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# THE SOLUTION OF SINGULAR LINEAR DIFFERENCE SYSTEMS UNDER RATIONAL EXPECTATIONS*

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Many linear rational expectations macroeconomic models can be cast in the first-order form,

$$AE_t y_{t+1} = By_t + CE_t x_t,$$

if the matrix $A$ is permitted to be singular. We show that there is a unique stable solution under two requirements: (i) the determinantal polynomial $|Az - B|$ is not zero for some value of $z$, and (ii) a rank condition. The unique solution is characterized using a familiar approach: a canonical variables transformation separating dynamics associated with stable and unstable eigenvalues. In singular models, however, there are new canonical variables associated with infinite eigenvalues. These arise from nonexpectational behavioral relations or dynamic identities present in the singular linear difference system.

## 1. INTRODUCTION

Linear rational expectations models are the workhorse of modern dynamic economics. We provide necessary and sufficient conditions for the solvability of a general rational expectations model called a singular linear difference system. These theoretical results provide insight into the nature of solutions in rational expectations models; they also provide a theoretical base for ongoing development of efficient and robust algorithms for computing solutions to quantitative macroeconomic models.

The early work of Blanchard and Kahn (1980) studied the linear difference system,

$$AE_t y_{t+1} = By_t + CE_t x_t, \tag{1}$$

where $y_t$ is a vector of endogenous variables, $x_t$ is a vector of exogenous variables and $A$, $B$, and $C$ are matrices of coefficients. Requiring $A$ to be nonsingular, they characterized solutions to (1) by producing an equivalent dynamic system involving canonical variables. In this transformed system, the distinction between stable and

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unstable canonical variables was central. In particular, a 'rank' condition was necessary and sufficient for the existence and uniqueness of a stable solution: there had to be as many unstable canonical variables as nonpredetermined variables, and it also had to be possible to associate these two sets of variables. In this solution, $y_t$ depended on predetermined variables and on an infinite distributed lead of expected exogenous variables.

Many economic models do not fit directly into the Blanchard–Kahn framework, but can be case in first-order form if $A$ is permitted to be singular. Extending the canonical variables approach to such singular linear difference equation systems, we show there exists a unique stable solution under two conditions: (i) there exists a number $z$ such that the determinant polynomial, $|Az - B|$, is nonzero; and (ii) a direct generalization of the Blanchard–Kahn rank condition (stated precisely in Section 4 below) is satisfied. If $A$ is singular, we find a new class of canonical variables—associated with infinite roots of the determinantal polynomial—whose solution introduces a finite order distributed lead of expected exogenous variables.[2]

The outline of this paper is as follows. Section 2 follows the Introduction and discusses the condition $|Az - B| \neq 0$. Section 3 presents two examples. Section 4 reviews the Blanchard–Kahn case of nonsingular $A$ and then provides the general canonical variables solution. We conclude by discussing the implications of our theoretical work for the development of efficient and robust numerical algorithms for computing solutions to (1).[3]

## 2. THE MODEL

To be more explicit about the specification (1), we assume that $y_t$ is an $m \times 1$ vector of endogenous variables, $x_t$ is an $n \times 1$ vector of exogenous variables, $A(m \times m)$, $B(m \times m)$, and $C(m \times n)$ are coefficients matrices. For any variable $w_t$, $E_t w_{t+s} = E(w_{t+s}|\Omega_t)$, where $\Omega_{t-1} \subseteq \Omega_t$, and $(y_t, x_t) \in \Omega_t$. Finally, we assume that the last $p$ elements of $y_t$ are predetermined, that is, do not respond to new information at period $t$, and that initial conditions for these variables are given. We partition $y_t$ into nonpredetermined variables $\Lambda$ and predetermined variables $k$,

$$y_t = \begin{bmatrix} \Lambda_t \\ k_t \end{bmatrix},$$

[2] In this paper, we extend the canonical variable approach to the important practical case of singular $A$. Models with singular $A$ matrices have been studied using 'undetermined coefficients' and 'martingale/martingale difference' methods by Pesaran (1987), Broze et al., (1990), Broze and Szafarz (1991), and Binder and Pesaran (1995).

[3] When we produced the early drafts of this paper, we were able to list a small group of additional references on numerical algorithms for this and related control problems: Anderson et al., (1996), Anderson and Moore (1985), King and Watson (1995), Sims (1989), and the previously mentioned studies using undetermined coefficient or martingale difference methods. However, in the last two years, at least a half dozen other researchers have attacked this problem.

with $k_0 = \bar{k}$, a known vector of constants. For our purposes, it is convenient to write the model as

$$(2) \qquad (A\mathbf{F} - B)E_t y_t = CE_t x_t,$$

using a lead operator, $\mathbf{F}$, defined so that $\mathbf{F}^h E_t w_{t+j} = E_t w_{t+j+h}$, that is, application of $\mathbf{F}$ changes the dating of the variable but not the dating of the expectation.[4]

One requirement for the solvability of this model is that $|Az - B| \neq 0$ identically in $z$. To exposit this condition, we begin by noting that there are two distinct sources of nonuniqueness or nonexistence of solutions in the model. The first arises because the model describes expectations of endogenous variables $E_t y_{t+1}$, rather than the values of the variables themselves. The literature on rational expectations models has focused on this source of nonuniqueness, as do we in Section 4 below. However, in our general model, there is a more mundane source that the condition $|Az - B| \neq 0$ rules out: nonuniqueness or nonexistence can arise because certain linear combinations of $y$ are unrestricted by $A$ and $B$.

To see why this is the natural condition, we consider a sequence of increasingly elaborate related models. To begin, when $A = B = 0$, the model places no restrictions on $y$. More generally, in the static model $Ay = By + Cx$, a unique solution for $y$ will exist (for all values of the variable $x$) if and only if $[A - B]$ is nonsingular. Equivalently, existence of a unique solution requires that $|A - B| \neq 0$. This condition is necessary because if $[A - B]$ has rank $k < m$, then there exists an $(m-k) \times m$ matrix $\alpha$ such that $\alpha[A - B] = 0$; multiplying the static equation by $\alpha$ implies $0 = \alpha Cx$. In turn, this latter condition implies that the system of equations is under-determined if $\alpha C = 0$ or inconsistent if $\alpha C \neq 0$.

The analogous restriction in a dynamic model is $|Az - B| \neq 0$ identically in $z$. If this condition fails to hold, then $\text{rank}(Az - B) \leq k < m$ for all $z$ and there exists an $(m-k) \times m$ matrix polynomial $\alpha(z)$, such that $\alpha(z)[Az - B] = 0$. Multiplying the dynamic equation (1) by $\alpha(\mathbf{F})$ implies that $0 = \alpha(\mathbf{F})CE_t x_{t+1}$. Since a solution must exist for all possible values of $E_t x_t$, the system of equations is under-determined if $\alpha(1)C = 0$ or inconsistent if $\alpha(1)C \neq 0$. The condition that $|Az - B| \neq 0$ identically in $z$ is obviously much weaker that $|A| \neq 0$ or $|B| \neq 0$, which are common assumptions in the literature.[5]

## 3. EXAMPLES

Our first example is motivated purely by the mathematics of singular systems. The second illustrates the kind of economic considerations which lead to a singular system.

[4] Note that $\mathbf{F}$ operates only on conditional expectations and in this sense differs from the usual 'forward' operator. That is, $\mathbf{F}E_t w_t = E_t w_{t+1}$ is well defined, but '$\mathbf{F}w_t$' is not defined since the conditioning set is not specified. This is the same operator as $B^{-1}$ defined in Sargent (1979), p. 269.

[5] Formally, $|Az - B| \neq 0$ makes $Az - B$ a nonsingular matrix pencil in Gantmacher's (1959) terminology.

*The Scalar Case.* With a single endogenous variable $y$ and a single exogenous variable $x$, our general model can be written:

$$aE_t y_{t+1} = by_t + cx_t \tag{3}$$

with $a$, $b$, and $c$ being scalars. With $a$ and $b$ nonzero, this is a textbook rational expectations example (see Sargent 1979). The root of $|az - b| = 0$ is $z = b/a$. It is well known that the existence and uniqueness of a stable rational expectations equilibrium depends on whether $|b/a| < 1$ and on whether $y$ is predetermined or not. In particular, if $y$ is predetermined then the root must be stable for a stable solution exit, while if $y$ is nonpredetermined then the root must be unstable for a unique rational expectations solution.

Singularities arise when $a$ or $b$ or both are zero. If $a = b = 0$, then the system implies that $cx_t = 0$, so that there is no solution for general forcing processes: our general condition $|Az - B| \neq 0$ is evidently necessary for a unique solution.

If $a = 0$ then the model is $by_t + cx_t = 0$: it is conventional to say that the root of $|az - b| = 0$ is infinite, since this is the limit of $b/a$ as $a \to 0$. For (3) with $a = 0$, the existence requirement is similar to when there is a finite, but unstable, root in the nonsingular case. If $y_t$ is predetermined, there is no solution because the equation is inconsistent. If $y_t$ is not predetermined, the solution is $y_t = -(c/b)x_t$. The solvability condition in this 'infinite' root case is thus the same as the unstable root case in the familiar nonsingular model.

If $b = 0$, then the equation becomes $aE_t y_{t+1} = cx_t$: there is a zero root. For (3) with $b = 0$, the uniqueness requirement is similar to a nonzero stable root in the nonsingular model. If $y_t$ is not predetermined, then there will be many solutions of the form $y_{t+1} = (c/a)x_t + \xi_{t+1}$, where $\xi_{t+1}$ is unpredictable given information at date $t$. If $y_t$ is predetermined, then there is a unique solution $y_{t+1} = (c/a)x_t$. The solvability condition in this zero root case is thus the same as when there is a stable root in the familiar nonsingular model.

In this simple example, then, we find that existence and uniqueness of a solution to the general singular model requires our determinant condition, $|Az - B| \neq 0$. In addition, we must be able to associate any unstable roots—including infinite roots—with nonpredetermined variables.

*A Version of the Cagan* (1956) *Monetary Model.* Let $R_t$ be the nominal interest rate, $P_t$ be the logarithm of the price level, and $M_t$ be the logarithm of the money stock. The model is comprised of the Fisher equation, $R_t = E_t P_{t+1} - P_t$, and the monetary equilibrium condition, $M_t - P_t = -\alpha R_t$. Writing these in first-order form, $AE_t y_{t+1} = By_t + CE_t x_t$, we have

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_t R_{t+1} \\ E_t P_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \alpha & -1 \end{bmatrix} \begin{bmatrix} R_t \\ P_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} E_t M_t. \tag{4}$$

Notice that $\text{rank}(A) = 1$, so that $A$ is singular. Solving out the Fisher identity yields $M_t - P_t = -\alpha(E_t P_{t+1} - P_t)$ or equivalently,

$$\alpha E_t P_{t+1} = (1+\alpha) P_t - M_t, \tag{5}$$

which can be uniquely solved if $\alpha > 0$. This example suggests that one way to characterize models with singular $A$ matrices, is that they contain identities, possibly dynamic in nature. It also suggests that eliminating the features that lead to the singularity of $A$ results in a system of smaller dimension.[6]

## 4. CANONICAL VARIABLES SOLUTIONS

Canonical variable solutions are constructed by transforming (1) into an equivalent system that is easier to solve than the original system. Specifically, the transformed system is constructed by pre-multiplying (1) by a nonsingular matrix $T$, and pre-multiplying $y_t$ by a nonsingular matrix $V$. This yields

$$(A^*\mathbf{F} - B^*) E_t y_t^* = C^* E_t x_t \tag{6}$$

where $A^* = TAV^{-1}$, $B^* = TBV^{-1}$, $C^* = TC$, and the change of variables is $y_t^* = Vy_t$.

For a particular choice of $V$ detailed in the Appendix, the transformed variables are:

$$y^* = Vy = \begin{bmatrix} i \\ u \\ s \end{bmatrix} = \begin{bmatrix} U \\ s \end{bmatrix}, \quad \text{with } U = \begin{bmatrix} i \\ u \end{bmatrix}$$

These three canonical variables have particularly simple dynamics. Two are those identified by Blanchard and Kahn (1980): the stable canonical variables ($s$) are associated with roots of $|Az - B|$ that have modulus less than unity and the unstable canonical variables ($u$) are associated with roots of $|Az - B|$ that have modulus greater than unity. However, in a singular linear difference equation, there is a new class of canonical variables ($i$) that are associated with the 'infinite' roots of $|Az - B|$. These new variables are usefully viewed as extreme versions of unstable canonical variables and so we allow for a partition of $y^*$ that collects $i$ and $u$ into a vector $U$.

We begin our discussion by reviewing the characterization of solutions in models with nonsingular $A$. Here there is no distinction between $U$ and $u$, that is, there are no elements of $i$. We then consider the more general model with nonsingular $A$.

### 4.1. *Analysis of Nonsingular Systems.*

With $A$ invertible, the dynamic system can be written as $E_t y_{t+1} = A^{-1} B y_t + A^{-1} C E_t x_t$. In this case of a nonsingular system, the determinantal condition $|Az - B| \neq 0$ identically in $z$ is always satisfied. To characterize the solution as in Blanchard and Kahn (1980), we begin by taking the

[6] Several other examples are given in King and Watson (1995). In all of these examples, including the well-known example $P_t = \gamma E_{t-1} P_t + M_t$ from Blanchard and Kahn (1980), $A$ and/or $B$ are singular and yet $|Az - B| \neq 0$; thus their solutions can be characterized using the methods discussed in the next section.

left eigenvectors of $W = A^{-1}B$ to be $L$; it follows that $LW = JL$, where $J$ is the Jordan matrix with the eigenvalues (ordered in declining absolute value) on its diagonal and zeros or ones (arranged in Jordan blocks) on the first super diagonal. Then, taking $T = LA^{-1}$ and $V = L$, we can write an equivalent dynamic system as:

$$(7) \qquad E_t\begin{bmatrix} u_{t+1} \\ s_{t+1} \end{bmatrix} = \begin{bmatrix} J_u & 0 \\ 0 & J_s \end{bmatrix}\begin{bmatrix} u_t \\ s_t \end{bmatrix} + \begin{bmatrix} C_u^* \\ C_s^* \end{bmatrix} E_t x_t$$

using the partitioning $y^* = [u' \ s']'$. As indicated above, the mnemonic is that the $u$ variables are unstable canonical variables and the $s$ variables are stable canonical variables. That is, the above partitioning is based on locating the unstable eigenvalues in $J_u$ and the stable eigenvalues in $J_s$. This equivalent dynamic system indicates that the observed variables $y$ are linear combinations of canonical variables $y^*$, since $y = V^{-1}y^*$.

There are many solutions to (1) for $\{y_t\}_{t=1}^T$ because initial conditions are only specified for a subset of the elements of $y$, namely the predetermined variables $k$. Blanchard and Kahn (1980) show that initial conditions on $\Lambda$ can be determined and a unique solution obtained if (i) the solution is restricted to be stable and (ii) a certain sub-matrix of $V$ has full rank.[7]

To summarize their argument, the stability requirement implies that the equations for the $u_t$ must be solved forward since eigenvalues of $J_u$ are greater than one in modulus, yielding the solution

$$(8) \qquad u_t = -[J_u - \mathbf{F}I]^{-1}C_u^* E_t x_t = -\sum_{h=0}^{\infty} J_u^{-h} C_u^* E_t x_{t+h}.$$

That is: the unstable canonical variables are an infinite 'distributed lead' of the exogenous variables. Any sequence of candidate perturbations $v_t$ from this solution will have the expected path $E_t v_{t+k} = J_u^k v_t$ and will thus be asymptotically explosive in expected value.

With $\{u_t\}_{t=0}^{\infty}$ determined by the stability condition, the predeterminedness condition yields a recursive solution for $\{\Lambda_t\}$ if the matrix $V$ satisfies a 'rank condition' developed in Blanchard and Kahn (1980).[8] To exposit this condition, we define the partitioned transformations,

$$(9) \qquad \begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} V_{u\Lambda} & V_{uk} \\ V_{s\Lambda} & V_{sk} \end{bmatrix}\begin{bmatrix} \Lambda \\ k \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Lambda \\ k \end{bmatrix} = \begin{bmatrix} R_{\Lambda u} & R_{\Lambda s} \\ R_{ku} & R_{ks} \end{bmatrix}\begin{bmatrix} u \\ s \end{bmatrix}$$

with $R = V^{-1}$ and we use standard notation for blocks of $V$ and $R$. These transformations imply that $u_t = V_{u\Lambda}\Lambda_t + V_{uk}k_t$, which links $\Lambda_t$ to $u_t$ (which is

[7] The stability condition can be written as the requirement that, if $|E_t x_{t+k}| < \bar{x}$ *a.s.* for some finite $\bar{x}$ and all $t$ and $k$, then $|E_t y_{t+k}| < \bar{y}$ *a.s.* for some finite $\bar{y}$ and all $t$ and $k$.

[8] The solvability requirement is sometimes described as 'as many unstable canonical variables as nonpredetermined variables.' Boyd and Dotsey (1994) give examples of models in which this counting rule works but in which a simple monetary model with an interest rate rule for monetary policy cannot be solved due to a failure of the invertibility of $V_{U\Lambda}$.

determined by the stability condition) and $k_t$ (which is predetermined). A unique solution requires that $V_{u\Lambda}$ is nonsingular, so that

$$\Lambda_t = V_{u\Lambda}^{-1} u_t - V_{u\Lambda}^{-1} V_{uk} k_t. \tag{10}$$

To complete the solution of the model, the predetermined variables evolve according to:[9]

$$k_{t+1} = R_{ku} E_t u_{t+1} + R_{ks} J_s V_{s\Lambda} V_{u\Lambda}^{-1} u_t + R_{ks} C_s^* x_t + R_{ks} J_s R_{ks}^{-1} k_t \tag{11}$$

where $R_{ks} = (V_{sk} - V_{s\Lambda} V_{u\Lambda}^{-1} V_{uk})^{-1}$ is nonsingular since $V$ and $V_{u\Lambda}$ are nonsingular. Equations (11) and (10) can be used recursively to construct $\{k_t, \Lambda_t, y_t\}_{t=1}^{\infty}$, given initial conditions for $k_0$ and $x_0$ together with the solution for $u_t$.

4.2. *Analysis of Singular Systems.* Our analysis of singular models is a natural generalization of the preceding approach. In the Appendix, we show how to construct matrices $T$ and $V$ such that there is an equivalent dynamic system to (1) of the form:

$$\begin{bmatrix} N & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} E_t \begin{bmatrix} i_{t+1} \\ u_{t+1} \\ s_{t+1} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & J_u & 0 \\ 0 & 0 & J_s \end{bmatrix} \begin{bmatrix} i_t \\ u_t \\ s_t \end{bmatrix} + \begin{bmatrix} C_i^* \\ C_u^* \\ C_s^* \end{bmatrix} E_t x_t. \tag{12}$$

In this expression, $N$ is upper triangular with zeros on its main diagonal. Thus, there is a number $l \le \text{rows}(N)$ such that $N^h$ is a matrix of zeros for all $h > l$.

The representation (12) contains the elements $u$ and $s$ stressed by Blanchard and Kahn (1980), but it also contains some new elements $i_t$. These variables are governed by the equations:

$$(N\mathbf{F} - I) E_t i_t = C_i^* E_t x_t. \tag{13}$$

In terms of the foregoing discussion, the crucial point is that $|(Nz - I)| = 1$ or $-1$ so that there are no finite eigenvalues and it is conventional to say that there are as many infinite eigenvalues as there are elements of $i$.[10] Successive forward substitution in (13) yields the unique solution:

$$i_t = (N\mathbf{F} - I)^{-1} C_i^* E_t x_t = -\sum_{j=0}^{l} N^j C_i^* E_t x_{t+j}. \tag{14}$$

That is, the variables with infinite eigenvalues depend on expected future exogenous variables in a finite order (moving average) manner. In contrast to the solution for

[9] To derive this condition, we use $k_{t+1} = R_{ku} E_t u_{t+1} + R_{ks} E_t s_{t+1}$. The stable canonical variables dynamics are $E_t s_{t+1} = J_s s_t + C_s^* x_t$, with the elements of this expression following from the fact that $s_t = V_{s\Lambda} \Lambda_t + V_{sk} k_t$ from (9) and the preceding solution for nonpredetermined variables (10).

[10] Since the matrix $N$ is nilpotent, its eigenvalues are all 0, so that the 'reciprocal' roots of $|(Nz - I)|$ are infinite.

the $u$ variables, which was determined on stability grounds, this solution simply involves solving out for some variables using some of the equations of the model.[11]

Taking the stable solution for the $u$ variables described in the previous subsection, we then have that the nonpredetermined variables of the model, $\Lambda_t$, are restricted by:

$$U_t = \begin{bmatrix} i_t \\ u_t \end{bmatrix} = V_{U\Lambda}\Lambda_t + V_{Uk}k_t$$

where the number of rows of $V_{U\Lambda}$ and $V_{Uk}$ is the sum of the number of elements of $i$ and the number of elements of $u$. Requiring that $V_{U\Lambda}$ is invertible, we can write the solution for the nonpredetermined variables as:

$$\Lambda_t = V_{U\Lambda}^{-1}U_t - V_{U\Lambda}^{-1}V_{Uk}k_t, \qquad t = 0, 1, 2, \ldots \tag{15}$$

Note that this expression for the singular model is the natural generalization of (10) for the nonsingular model. Essentially, the canonical variables attached to infinite (and hence unstable) eigenvalues are treated in exactly the same manner as the unstable eigenvalues were previously once the solution (14) has been determined. The solution for $k_{t+1}$ then also proceeds in exactly the same fashion as above, taking into account the fact that there is a larger vector $U$ that contains the solutions for both $i$ and $u$.

In this section, we have established that the solvability conditions for the singular linear expectations difference equation are twofold. First, we must require that $|Az - B| \neq 0$ identically in $z$. This condition insures a unique solution for the difference equation $Ay_{t+1} = By_t + Cx_t$, given initial conditions for $k_t$. Second, we require that $V_{U\Lambda}$ be nonsingular. As explained above, this requirement is necessary to deduce the values of the nonpredetermined variables, $\Lambda$.

## 5. CONCLUSIONS AND IMPLICATIONS

This paper provides a theoretical characterization of the solution to a singular linear difference system, but there are useful implications for applied researchers. The first major point is that one necessary condition for solvability is that there must exist a number $z$ such that the determinant polynomial, $|Az - B|$, is nonzero. This practical condition can be checked as a precondition to attempting the solution of a model: if it is violated, then the model is ill-specified. The second point is that additional conditions are required that can be validated only as part of the process of model solution: in a literal application of our approach, one would need to verify that $|V_{U\Lambda}| \neq 0$ after computing the canonical variables decomposition.[12]

The theoretical characterization also provides insight into why there are two general approaches to computing solutions in singular models. Proceeding directly

[11] Broze and Szafarz (1991) and Binder and Pesaran (1995) display similar forward-looking components.

[12] However, a direct application of this paper's approach is not generally advisable, because the Jordan canonical form cannot be accurately computed numerically.

from the theory, one approach—as in Sims [1989]—is to find transformation matrices like $T$ and $V$ that allow the decoupling of stable and unstable dynamics, while allowing for possibly infinite roots of the determinant polynomial $|Az - B|$. However, the theoretical characterization also explains why there are 'system reduction' approaches like those of Anderson and Moore (1985) and King and Watson (1995): the structure of singular linear rational expectations makes it possible to reduce the dimension of the system by solving out the identities responsible for the singularity. Specifically, for any solvable model, our canonical variables analysis can be used to show that it is always possible to decompose the $y$ vector into subvectors $f$ and $d$, with these elements evolving as:

$$0 = f_t + Kd_t + \Psi_f(\mathbf{F}) E_t x_t \tag{16}$$

$$E_t d_{t+1} = Wd_t + \Psi_d(\mathbf{F}) E_t x_t \tag{17}$$

In this transformed system, the variables $f_t$ are a subset of the nonpredetermined elements of $y_t$, and the variables $d_t$ are the remaining nonpredetermined elements of $y_t$ and all predetermined elements of $y_t$. The number of elements in $f$ is equal to the number of elements of the canonical variables $i$, from Section 4. In this transformation, the system's intrinsic dynamics can be described without reference to $f_t$ by focusing on the evolution of $d_t$.[13] Such system reduction approaches explicitly eliminate the dynamic identities that lead to singularities, in much the same way that a researcher would working with pencil and paper.

In summary, we have provided a theoretical characterization of solutions to the general linear rational expectations model when $A$ is singular, but $|Az - B| \neq 0$. We established that a generalization of the familiar canonical variables approach of Blanchard and Kahn (1980) can be applied to this model, if canonical variables are admitted that have 'infinitely unstable' forward dynamics. Our work provides an essential underpinning to ongoing efforts to develop generally applicable efficient and robust algorithms for use in quantitative studies.

## APPENDIX

### CONSTRUCTION OF CANONICAL VARIABLES SOLUTION

As necessary for the canonical system (12), this Appendix constructs matrices $T$ and $V$ such that

$$TAV^{-1} = A^* = \begin{bmatrix} N & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad TBV^{-1} = B^* = \begin{bmatrix} I & 0 & 0 \\ 0 & J_u & 0 \\ 0 & 0 & J_s \end{bmatrix}$$

[13] This system reduction implication was proved in early drafts of the current paper. King and Watson (1995) details the construction of (16) and (17) as steps in a numerical algorithm for solving (1). It also shows that system reduction is feasible for some models that display dynamic indeterminacies as well.

in four steps, which follow Gantmacher's (1959) construction of the canonical form of a nonsingular matrix pencil.

*Step* 1. Select a number $\alpha$ such that $|A\alpha - B| \neq 0$ and write

$$\begin{aligned}[A\mathbf{F} - B] &= [A(\mathbf{F} - \alpha) - (B - A\alpha)] \\ &= \left[A(B - A\alpha)^{-1}(\mathbf{F} - \alpha) - I\right][B - A\alpha]\end{aligned}$$

The existence of this number is assured by the requirement that $|Az - B| \neq 0$.

*Step* 2. Construct the Jordan decomposition of $M \equiv A(B - A\alpha)^{-1}$ and write it as $QM = \mu Q$ with

$$\mu = \begin{bmatrix} \mu_0 & 0 \\ 0 & \mu_+ \end{bmatrix}.$$

The diagonal of $\mu_0$ contains the zero eigenvalues of $M$ and the nonzero eigenvalues are on the diagonal of $\mu_+$. Since $\mu_0$ is a Jordan matrix with a zero diagonal, it is nilpotent, that is, there is a number $l$ such that $\mu_0^l$ is a matrix of zeros. Using this decomposition of $\mu$, write

$$[A\mathbf{F} - B] = Q^{-1}[\mu(\mathbf{F} - \alpha) - I]Q[B - \alpha A],$$

where

$$\mu(\mathbf{F} - \alpha) - I = \begin{bmatrix} \mu_0 & 0 \\ 0 & \mu_+ \end{bmatrix}\mathbf{F} - \begin{bmatrix} I + \alpha\mu_0 & 0 \\ 0 & I + \alpha\mu_+ \end{bmatrix}.$$

This construction insures that $I + \alpha\mu_0$ is invertible and that $\mu_+$ is invertible.

*Step* 3. Define the matrix $S$ such that

$$S^{-1} = \begin{bmatrix} I + \alpha\mu_0 & 0 \\ 0 & \mu_+ \end{bmatrix}$$

and write $(A\mathbf{F} - B) = Q^{-1}S^{-1}\{S[\mu(\mathbf{F} - \alpha) - I]\}Q(B - \alpha A)$, where the term in braces is

$$\begin{bmatrix} (I + \alpha\mu_0)^{-1}\mu_0 & 0 \\ 0 & I \end{bmatrix}\mathbf{F} - \begin{bmatrix} I & 0 \\ 0 & \mu_+^{-1}(I + \alpha\mu_+) \end{bmatrix}$$

Note that $(I + \alpha\mu_0)^{-1}\mu_0$ is upper triangular with all zeros on its main diagonal and is nilpotent.

*Step* 4. Construct the Jordan decomposition of $\mu_+^{-1}(I+\alpha\mu_+)$ and write this as

$$G_+^{-1}\left[\mu_+^{-1}(I+\alpha\mu_+)\right]G_+=\begin{bmatrix} J_u & 0 \\ 0 & J_s \end{bmatrix}$$

where the diagonal elements of $J_u$ are those eigenvalues whose modulus exceeds unity. Let

$$G=\begin{bmatrix} I & 0 \\ 0 & G_+ \end{bmatrix}$$

be a matrix with the same dimension as $A$ and $B$. Then, it follows that

$$(A\mathbf{F}-B)=Q^{-1}S^{-1}G^{-1}[A^*\mathbf{F}-B^*]GQ(B-\alpha A).$$

where

$$A^*=\begin{bmatrix} N & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad B^*=\begin{bmatrix} I & 0 & 0 \\ 0 & J_u & 0 \\ 0 & 0 & J_s \end{bmatrix}$$

with $N=(I+\alpha\mu_0)^{-1}\mu_0$. Our desired result thus is obtained by setting

$$T=GSQ \quad \text{and} \quad V=GQ(B-\alpha A).$$

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