_{x_t} = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots \\ \ldots & & & & \ldots & \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} k_{1t} \\ \ldots \\ k_{Nt} \\ \delta_{1t} \\ \ldots \\ \delta_{Nt} \end{bmatrix},$$

and

$$\begin{bmatrix} k_{1t+1} \\ \ldots \\ k_{Nt+1} \\ \delta_{1t+1} \\ \ldots \\ \delta_{Nt+1} \end{bmatrix} = \begin{bmatrix} M_k & M_a \\ 0 & I \end{bmatrix} \begin{bmatrix} k_{1t} \\ \ldots \\ k_{Nt} \\ \delta_{1t} \\ \ldots \\ \delta_{Nt} \end{bmatrix} + H\varepsilon_{t+1}.$$

More generally, we can write these equations as follows:

$$\begin{bmatrix} c_t \\ k_t \end{bmatrix} = \begin{bmatrix} \Pi_{ck} & \Pi_{ca} \\ I & 0 \end{bmatrix} \begin{bmatrix} k_t \\ \delta_t \end{bmatrix},$$

$$a_t = \begin{bmatrix} 0 & Q \end{bmatrix} \begin{bmatrix} k_t \\ \delta_t \end{bmatrix},$$

and

$$\begin{bmatrix} k_{t+1} \\ \delta_{t+1} \end{bmatrix} = \begin{bmatrix} M_k & M_a \\ 0 & I \end{bmatrix} \begin{bmatrix} k_t \\ \delta_t \end{bmatrix} + H\varepsilon_t.$$

7 Obtaining the Model Filter

Since we assume that the logarithm of sectoral productivity follows a random walk, $Q = I$ in the procedure governing the driving process (i.e. drp.gss) of King and Watson (2002).

Therefore, we have

$$k_{t+1} = M_k k_t + M_a a_t,$$

and

$$c_t = \Pi_{ck} k_t + \Pi_{ca} a_t.$$

Recall from equation (33) that

$$y_t = \alpha_d^{-1} a_t + k_t + \alpha_d^{-1} Q_c c_t.$$

Therefore,

$$\begin{aligned} y_t &= \alpha_d^{-1} a_t + k_t + \alpha_d^{-1} Q_c [\Pi_{ck} k_t + \Pi_{ca} a_t] \\ &= \underbrace{\alpha_d^{-1}[I + Q_c \Pi_{ca}]}_{\Pi_a} a_t + \underbrace{[I + \alpha_d^{-1} Q_c \Pi_{ck}]}_{\Pi_k} k_t \end{aligned}$$

so that

$$k_t = \Pi_k^{-1} y_t - \Pi_k^{-1} \Pi_a a_t.$$

Using these equations, we have that

$$\begin{aligned} y_{t+1} &= \Pi_k k_{t+1} + \Pi_a a_{t+1} \\ &= \Pi_k (M_k k_t + M_a a_t) + \Pi_a a_{t+1} \\ &= \Pi_k M_k (\Pi_k^{-1} y_t - \Pi_k^{-1} \Pi_a a_t) + \Pi_k M_a a_t + \Pi_a a_{t+1} \end{aligned}$$

or

$$y_{t+1} = \underbrace{\Pi_k M_k \Pi_k^{-1}}_{\varrho} y_t + \underbrace{\Pi_k (M_a - M_k \Pi_k^{-1} \Pi_a)}_{\Xi} a_t + \Pi_a a_{t+1}.$$

Under the assumptions made in the paper regarding the process for a_t, it follows that

$$\Delta y_{t+1} = \varrho \Delta y_t + \Xi \varepsilon_t + \Pi_a \varepsilon_{t+1},$$

so that the filtering is carried out according to

$$\varepsilon_{t+1} = \Pi_a^{-1} \Delta y_{t+1} - \Pi_a^{-1} \varrho \Delta y_t - \Pi_a^{-1} \Xi \varepsilon_t. \tag{38}$$

where ε_0 is set to zero. In order that the implied sectoral productivity growth rates be stationary, the filtering process (38) must satisfy the condition that the roots of $|I - \Pi_a^{-1} \Xi z|$ lie outside the unit circle.

Let

$$\eta_{t+1} = \Xi \varepsilon_t + \Pi_a \varepsilon_{t+1},$$

Then, if $\mathrm{var}(\varepsilon_t) = I$,

$$\Sigma_{\eta\eta} = \Xi \Xi' + \Pi_a \Pi_a'.$$