Supplementary Appendix to

Long-Run Covariability

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This appendix provides supplemental material. Section A.1 discusses the form of the confidence sets; Section A.2 derives the necessary densities; Section A.3 discusses the numerically determined approximate least favorable distributions; the data are described in Section A.4; Section A.5 compares low-pass and low-frequency projections for GDP and consumption; Section A.6 discusses alternative versions of Ω_T constructed from projections onto a subset of the columns of Ψ_T and summarizes the resulting empirical results.

A.1 Form of Confidence Sets

For each of the three sets H^ρ, H^β and H^σ, we exogenously impose that they contain the $(1-\alpha)$ equal-tailed invariant credible set relative to the prior F, as suggested by Müller and Norets (2016). Denote this credible set by H_0^i, $i \in \{\rho, \beta, \sigma\}$. Specializing Theorem 3 of Müller and Norets (2016) to the three problems considered here yields the following form for the three type of confidence sets:

H^ρ: Let $X^s = X/\sqrt{X'X}$ and $Y^s = Y/\sqrt{Y'Y}$, and let $f^s(x^s, y^s|\theta)$ be the density of (X^s, Y^s) under $\theta \in \Theta$.[16] Then

$$\begin{aligned} H_0^\rho(x,y) &= \left\{ r : \alpha/2 \leq \frac{\int \mathbf{1}[g^\rho(\theta) \leq r] f^s(x^s, y^s|\theta) dF(\theta)}{\int f^s(x^s, y^s|\theta) dF(\theta)} \leq 1 - \alpha/2 \right\} \quad (A.1) \\ H^\rho(x,y) &= \left\{ r : \int f^s(x^s, y^s|\theta) dW(\theta) \geq \int f^s(x^s, y^s|\theta) d\Lambda_r^\rho(\theta) \right\} \cup H_0^\rho(x,y) \end{aligned}$$

where W is the weighting function over which expected length is minimized and the family of positive measures Λ_r^ρ on Θ, indexed by $r \in (-1, 1)$, are such that $\Lambda_r^\rho(\{\theta : g^\rho(\theta) \neq r$ or $P_\theta(g^\rho(\theta) \in H^\rho(X,Y)) > 1-\alpha\}) = 0$ and $P_\theta(g^\rho(\theta) \in H^\rho(X,Y)) \geq 1 - \alpha$ for all $\theta \in \Theta$.

[16] Here and in the following, we distinguish between random variables and generic real numbers by the usual upper case / lower case convention. We also implicitly assume the same functional relationship between the random variables and their corresponding real variables, if appropriate. For example, (x^s, y^s) on the right hand side of (A.1) is implicitly thought of as a function of (x, y).

H^β: Let the $q-2$ vectors X^* and Y^*, and $X_0^*, U_{11}, U_{12}, U_{22} \in \mathbb{R}$ be such that

$$(X,Y) = \left(\left(\begin{array}{c} 1 \\ X_0^* \\ X^* \end{array} \right), \left(\begin{array}{c} 1 \\ 0 \\ Y^* \end{array} \right) \right) \left(\begin{array}{cc} U_{11} & U_{12} \\ 0 & U_{22} \end{array} \right),$$

that is, perform the LDU decomposition of the upper 2×2 block of the $q \times 2$ matrix (X,Y). Let $Z^* = (X_0^*, X^{*\prime}, Y^{*\prime})'$. Then

$$\begin{aligned} H_0^\beta(x,y) &= \left\{ b : \alpha/2 \leq \frac{\int\int \mathbf{1}[\frac{u_{11}b-u_{12}}{u_{22}} \leq w] f_0^\beta(z^*, w|\theta) dw dF(\theta)}{\int f_1^*(z^*|\theta)dF(\theta)} \leq 1-\alpha/2 \right\} \\ H^\beta(x,y) &= \left\{ b : \int h^\beta(z^*|\theta) f_1^*(z^*|\theta) dW(\theta) \geq \int f_0^\beta(z^*, \frac{u_{11}b-u_{12}}{u_{22}}|\theta) d\Lambda^\beta(\theta) \right\} \cup H_0^\beta(x,y) \end{aligned}$$

where $f_1^*(z^*|\theta)$ is the density of Z^* under θ, $h^\beta(z^*|\theta) = E_\theta[|U_{22}/U_{11}||Z^* = z^*]$, $f_0^\beta(z^*, w|\theta)$ is the density of the $2q-2$ vector $(Z^{*\prime}, (U_{11}g^\beta(\theta) - U_{12})/U_{22})'$ under θ, and Λ^β is a positive measure on Θ such that $\Lambda^\beta(\{\theta : P_\theta(g^\beta(\theta) \in H^\beta(X,Y)) > 1-\alpha\}) = 0$ and $P_\theta(g^\beta(\theta) \in H^\beta(X,Y)) \geq 1-\alpha$ for all $\theta \in \Theta$.

H^σ:

$$\begin{aligned} H_0^\sigma(x,y) &= \left\{ s : \alpha/2 \leq \frac{\int\int \mathbf{1}[\frac{s}{u_{22}} \leq w] f_0^\sigma(z^*, w|\theta) dw dF(\theta)}{\int f_1^*(z^*|\theta)dF(\theta)} \leq 1-\alpha/2 \right\} \\ H^\sigma(x,y) &= \left\{ s : \int h^\sigma(z^*|\theta) f_1^*(z^*|\theta) dW(\theta) \geq \int f_0^\sigma(z^*, \frac{s}{u_{22}}|\theta) d\Lambda^\sigma(\theta) \right\} \cup H_0^\sigma(x,y) \end{aligned}$$

where $h^\sigma(z^*|\theta) = E[|U_{22}||Z^* = z^*]$ under θ, $f_0^\sigma(z^*, w|\theta)$ is the density of the $2q-2$ vector $(Z^{*\prime}, g^\sigma(\theta)/|U_{22}|)'$ under θ, and Λ^σ is a positive measure on Θ such that $\Lambda^\sigma(\{\theta : P_\theta(g^\sigma(\theta) \in H^\sigma(X,Y)) > 1-\alpha\}) = 0$ and $P_\theta(g^\sigma(\theta) \in H^\sigma(X,Y)) \geq 1-\alpha$ for all $\theta \in \Theta$.

It remains to derive f^s, f_1^*, $f_1^* h^\beta$, $f_1^* h^\sigma$, f_0^β and f_0^σ, and to determine Λ_r^ρ, Λ^β and Λ^σ.

A.2 Densities of Maximal Invariants and Related Results

A.2.1 Preliminaries

As we show below, most densities of interest involve integrals of the form

$$Q(r) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty s^{p_1} t^{p_2} \exp[-\tfrac{1}{2} \left(\begin{array}{c} s \\ t \end{array} \right)' \left(\begin{array}{cc} a^2 & abr \\ abr & b^2 \end{array} \right) \left(\begin{array}{c} s \\ t \end{array} \right)] ds dt$$

$$
\begin{aligned}
&= a^{p_1-1}b^{p_2-1}\frac{1}{2\pi}\int_0^\infty\int_0^\infty s^{p_1}t^{p_2}\exp[-\tfrac{1}{2}(s^2-2rst+t^2)]dsdt \\
&= a^{p_1-1}b^{p_2-1}\int_0^\infty t^{p_2}\phi(\sqrt{1-r^2}t)\int_0^\infty s^{p_1}\phi(s-rt)dsdt
\end{aligned}
$$

for nonnegative integers p_1 and p_2, positive reals a, b, and $-1 < r < 1$, with ϕ the p.d.f. of a standard normal distribution. Note that

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty |s|^{p_1}|t|^{p_2}\exp[-\tfrac{1}{2}\begin{pmatrix} s \\ t\end{pmatrix}'\begin{pmatrix} a^2 & abr \\ abr & b^2\end{pmatrix}\begin{pmatrix} s \\ t\end{pmatrix}]dsdt \\
= \; & 2Q(r)+2Q(-r).
\end{aligned}
$$

We initially discuss how to obtain closed-form expressions for $Q(r)$. The resulting explicit formulae for densities, even after simplification with a computer algebra system, are long and uninformative, and they are relegated to the replication files.

Lemma A.1 *Let Φ be the c.d.f. of a standard normal random variable.*

(a) With $B_p(m) = \int_{-\infty}^\infty \phi(s-m)s^p ds$, we have $B_0(m) = 1$, $B_1(m) = m$ and

$$B_{p+2}(m) = (p+1)B_p(m) + mB_{p+1}(m);$$

(b) With $I_l(h) = \frac{1}{\Phi(h)}\int_{-\infty}^h \phi(s)s^l ds$,

$$\int_{-\infty}^0 \phi(s+h)s^p ds = \Phi(h)\sum_{l=0}^p \binom{p}{l}(-h)^{p-l}I_l(h)$$

and $I_0(h) = 1$, $I_1(h) = -\phi(h)/\Phi(h)$ and $I_p(h) = -h^{p-1}\phi(h)/\Phi(h) + (p-1)I_{p-2}(h)$;

(c) $\sqrt{2\pi}\int_0^\infty \phi(\sqrt{1+c^2}s)s^{p+1}ds = 2^{\frac{p}{2}}\Gamma(1+p/2)(1+c^2)^{-p/2-1}$;

(d) With $A_p(r) = 2\pi\int_0^\infty \phi(s)\Phi(\frac{r}{\sqrt{1-r^2}}s)s^p ds$, $A_0(r) = \pi - \arccos(r)$, $A_1(r) = \sqrt{\pi/2}(1+r)$, and

$$A_{p+2}(r) = (p+1)A_p(r) + \Gamma(1+p/2)2^{p/2}r(1-r^2)^{(1+p)/2}.$$

Proof. (a) By integration by parts and $\phi'(s) = -s\phi(s)$

$$\int_{-\infty}^\infty \phi(s-m)s^p ds = \int_{-\infty}^\infty (s-m)\phi(s-m)\frac{s^{p+1}}{p+1}ds$$

and the result follows.

(b) See Dhrymes (2005).

(c) Immediate after substituting $s^2 \to u$ from the definition of the Gamma function.

(d) Define $\tilde{A}_p(c) = 2\pi \int_0^\infty \phi(s)\Phi(cs)s^p ds$, so that $A_p(r) = \tilde{A}_p(r/\sqrt{1-r^2})$. Note that $\tilde{A}_p(0) = \pi \int_0^\infty \phi(s)s^p ds$, and $\tilde{A}_p'(c) = d\tilde{A}_p(c)/dc = 2\pi \int_0^\infty \phi(s)\phi(cs)s^{p+1}ds = \sqrt{2\pi}\int_0^\infty \phi(\sqrt{1+c^2}s)s^{p+1}ds$. Now $\tilde{A}_p(c) = \tilde{A}_p(0) + \int_0^c \tilde{A}_p'(u)du$. The results for $A_0(r)$ and $A_1(r)$ now follow by applying (c) and a direct calculation. For the iterative expression, by integration by parts and $\phi'(s) = -s\phi(s)$

$$\begin{aligned}\tilde{A}_p(c) &= \left[2\pi\phi(s)\Phi(cs)\frac{s^{p+1}}{p+1}\right]_0^\infty - 2\pi\int_0^\infty \frac{s^{p+1}}{p+1}(c\phi(cs)\phi(s) - s\phi(s)\Phi(cs))ds \\ &= \frac{1}{p+1}\left(\tilde{A}_{p+2}(c) - c\sqrt{2\pi}\int_0^\infty \phi(\sqrt{1+c^2}s)s^{p+1}ds\right),\end{aligned}$$

and the result follows from applying part (c). ■

Now by Lemma A.1 (a),

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty s^{p_1}\exp[-\tfrac{1}{2}(s-rt)^2]ds = C_0(rt) - \int_{-\infty}^0 \phi(s-rt)s^{p_1}ds$$

for some polynomial C_0 whose coefficients may be determined explicitly by the formula in Lemma A.1 (a). Furthermore,

$$\int_{-\infty}^0 \phi(s-rt)s^{p_1}ds = \phi(rt)C_1(rt) + \Phi(rt)C_2(t)$$

for some polynomials C_1 and C_2 that may be determined explicitly by the formula in Lemma A.1 (b). The remaining integral over dt is of the form

$$\begin{aligned}&\int_0^\infty t^{p_2}\phi(\sqrt{1-r^2}t)[C_0(rt) - \phi(rt)C_1(rt) - \Phi(rt)C_2(t)]dt \\ = \; &(1-r^2)^{p_2/2-1}\int_0^\infty \phi(t)t^{p_2}C_0(\frac{r}{\sqrt{1-r^2}}t)dt - \frac{1}{\sqrt{2\pi}}\int_0^\infty \phi(t)t^{p_2}C_1(rt)dt \\ &-(1-r^2)^{p_2/2-1}\int_0^\infty \phi(t)\Phi(\frac{r}{\sqrt{1-r^2}}t)t^{p_2}C_2(\frac{r}{\sqrt{1-r^2}}t)dt\end{aligned}$$

which can be determined explicitly by applying Lemma A.1 (c)-(d).

In the following, we simply write Σ for the covariance matrix of $\mathrm{vec}(X,Y)$, keeping the dependence on θ implicit. If not specified otherwise, all integrals are over the entire real line. Also, denote the four $q \times q$ blocks of the inverse of Σ as

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{xx}^- & \Sigma_{xy}^- \\ \Sigma_{yx}^- & \Sigma_{yy}^- \end{pmatrix}.$$

A.2.2 Derivation of f^s

Let $S_x = \sqrt{X'X}$ and $S_y = \sqrt{Y'Y}$. Write μ_l for Lebesgue measure on $\mathbb{R}^l$, and ν_q for the surface measure of a q dimensional unit sphere. For $x \in \mathbb{R}^q$, let $x = x^s s_x$, where x^s is a point on the surface of a q dimensional unit sphere, and $s_x \in \mathbb{R}^+$. By Theorem 2.1.13 of Muirhead (1982), $d\mu_q(x) = s_x^{q-1} d\nu_q(x^s) d\mu_1(s_x)$. We thus can write the joint density of (X^s, Y^s, S_x, S_y) with respect to $\nu_q \times \nu_q \times \mu_1 \times \mu_1$ as

$$
\begin{aligned}
& (2\pi)^{-q} (\det \Sigma)^{-1/2} \exp[-\tfrac{1}{2} \begin{pmatrix} x^s s_x \\ y^s s_y \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} x^s s_x \\ y^s s_y \end{pmatrix}] s_x^{q-1} s_y^{q-1} \\
= \; & (2\pi)^{-q} (\det \Sigma)^{-1/2} \exp[-\tfrac{1}{2} \begin{pmatrix} s_x \\ s_y \end{pmatrix}' \begin{pmatrix} x^{s\prime} \Sigma_{xx}^{-} x^s & x^{s\prime} \Sigma_{xy}^{-} y^s \\ y^{s\prime} \Sigma_{yx}^{-} x^s & y^{s\prime} \Sigma_{yy}^{-} y^s \end{pmatrix} \begin{pmatrix} s_x \\ s_y \end{pmatrix}] s_x^{q-1} s_y^{q-1}
\end{aligned}
$$

and the marginal density of $(X^{s\prime}, Y^{s\prime})'$ with respect to $\nu_q \times \nu_q$ is

$$
(2\pi)^{-q} (\det \Sigma)^{-1/2} \int_0^\infty \int_0^\infty \exp[-\tfrac{1}{2} \begin{pmatrix} s_x \\ s_y \end{pmatrix}' \begin{pmatrix} x^{s\prime} \Sigma_{xx}^{-} x^s & x^{s\prime} \Sigma_{xy}^{-} y^s \\ y^{s\prime} \Sigma_{yx}^{-} x^s & y^{s\prime} \Sigma_{yy}^{-} y^s \end{pmatrix} \begin{pmatrix} s_x \\ s_y \end{pmatrix}] s_x^{q-1} s_y^{q-1} ds_x ds_y.
$$

A.2.3 Derivation of f_1^*

With $X^\dagger = (1, X_0^*, X^{*\prime})'$, $Y^\dagger = (1, 0, Y^{*\prime})'$ and $U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$, we have

$$
\begin{aligned}
(X, Y) &= (X^\dagger, Y^\dagger) U \\
&= \begin{pmatrix} U_{11} & U_{12} \\ U_{11} X_1^* & U_{12} X_1^* + U_{22} \\ U_{11} X^* & U_{12} X^* + U_{22} Y^* \end{pmatrix}.
\end{aligned}
$$

This equation, viewed as a $\mathbb{R}^{2q} \to \mathbb{R}^{2q}$ function of $T^* = (X^*, Y^*, X_0^*, U_{11}, U_{12}, U_{22})$ has Jacobian determinant $U_{11}^{q-1} U_{22}^{q-2}$, so that the density of T^* is

$$
f_{T^*}(t^*) = (2\pi)^{-q} (\det \Sigma)^{-1/2} |u_{11}|^{q-1} |u_{22}|^{q-2} \exp[-\tfrac{1}{2} (\operatorname{vec} z^\dagger u)' \Sigma^{-1} (\operatorname{vec} z^\dagger u)] \qquad \text{(A.2)}
$$

with $z^\dagger = (x^\dagger, y^\dagger)$, and we are left to integrate out u_{11}, u_{12} and u_{22}. Using $\operatorname{vec}(z^\dagger u) = (I_2 \otimes z^\dagger) \operatorname{vec}(u)$, we have

$$
\operatorname{vec}(z^\dagger u)' \Sigma^{-1} \operatorname{vec}(z^\dagger u) = \operatorname{vec}(u)' [(I_2 \otimes z^\dagger)' \Sigma^{-1} (I_2 \otimes z^\dagger)] \operatorname{vec}(u)
$$

$$= \begin{pmatrix} u_{11} \\ 0 \\ u_{12} \\ u_{22} \end{pmatrix}' \begin{pmatrix} z^\dagger & 0 \\ 0 & z^\dagger \end{pmatrix}' \begin{pmatrix} \Sigma^-_{xx} & \Sigma^-_{xy} \\ \Sigma^-_{yx} & \Sigma^-_{yy} \end{pmatrix} \begin{pmatrix} z^\dagger & 0 \\ 0 & z^\dagger \end{pmatrix} \begin{pmatrix} u_{11} \\ 0 \\ u_{12} \\ u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} \\ u_{22} \\ u_{12} \end{pmatrix}' \begin{pmatrix} x^{\dagger\prime}\Sigma^-_{xx}x^\dagger & \cdot & \cdot \\ y^{\dagger\prime}\Sigma^-_{yx}x^\dagger & y^{\dagger\prime}\Sigma^-_{yy}y^\dagger & \cdot \\ x^{\dagger\prime}\Sigma^-_{yx}x^\dagger & y^{\dagger\prime}\Sigma^-_{yy}x^\dagger & x^{\dagger\prime}\Sigma^-_{yy}x^\dagger \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{22} \\ u_{12} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{u} \\ u_{12} \end{pmatrix}' \begin{pmatrix} V & v \\ v' & v_0^2 \end{pmatrix} \begin{pmatrix} \hat{u} \\ u_{12} \end{pmatrix}$$

where $\hat{u} = (u_{11}, u_{22})'$, $V = \begin{pmatrix} x^{\dagger\prime}\Sigma^-_{xx}x^\dagger & y^{\dagger\prime}\Sigma^-_{yx}x^\dagger \\ y^{\dagger\prime}\Sigma^-_{yx}x^\dagger & y^{\dagger\prime}\Sigma^-_{yy}y^\dagger \end{pmatrix}$, $v' = (x^{\dagger\prime}\Sigma^-_{yx}x^\dagger, y^{\dagger\prime}\Sigma^-_{yy}x^\dagger)$ and $v_0^2 = x^{\dagger\prime}\Sigma^-_{yy}x^\dagger$. Furthermore, by "completing the square",

$$\int \exp[-\tfrac{1}{2} \begin{pmatrix} \hat{u} \\ u_{12} \end{pmatrix}' \begin{pmatrix} V & v \\ v' & v_0^2 \end{pmatrix} \begin{pmatrix} \hat{u} \\ u_{12} \end{pmatrix}] du_{12} = \sqrt{2\pi} v_0^{-1} \exp[-\tfrac{1}{2}\hat{u}'(V - vv'/v_0)\hat{u}]$$

and with $\tilde{V} = V - vv'/v_0$, we obtain

$$f_1^*(z^*) = (2\pi)^{-q+1/2}(\det \Sigma)^{-1/2} v_0^{-1} \int\int |u_{11}|^{q-1}|u_{22}|^{q-2} \exp[-\tfrac{1}{2} \begin{pmatrix} u_{11} \\ u_{22} \end{pmatrix}' \tilde{V} \begin{pmatrix} u_{11} \\ u_{22} \end{pmatrix}] du_{11} du_{22}.$$

A.2.4 Derivation of $f_1^* h^\beta$

We have

$$\begin{aligned} h^\beta(z^*|\theta) &= E_\theta[|U_{22}/U_{11}||Z^* = z^*] \\ &= \frac{\int\int\int |\frac{u_{22}}{u_{11}}| f_{T^*}(t^*) du_{12} du_{11} du_{22}}{f_1^*(z^*)}. \end{aligned}$$

Thus, proceeding as in the derivation of f_1^* yields

$$\begin{aligned} & h^\beta(z^*|\theta) f_1^*(z^*|\theta) \\ = & \int\int\int |\frac{u_{22}}{u_{11}}| f_{T^*}(t^*) du_{12} du_{11} du_{22} \\ = & (2\pi)^{-q+1/2}(\det \Sigma)^{-1/2} v_0^{-1} \int\int |u_{11}|^{q-2}|u_{22}|^{q-1} \exp[-\tfrac{1}{2} \begin{pmatrix} u_{11} \\ u_{22} \end{pmatrix}' \tilde{V} \begin{pmatrix} u_{11} \\ u_{22} \end{pmatrix}] du_{11} du_{22}. \end{aligned}$$

A.2.5 Derivation of $f_1^* h^\sigma$

Proceeding analogously to the derivation of $f_1^* h^\beta$, we obtain

$$\begin{aligned}
& h^\sigma(z^*|\theta) f_1^*(z^*|\theta) \\
= & \int\int\int |u_{22}| f_{T^*}(t^*) du_{12} du_{11} du_{22} \\
= & (2\pi)^{-q+1/2} (\det \Sigma)^{-1/2} v_0^{-1} \int\int |u_{11}|^{q-1} |u_{22}|^{q-1} \exp[-\tfrac{1}{2} \begin{pmatrix} u_{11} \\ u_{22} \end{pmatrix}' \tilde{V} \begin{pmatrix} u_{11} \\ u_{22} \end{pmatrix}] du_{11} du_{22}.
\end{aligned}$$

A.2.6 Derivation of f_0^β

With $W^\beta = (U_{11} g^\beta(\theta) - U_{12})/U_{22}$, we have

$$\begin{pmatrix} U_{11} \\ U_{22} \\ U_{12} \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{22} \\ U_{11} g^\beta(\theta) - U_{22} W^\beta \end{pmatrix} = \begin{pmatrix} \hat{U} \\ \lambda_W' \hat{U} \end{pmatrix}$$

with $\hat{U} = (U_{11}, U_{22})'$ and $\lambda_W = (g^\beta(\theta), -W^\beta)'$. This equation, viewed as $\mathbb{R}^3 \mapsto \mathbb{R}^3$ function of $(U_{11}, U_{22}, W^\beta)$, has Jacobian determinant equal to $-U_{22}$. Thus, with $u_w = \begin{pmatrix} u_{11} & \lambda_w' \hat{u} \\ 0 & u_{22} \end{pmatrix}$, the joint density of (Z^*, W^β) can be written as

$$\int\int (2\pi)^{-q} (\det \Sigma)^{-1/2} |u_{11}|^{q-1} |u_{22}|^{q-1} \exp[-\tfrac{1}{2} (\operatorname{vec} z^\dagger u_w)' \Sigma^{-1} (\operatorname{vec} z^\dagger u_w)] du_{11} du_{22}.$$

Now similar to the derivation of f_1^*,

$$\begin{aligned}
(\operatorname{vec} z^\dagger u_w)' \Sigma^{-1} (\operatorname{vec} z^\dagger u_w) &= \begin{pmatrix} \hat{u} \\ \lambda_w' \hat{u} \end{pmatrix}' \begin{pmatrix} V & v \\ v' & v_0^2 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \lambda_w' \hat{u} \end{pmatrix} \\
&= \hat{u}' \begin{pmatrix} I_2 \\ \lambda_w' \end{pmatrix}' \begin{pmatrix} V & v \\ v' & v_0^2 \end{pmatrix} \begin{pmatrix} I_2 \\ \lambda_w' \end{pmatrix} \hat{u}.
\end{aligned}$$

Thus, with $V_w = \begin{pmatrix} I_2 \\ \lambda_w' \end{pmatrix}' \begin{pmatrix} V & v \\ v' & v_0^2 \end{pmatrix} \begin{pmatrix} I_2 \\ \lambda_w' \end{pmatrix}$,

$$f_0^\beta(z^*, w|\theta) = (2\pi)^{-q} (\det \Sigma)^{-1/2} \int\int |u_{11}|^{q-1} |u_{22}|^{q-1} \exp[-\tfrac{1}{2} \begin{pmatrix} u_{11} \\ u_{22} \end{pmatrix}' V_w \begin{pmatrix} u_{11} \\ u_{22} \end{pmatrix}] du_{11} du_{22}.$$

A.2.7 Derivation of f_0^σ

Let $\tilde{W}^\sigma = g^\sigma(\theta)/U_{22}$, so that $W^\sigma = |\tilde{W}^\sigma|$. Let $\tilde{f}_0^\sigma(z^*, \tilde{w}^\sigma|\theta)$ be the joint density of $(Z^*, \tilde{W}^\sigma)$. Then $f_0^\sigma(z^*, w^\sigma|\theta) = \tilde{f}_0^\sigma(z^*, w^\sigma|\theta) + \tilde{f}_0^\sigma(z^*, -w^\sigma|\theta)$, so it suffices to derive an expression for $\tilde{f}_0^\sigma$.

We have

$$\begin{pmatrix} U_{11} \\ U_{22} \\ U_{12} \end{pmatrix} = \begin{pmatrix} U_{11} \\ g^\sigma(\theta)/\tilde{W}^\sigma \\ U_{12} \end{pmatrix}.$$

This equation, viewed as $\mathbb{R}^3 \mapsto \mathbb{R}^3$ function of $(U_{11}, U_{12}, \tilde{W}^\sigma)$, has Jacobian determinant equal to $-g^\sigma(\theta)/\tilde{W}^{\sigma 2}$. From (A.2), with $u_w^\sigma = \begin{pmatrix} u_{11} & u_{12} \\ 0 & g^\sigma(\theta)/\tilde{w}^\sigma \end{pmatrix}$, the joint density of $(Z^*, \tilde{W}^\sigma)$ can thus be written as

$$(2\pi)^{-q}(\det\Sigma)^{-1/2}|\tilde{w}^\sigma|^{-q}|g^\sigma(\theta)|^{q-1}\int\int |u_{11}|^{q-1}\exp[-\tfrac{1}{2}(\operatorname{vec} z^\dagger u_w^\sigma)'\Sigma^{-1}(\operatorname{vec} z^\dagger u_w^\sigma)]du_{12}du_{11}.$$

Now similar to the derivation of f_1^*,

$$(\operatorname{vec} z^\dagger u_w^\sigma)'\Sigma^{-1}(\operatorname{vec} z^\dagger u_w^\sigma) = \begin{pmatrix} u_{11} \\ g^\sigma(\theta)/\tilde{w}^\sigma \\ u_{12} \end{pmatrix}' \begin{pmatrix} V & v \\ v' & v_0^2 \end{pmatrix} \begin{pmatrix} u_{11} \\ g^\sigma(\theta)/\tilde{w}^\sigma \\ u_{12} \end{pmatrix}$$

and

$$\begin{aligned} &\int \exp[-\tfrac{1}{2}\begin{pmatrix} u_{11} \\ g^\sigma(\theta)/\tilde{w}^\sigma \\ u_{12} \end{pmatrix}' \begin{pmatrix} V & v \\ v' & v_0^2 \end{pmatrix} \begin{pmatrix} u_{11} \\ g^\sigma(\theta)/\tilde{w}^\sigma \\ u_{12} \end{pmatrix}]du_{12} \\ = &\sqrt{2\pi}v_0^{-1}\exp[-\tfrac{1}{2}\begin{pmatrix} u_{11} \\ g^\sigma(\theta)/\tilde{w}^\sigma \end{pmatrix}'(V - vv'/v_0)\begin{pmatrix} u_{11} \\ g^\sigma(\theta)/\tilde{w}^\sigma \end{pmatrix}] \end{aligned}$$

so that

$$\begin{aligned} \tilde{f}_0^\sigma(z^*, \tilde{w}^\sigma|\theta) = &(2\pi)^{-q+1/2}(\det\Sigma)^{-1/2}|g^\beta(\theta)|^{q-1}|\tilde{w}^\sigma|^{-q}v_0^{-1} \\ &\times \int |u_{11}|^{q-1}\exp[-\tfrac{1}{2}\begin{pmatrix} u_{11} \\ g^\sigma(\theta)/\tilde{w}^\sigma \end{pmatrix}' \tilde{V} \begin{pmatrix} u_{11} \\ g^\sigma(\theta)/\tilde{w}^\sigma \end{pmatrix}]du_{11}. \end{aligned}$$

Furthermore, with $\tilde{v}_{ij}^2$ the i,jth element of $\tilde{V}$, $b_w = \tilde{v}_{12} g^\beta(\theta)/(\tilde{w}^\sigma \tilde{v}_{11})$ and $\tilde{a}_w = \tilde{v}_{22}(g^\beta(\theta)/\tilde{w}^\sigma)^2$

$$\begin{aligned}
&\int |u_{11}|^{q-1} \exp[-\tfrac{1}{2} \begin{pmatrix} u_{11} \\ g^\sigma(\theta)/\tilde{w}^\sigma \end{pmatrix}' \tilde{V} \begin{pmatrix} u_{11} \\ g^\sigma(\theta)/\tilde{w}^\sigma \end{pmatrix}] du_{11} \\
&= \int |u_{11}|^{q-1} \exp[-\tfrac{1}{2}(\tilde{v}_{11}^2 u_{11}^2 + 2 b_w \tilde{v}_{11} u_{11} + a_w^2)] du_{11} \\
&= \tilde{v}_{11}^{-q} \int |\omega|^{q-1} \exp[-\tfrac{1}{2}(\omega^2 + 2 b_w \omega + a_w^2)] d\omega \\
&= \tilde{v}_{11}^{-q} \exp[-\tfrac{1}{2}(a_w^2 - b_w^2)] \int |\omega|^{q-1} \exp[-\tfrac{1}{2}(\omega + b_w)^2] d\omega.
\end{aligned}$$

For $q-1$ even, a closed-form expression for the integral follows from Lemma A.1 (a). For $q-1$ odd, note that

$$\int_{-\infty}^{\infty} |\omega|^{q-1} \exp[-\tfrac{1}{2}(\omega+b_w)^2] d\omega = \int_{-\infty}^{\infty} \omega^{q-1} \exp[-\tfrac{1}{2}(\omega+b_w)^2] d\omega - 2 \int_{-\infty}^{0} \omega^{q-1} \exp[-\tfrac{1}{2}(\omega+b_w)^2] d\omega$$

so that a closed-form expression can be deduced from Lemma A.1 (a) and (b).

A.3 Determination of Approximate Least Favorable Distributions

A.3.1 Overview

The algorithm is a modified version of what is suggested in Elliott, Müller, and Watson (2015). Let the set $\Theta_c = \{\theta_1, \ldots, \theta_m\} \subset \Theta$ be a candidate support for the least favorable measure Λ_c, which is fully characterized by the m nonnegative values λ_j it assigns to $\theta_j \in \Theta_c$. Denote by H_{Λ_c} the corresponding confidence set of the form described in Section A.1.[17] We determine λ_j by an iterative procedure, starting with equal mass on all m points, and then adjusting λ_j as a function of $P_{\theta_j}(g(\theta_j) \notin H_{\Lambda_c}(X,Y))$. In each iteration, λ_j is increased if $P_{\theta_j}(g(\theta_j) \notin H_{\Lambda_c}(X,Y)) > \alpha - \epsilon$ and decreased if $P_{\theta_j}(g(\theta_j) \notin H_{\Lambda_c}(X,Y)) < \alpha - \epsilon$, until numerical convergence to the measure Λ_c^*. The parameter $\epsilon > 0$ induces slight overcoverage of $H_{\Lambda_c^*}$ on Θ_c, so that even with an imperfect candidate Θ_c (and numerically determined Λ_c^*),

[17] Here and below, we omit the superscripts ρ, β and σ if the statement applies to all three types of confidence sets.

it is possible that $H_{\Lambda_c^*}$ has coverage uniformly on Θ. This is checked by a numerical search for the maximum of the $\Theta \mapsto [0,1]$ non-coverage function $RP(\theta) = P_\theta(g(\theta) \notin H_{\Lambda_c^*}(X,Y))$. To this end, it is particularly convenient to employ an importance sampling approximation to $RP(\theta)$, which generates a continuously differentiable approximation, so that standard gradient search algorithms can be employed. If these searches (using random starting points) do not yield a maximum above α, a nearly (up to the paramter $\epsilon > 0$) optimal least favorable measure Λ_c^* has been determined. If the searches yield a θ_0 for which $RP(\theta_0) > \alpha$, then this θ_0 is added to the candidate set Θ_c, and the algorithm iterates.

For the confidence set H^ρ, we seek a family of measures Λ_r^ρ that, for each $r \in (-1,1)$, have support on the subspace of $\Theta_r = \{\theta : g^\rho(\theta) = r\}$. We discretize this problem into a finite number of values of r. For each given r, we apply the above algorithm, except that the non-coverage function $RP(\theta)$ now only needs to be searched over Θ_r.

We discuss details in the following subsections.

A.3.2 Parameterization

Since the algorithm involves optimization over Θ (or Θ_r), it is convenient to introduce a reparameterization so that this search can be conducted in a unit hypercube. The (A,B,c,d) model is described by 11 parameters. The restriction to invariant sets reduces the number of effective parameters to $11-3=8$ for H^β and H^σ, and the combination of the bivariate scale invariance and the restriction $\Theta_r = \{\theta : g^\rho(\theta) = r\}$ also makes Θ_r effectively 8 dimensional. The effective parameter space can hence be covered by a $[0,1]^8 \mapsto \Theta$ function. In particular, given $\eta = (\eta_1, \ldots, \eta_8) \in [0,1]^8$, we set

$$\begin{aligned}
c_i &= 2(200)^{2\eta_i - 1},\ d_i = -0.4 + 1.4\eta_{2+i} \\
r_\eta &= (2\eta_5 - 1)\min(\sqrt{\eta_6\eta_7}, \sqrt{(1-\eta_6)(1-\eta_7)}),\ \phi_\eta = \pi\eta_8 \\
B &= R\,\mathrm{chol}\begin{pmatrix} \eta_6 & r_\eta \\ r_\eta & \eta_7 \end{pmatrix},\ A = R\,\mathrm{chol}(I_2 - BB')O(\phi_\eta)S_{c,d}
\end{aligned}$$

where $\mathrm{chol}(\cdot)$ is the Choleski decomposition of a matrix, $O(\phi_\eta)$ is the 2×2 rotation matrix for the angle ϕ_η, and $S_{c,d} = \mathrm{diag}(\sqrt{q/\operatorname{tr}\Sigma_X(c_1,d_1)}, \sqrt{q/\operatorname{tr}\Sigma_X(c_2,d_2)})$, with $\Sigma_X(c_0,d_0)$ the $q \times q$ covariance matrix of X in the (A,B,c,d) model when $A = I_2$, $B = 0$, $c_1 = c_0$ and $d_1 = d_0$ (so $\Sigma_X(c_0,d_0)$ is the covariance matrix in the scalar c,d model employed in Müller and Watson (2016) without additional white noise). For H^β and H^σ, we set $R = I_2$. For

H^ρ, we enforce $\theta \in \Theta_r$ by setting $R = \text{chol}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$. The lower and upper bounds for c_1 and c_2 of 0.01 and 400 are such that the distribution of (X, Y) from the resulting $\Sigma_X(c_j, d_0)$s is nearly indistinguishable from the distribution under the limits $c_j \to 0$ and $c_j \to \infty$.

The rationale of this parameterization is that under the equivariance governing H^β and H^σ, it is without loss of generality to consider the case where $\Omega(\theta) = qI_2$. Now both $A = O(\phi_\eta)S_{c,d}$ and $B = 0$, as well as $A = 0$ and $B = I_2$, induce $\Omega(\theta) = qI_2$ with $(2\pi)^{-1}$ as the factor of proportionality for the local-to-zero spectrum $S_z(\omega)$ given in the text. The parameterization of BB' in terms of (η_5, η_6, η_7) exhaustively describes all decompositions of $I_2 = BB' + (I_2 - BB')$ into two positive semidefinite matrices BB' and $(I_2 - BB')$. Under the bivariate scale invariance governing H^ρ, it is without loss of generality to consider the case where $\Omega(\theta) = \begin{pmatrix} 1 & g^\rho(\theta) \\ g^\rho(\theta) & 1 \end{pmatrix}$, and on Θ_r, $g^\rho(\theta) = r$.

A.3.3 Computation of $\Sigma(\theta)$

Gradient methods require fast evaluation of the likelihood for generic θ, which depends on $\Sigma(\theta)$. We initially compute and store the $q \times q$ matrices $\Sigma_X(c_0, d_0)$ introduced in the last subsection for all combinations of the values $c_0 \in \{2(200)^{2i/50-1}\}_{i=0}^{50}$ and $d_0 \in \{-0.4 + 1.4i/40\}_{i=0}^{40}$ using the algorithm developed in Müller and Watson (2016). For a general θ, we then compute $\Sigma_X(c_1, d_1)$ and $\Sigma_X(c_2, d_2)$ by two-dimensional quadratic interpolation of the matrix elements, and construct $\Sigma(\theta)$ via

$$\Sigma(\theta) = (A \otimes I_q)\begin{pmatrix} \Sigma_X(c_1, d_1) & 0 \\ 0 & \Sigma_X(c_2, d_2) \end{pmatrix}(A \otimes I_q)' + (BB' \otimes I_q).$$

A.3.4 Importance Sampling

For H^ρ we employ the importance sampling approximation

$$P_\theta(g^\rho(\theta) \notin H^\rho_{\Lambda_c^*}(X, Y)) \approx N^{-1}\sum_{i=1}^{N} \frac{f^s(X^s_{(i)}, Y^s_{(i)}|\theta)}{f^s_p(X^s_{(i)}, Y^s_{(i)})}\mathbf{1}[g^\rho(\theta) \notin H^\rho_{\Lambda_c^*}(X^s_{(i)}, Y^s_{(i)})] \qquad \text{(A.3)}$$

for some proposal density f^s_p, where $(X^s_{(i)}, Y^s_{(i)})$ are i.i.d. draws from f^s_p. For given r, this obviously induces a smooth approximating function on Θ_r, since for all $\theta \in \Theta_r$, $g(\theta) = r$, so that the indicator function does not vary with θ. In fact, for given $H^\rho_{\Lambda_c^*}$, it suffices to compute the sum over those i where $r \notin H^\rho_{\Lambda_c^*}(X^s_{(i)}, Y^s_{(i)})$, no matter the value of $\theta \in \Theta_r$.

For H^β, note that by equivariance, the event $g^\beta(\theta) \in H^\beta_{\Lambda_c^*}(X,Y)$ is equivalent to $W^\beta = (U_{11}g^\beta(\theta) - U_{12})/U_{22} \in H^\beta_{\Lambda_c^*}(X^\dagger, Y^\dagger)$, where $X^\dagger = (1, X_0^*, X^{*\prime})'$ and $Y^\dagger = (1, 0, Y^{*\prime})'$. Thus, given that $(X^\dagger, Y^\dagger)$ are functions of Z^*, we have

$$P_\theta(g^\beta(\theta) \notin H^\beta_{\Lambda_c^*}(X,Y)) \approx N^{-1}\sum_{i=1}^N \frac{f_0^\beta(Z^*_{(i)}, W^\beta_{(i)}|\theta)}{f_p^\beta(Z^*_{(i)}, W^\beta_{(i)})}\mathbf{1}[W^\beta_{(i)} \notin H^\beta_{\Lambda_c^*}(X^\dagger_{(i)}, Y^\dagger_{(i)})] \tag{A.4}$$

for some proposal density f_p^β, where $(Z^*_{(i)}, W^\beta_{(i)})$ are i.i.d. draws from f_p^β. Analogously, for H^σ,

$$P_\theta(g^\sigma(\theta) \notin H^\sigma_{\Lambda_c^*}(X,Y)) \approx N^{-1}\sum_{i=1}^N \frac{f_0^\sigma(Z^*_{(i)}, W^\sigma_{(i)}|\theta)}{f_p^\sigma(Z^*_{(i)}, W^\sigma_{(i)})}\mathbf{1}[W^\sigma_{(i)} \notin H^\sigma_{\Lambda_c^*}(X^\dagger_{(i)}, Y^\dagger_{(i)})] \tag{A.5}$$

with $W^\sigma = g^\sigma(\beta)/|U_{22}|$. These approximation functions are again continuously differentiable in θ, and for given $H^j_{\Lambda_c^*}$, it suffices to perform the summation over those i where $W^j_{(i)} \notin H^j_{\Lambda_c^*}(X^\dagger_{(i)}, Y^\dagger_{(i)})$, $j \in \{\beta, \sigma\}$.

For the importance sampling approximations to work well, it is crucial that the proposal density f_p is never much smaller than f_0 for all $\theta \in \Theta$ (or never much smaller than f^s over Θ_r in the case of H^ρ). Otherwise, a single large weight $f_0(Z^*, W|\theta)/f_p(Z^*, W)$ may dominate the sums in (A.3), (A.4) and (A.5), inducing imprecise approximations. It is not *a priori* obvious how to construct such a proposal, though, since the densities depend on the fairly high dimensional θ in a complicated way.

To overcome this difficulty, we numerically construct a mixture proposal f_p such that $f_0(z^*, w|\theta)/f_p(z^*, w) \leq \text{IS}_{\max}$ uniformly in θ and (z^*, w). For simplicity, we describe the approach only in the notation that is relevant for H^β and H^σ:

1. Randomly choose θ_p^1 and set $M = 1$.

2. Using a gradient search algorithm, numerically solve

$$\max_{(z^*, w), \theta} \frac{f_0^j(z^*, w|\theta)}{\sum_{i=1}^M f_0^j(z^*, w|\theta_p^i)}$$

   using up to 250 BFGS searches with randomly chosen starting points.

   (a) If the numerically obtained maximum is no larger than $\text{IS}_{\max}$, then set $f_p(z^*, w) = M^{-1}\sum_{i=1}^M f_0^j(z^*, w|\theta_p^i)$ and conclude.

(b) Otherwise, set θ_p^{M+1} equal to the maximizing value of θ, increase M by one, and iterate step 2.

We set $\text{IS}_{\max}$ equal to 2000, 6000 and 12000 for $q \leq 12$, $12 < q \leq 20$ and $q > 20$, respectively.

A.3.5 Computation of Credible Sets and Integrals over F

We approximate integrals over F by a discrete sum over 1000 points θ_j^F, where jointly uniformly distributed random variables are approximated by a low-discrepancy sequence. To ensure that (X, Y) and $(X, -Y)$ have the exact same distribution under our approximation of $\int f(X, Y|\theta)dF(\theta)$, the 1000 points are split into 500 corresponding pairs.

Note that it is not necessary to compute the credible sets H_0 for each realization of $(X_{(i)}^s, Y_{(i)}^s)$ or $Z_{(i)}^*$. Rather, it suffices to determine whether $r \in H_0^\rho(X_{(i)}^s, Y_{(i)}^s)$ or $W_{(i)} \in H_0(X_{(i)}^\dagger, Y_{(i)}^\dagger)$, respectively. Under the discrete approximation to F, it hence suffices to check whether or not

$$\frac{\sum_{j=1}^{1000} \mathbf{1}[g^\rho(\theta_j^F) \leq r] f^s(X_{(i)}^s, Y_{(i)}^s|\theta_j^F)}{\sum_{j=1}^{1000} f^s(X_{(i)}^s, Y_{(i)}^s|\theta_j^F)} \tag{A.6}$$

and, for $j \in \{\beta, \sigma\}$,

$$\frac{\int \sum_{j=1}^{1000} \mathbf{1}[w \leq W_{(i)}^j] f_0(Z_{(i)}^*, w|\theta_j^F)dw}{\sum_{j=1}^{1000} f_1^*(Z_{(i)}^*|\theta_j^F)} \tag{A.7}$$

take on values in the interval $[\alpha/2, 1-\alpha/2]$, respectively. We compute the integral in (A.7) by numerical quadrature.

Since all three type of confidence sets always contain H_0, the realizations of $(X_{(i)}^s, Y_{(i)}^s)$ and $(Z_{(i)}^*, W_{(i)}^j)$ for which (A.6) and (A.7) take on values between $[\alpha/2, 1-\alpha/2]$ never enter the sums (A.3), (A.4) and (A.5) that approximate the non-rejection probabilities. The effective number of terms in the sums is thus greatly reduced, which correspondingly facilitates computations. With this in mind, we modify the determination of the importance sampling proposal by maximizing the (empirical analogue of the) variance of the importance sampling weights conditional on the event $g(\theta) \notin H_0$.

A.3.6 Approximate Least Favorable Distributions and Size Control

The initial candidate set Θ_c consists of 10 randomly selected points in Θ (or in Θ_r in the case of H^r). For given Θ_c, Λ_c^* is computed by the algorithm described in Elliott, Müller, and Watson (2015), using a target value the level of $1 - \alpha + \epsilon$. We set ϵ to 0.3%, 0.6% and 1.0% for $\alpha = 5\%$, 10% and 33% for $q \leq 12$, respectively, double these values for $12 < q \leq 20$, and triple them for $q > 20$. We search for coverage violating points by BFGS maximizations over the importance sampling approximation to the non-coverage probability function $RP(\theta)$, using numerical derivatives and random starting values. We collect up to 5 coverage violating points in up to 50 BFGS searches before augmenting Θ_c and recomputing Λ_c^*, which is fairly time consuming, especially if Θ_c consists of many points. The algorithm stops once 500 consecutive BFGS searches do not yield a coverage violating point.

A.3.7 Quality of Approximation and Time to Compute

With $N = 250,000$ importance sampling draws and the baseline case of $q = 12$, the Monte Carlo standard errors of non-coverage probabilities are approximately 0.1%-0.25% at the 5% level, 0.1%-0.35% at the 10% level, and 0.3%-0.5% at the 33% level. Using results in Elliott, Müller, and Watson (2015) and Müller and Watson (2016), it is straightforward to use the approximately least favorable distributions to obtain lower bounds on the F-weighted average expected length of any confidence set of nominal level. We find that our sets are within approximately 1-3% of this lower bound, so they come reasonably close to being as short as possible under that criterion.

For $q = 12$, a specific level α and problem, the determination of the approximately least favorable measure Λ^* takes approximately 5 minutes using a Fortran implementation on a dual 10-core PC, and yields an approximate least favorable measure Λ^* with approximately 30-100 points of support. Running times are roughly quadratic in q due the $4q^2$ elements in the quadratic forms of the likelihoods. Larger q also lead to bigger Monte Carlo standard errors of rejection probabilities, as the importance sampling now must cover an effectively larger set of distributions of (X^s, Y^s) and (Z^*, W), respectively.

A.4 Data Used

The data and sources are listed in Table A.1.

A.5 Long-run projections and low-pass filters

Figure A.1 plots growth rates of GDP and consumption growth rates along with low-pass moving averages designed to isolate variation in the series with periods longer than 11 years and the long-run projections using $q = 12$ shown in Figure 1 of the paper. The low-pass moving averages were computed using an ideal low-pass filter for periods longer than $T/6$ ($\approx$ 11 years) truncated after $T/2$ terms. The series were padded with pre- and post-sample backcasts and forecasts constructed from an AR(4) model. The long-run projection were computed as the projections of GDP growth rates onto Ψ_T with $q = 12$, including constant term.

A.6 Long-run covariances using a subset of the columns of Ψ_T

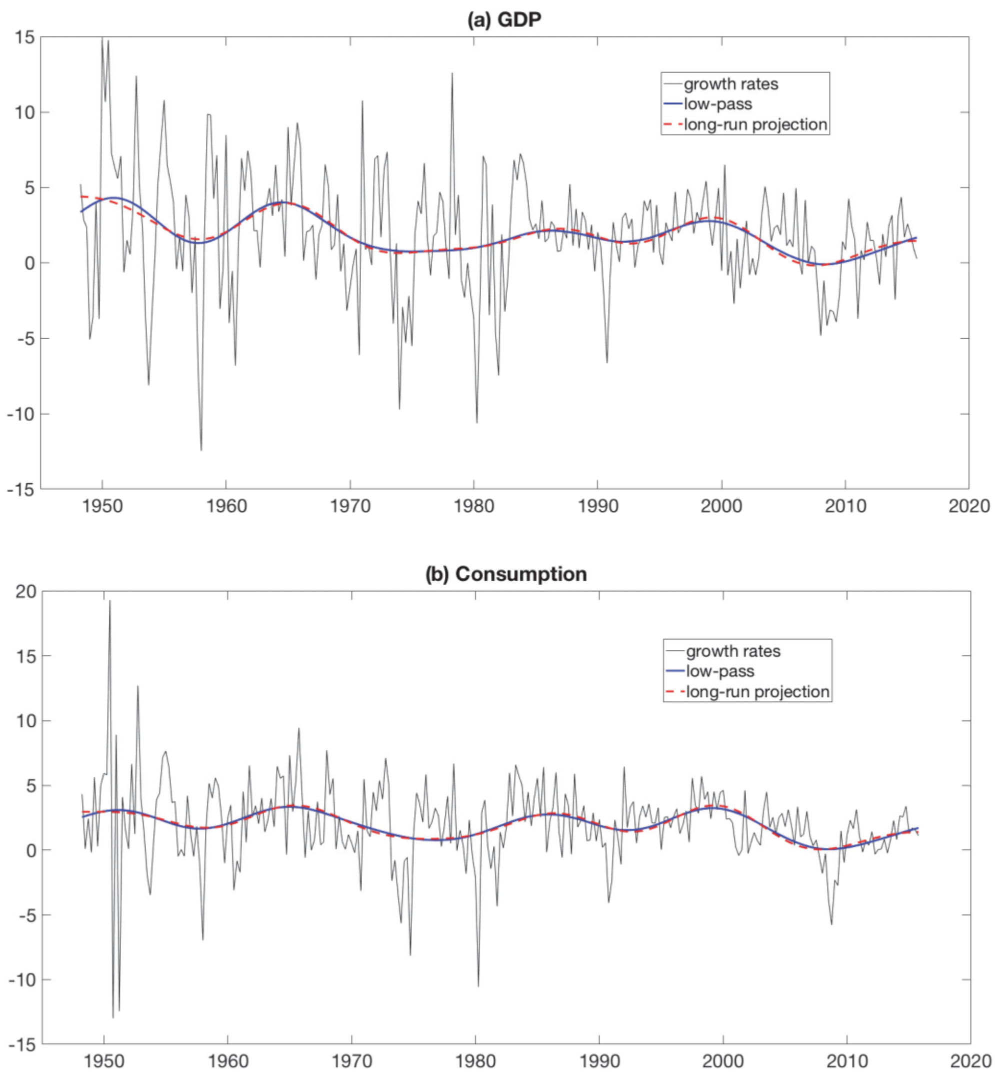
The empirical results in the body of the paper rely on covariance measures associated with projections of the data onto q cosine functions capturing periodicities of between $2T$ and $2T/q$, where T is the length of the sample. Using data from 1948-2015 ($T = 68$ years) this analysis used periods longer than 11 years to define "long-run" variation and covariation, so $q = 12$. While 11 years is longer than typical business cycles, it does incorporates periods corresponding to what some researchers refer to as the "medium run" (Blanchard (1997), Comin and Gertler (2006)). In this appendix we consider measures of long-run covariability that focus on a subset of the q periods. This allows a comparison of, say, results from periods corresponding to the "medium-long run" and to those from the "longer-long run."

To motivate the new measures, look at Figure 1 in the text which plots the projections of GDP and consumption growth rates onto $q = 12$ cosine regressors with periods that range from $T/6$ ($\approx$ 11 years) to $2T$ (136 years). Figure A.2 show the corresponding projections onto the first $q_1 = 6$ of these cosine terms (with periods from $T/3 \approx 23$ years to $2T = 136$ years) and last $q_2 = 6$ cosine terms (with periods $T/12 \approx 11$ years to $2T/7 \approx 19$ years). The

Table A.1: Data series used

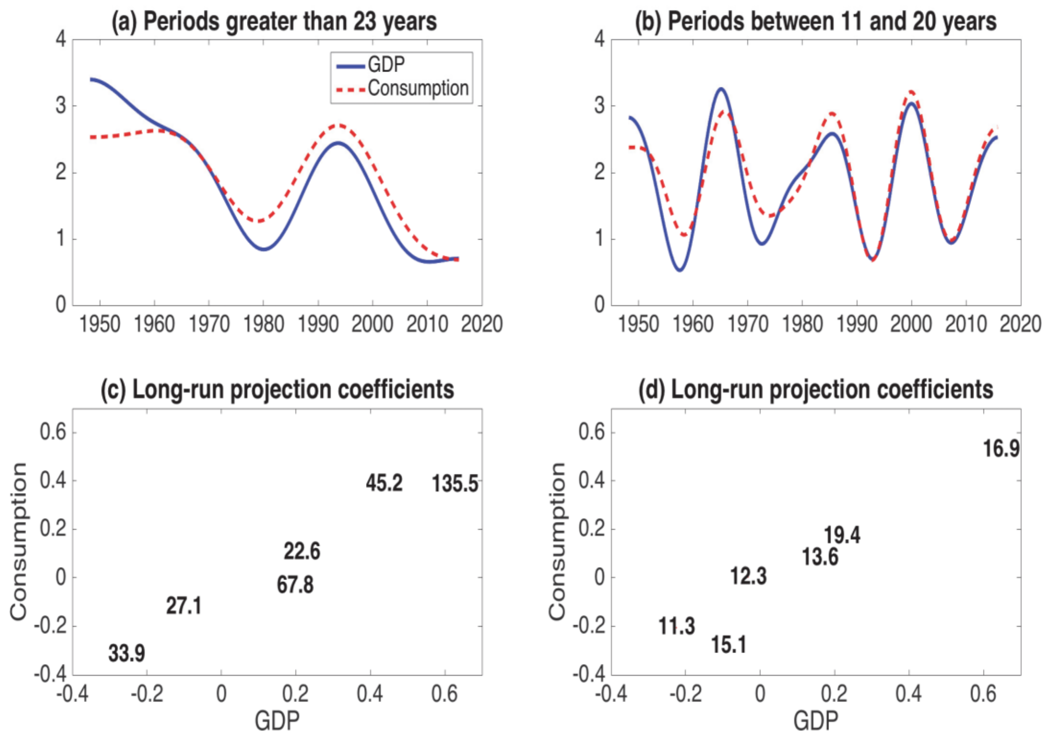
Series	Sources and Notes (FRED Codes)
GDP, consumption, investment, and employee compensation	NIPA nominal values (GDP, PCDG, PCND, PCESV, GDPI, PNFI, PRFI, Y033RC1Q027SBEA, COE) deflated by the price index for personal consumer expenditures (PCECTPI). The variables are expressed in per-capita terms using the $q = 12$ low-frequency projection of civilian non-institutionalized population (CNP16OV).
TFP	Growth rate for TFP from Fernald (2014), updated from his webpage.
Interest rates	3-Month Treasury bill rate (TB3MS) and 10-Year Treasury bond rate (GS10)
Inflation	Inflation from the personal consumption deflator (PCECTPI) and consumer price index (CPIAUCSL)
Money supply	M1 money supply (M1) from FRB beginning in 1959:M1. This is linked to M1 (currency + demand deposits) from Friedman and Schwartz (1963, Table A-1, Col. 7)
Unemployment rate	Bureau of Labor Statistics (UNRATE)
Stock returns	CRSP Nominal Monthly Returns are from WRDS. Monthly real returns were computed by subtracting the change in the logarithm in the CPI from the nominal returns, which were then compounded to yield quarterly returns. Values are 400×the logarithm of gross quarterly real returns.
Stock prices, dividends, and earnings	S&P composite prices, dividends, and earnings from Robert Shiller's webpage (file IE.XLS).
Exchange rates and relative CPIs	Nominal exchange rate (EXUSUK) from the FRB, CPI for the UK from the Bank of England (CPIUKQ) and U.S. CPI (CPIAUCSL) from the BLS.

Figure A.1: Long-run average growth rates of GDP and consumption



Notes: The first show growth rates, low-pass moving averages (periods 11 years and greater) and projections onto q =12 cosine terms.

Figure A2: Long-run projections for GDP and consumption growth rates for different periodicities



Notes: Panel (a) plots the projections of the data onto six cosine terms with periods 23-136 years. Panel (b) shows the projections onto six low-frequency terms with periods 11-19 years. Sample means have been added to both sets of projections. Panels (c) and (d) are scatterplots of the coefficients (cosine transforms) from panels (a) and (b) where the plot symbols are the periods (in years) of the associated cosine function.

first of these captures the longer-long-run variation in the data, and the second captures the medium-long-run variability. Each can be studied separately. To differentiate these periodicities, we replace equation (4) with

$$\Omega_{q_1:q_2,T} = T^{-1}\sum_{t=1}^{T} E\left[\begin{pmatrix} \widehat{x}_{q_1:q_2,t} \\ \widehat{y}_{q_1:q_2,t} \end{pmatrix}\begin{pmatrix} \widehat{x}_{q_1:q_2,t} & \widehat{y}_{q_1:q_2,t} \end{pmatrix}\right] = E\begin{bmatrix} X'_{q_1:q_2,T}X_{q_1:q_2,T} & X'_{q_1:q_2,T}Y_{q_1:q_2,T} \\ Y'_{q_1:q_2,T}X_{q_1:q_2,T} & Y'_{q_1:q_2,T}Y_{q_1:q_2,T} \end{bmatrix} \tag{A.8}$$

where the subscript "$q_1 : q_2$" notes that the projection is computed using the q_1 through q_2 cosine terms (i.e., the q_1 through q_2 columns of Ψ_T) corresponding to periods $2T/q_2$ through $2T/q_1$. Thus the longer-long-run periodicities shown in Figure A.2.a correspond to the covariance matrix $\Omega_{1:6,T}$ (the first 6 cosine terms) and the medium-long-run periodicities in Figure A.2.b correspond to $\Omega_{7:12,T}$ (the 7-12th cosine terms).

Throughout the paper we have used q to denote the number of low-frequency cosine terms that define the long-run periods of interest (perhaps divided further into longer-long and medium-long). But q plays another important role in the analysis. The value of Ω (or now $\Omega_{q_1:q_2}$) ultimately depends on the variability and persistence in the stochastic process as exhibited in the local-to-zero (pseudo-) spectrum S_z. This spectrum is parameterized by (A, B, c, d); see equation (11). We learn about the value of these parameters (and therefore the value of Ω) using the data $(X_{1:q,T}, Y_{1:q,T})$. Thus, q also denotes the sample variability in the data that is used to infer the value of the long-run covariance matrix Ω. So, while our interest might lie in the longer-long-run covariability captured in $\Omega_{1:6}$, the sample variability in $(X_{1:12,T}, Y_{1:12,T})$ might be used to learn about $\Omega_{1:6}$. While it is arguably most natural to match the variability in the data used for inference to the variability of interest, for example using $(X_{1:q,T}, Y_{1:q,T})$ to learn about $\Omega_{1:q}$, if the (A, B, c, d) model accurately characterizes the spectrum over a wider frequency band, then variability over this wider band can improve inference. But of course using a wider frequency band runs the risk of misspecification if the (A, B, c, d) model is a poor characterization of the spectrum over this wider range of frequencies. This is the standard trade-off of robustness and efficiency.

With these ideas in mind, Table A.2 shows results for long-run correlation and regression parameters from $\Omega_{1:12}$, $\Omega_{1:6}$, and $\Omega_{7:12}$, corresponding the periods $T/6$ and higher, $T/3$ and higher, and $T/6$ through $2T/7$. Results are shown using inference based on the same $q = 12$ cosine transforms used in the body of the paper, but also using $q = 6$, so only lower frequency variability in the data is used to learn about (A, B, c, d), and with $q = 18$, so higher frequency variability is also used. (For simplicity, we use these values of q for all series, even though the sample period is shorter for long-term interest rates and exchange rates.)

The first block of results in Table A.2 are for consumption and GDP. The first row repeats earlier results using the $q = 12$ cosine terms to learn about $\Omega_{q_1:q_2}$ with $q_1 = 1$ and $q_2 = 12$. The other rows are for other values of q, q_1, and q_2. The results suggest remarkable stability across the different values of q, q_1, and q_2. Figure A.2.c and A.2.d provides hints at this stability. It shows the scatter plot of $(X_{1:6,T}, Y_{1:6,T})$ and $(X_{7:12,T}, Y_{7:12,T})$ corresponding to the projections plotted in panels (a) and (b). The scatter plots corresponding to the different periodicities are quite similar, and this is reflected in the stability of the results shown in Table A.2. This same stability across q, q_1, and q_2 is evident for many of the other pairs of variables. Looking closely at Table A.2, there are subtle differences in the rows. For example, the confidence intervals for the parameters from $\Omega_{1:12}$ tend to be somewhat narrower using $q = 18$ than using $q = 12$, consistent with a modest amount of additional information using a larger value of q. The same result holds for results for $\Omega_{1:6}$ computed using $q = 6$ and $q = 12$.

Table A.2: Long-run covariation measures for selected variables with $\Omega_{q_1:q_2}$
Inference based on q cosine transforms.

Y	X	q	q_1:q_2	$\rho_{q_1:q_2}$			$\beta_{q_1:q_2}$			$\hat{\sigma}_{q_1:q_2,y\|x}$
				$\hat{\rho}$	67% CI	90% CI	$\hat{\beta}$	67% CI	90% CI	
Cons.	GDP	12	1:12	0.91	0.83, 0.96	0.71, 0.97	0.77	0.66, 0.87	0.48, 0.95	0.41
		12	1:6	0.92	0.82, 0.96	0.60, 0.98	0.74	0.57, 0.87	0.36, 0.97	0.31
		12	7:12	0.92	0.84, 0.96	0.74, 0.98	0.80	0.69, 0.92	0.60, 1.04	0.23
		18	1:12	0.91	0.84, 0.95	0.76, 0.97	0.75	0.66, 0.84	0.52, 0.91	0.40
		6	1:6	0.89	0.71, 0.95	0.50, 0.98	0.73	0.57, 0.89	0.36, 1.08	0.31
Inv.	GDP	12	1:12	0.53	0.29, 0.72	0.02, 0.81	1.24	0.64, 1.79	0.21, 2.21	2.18
		12	1:6	0.54	0.27, 0.76	-0.02, 0.91	1.18	0.58, 1.79	-0.46, 2.27	1.54
		12	7:12	0.55	0.25, 0.73	0.04, 0.82	1.36	0.70, 2.03	-0.15, 2.88	1.36
		18	1:12	0.63	0.42, 0.75	0.29, 0.81	1.60	1.05, 2.16	0.61, 2.59	2.39
		6	1:6	0.56	0.21, 0.79	-0.04, 0.91	1.07	0.52, 1.67	-0.14, 2.28	1.13
$w \times n$	GDP	12	1:12	0.98	0.96, 0.99	0.94, 0.99	1.28	1.20, 1.36	1.14, 1.42	0.31
		12	1:6	0.98	0.96, 0.99	0.93, 0.99	1.28	1.19, 1.36	1.12, 1.42	0.21
		12	7:12	0.97	0.95, 0.98	0.89, 0.99	1.29	1.20, 1.39	1.08, 1.54	0.22
		18	1:12	0.95	0.91, 0.97	0.87, 0.98	1.21	1.12, 1.31	1.04, 1.38	0.46
		6	1:6	0.97	0.92, 0.98	0.83, 0.99	1.26	1.14, 1.38	1.04, 1.48	0.23
GDP	TFP	12	1:12	0.78	0.64, 0.89	0.45, 0.95	1.22	0.92, 1.48	0.72, 1.72	0.74
		12	1:6	0.79	0.63, 0.94	0.44, 0.97	1.22	0.92, 1.54	0.69, 1.78	0.54
		12	7:12	0.75	0.58, 0.86	0.38, 0.91	1.16	0.84, 1.48	0.60, 1.72	0.44
		18	1:12	0.76	0.67, 0.87	0.50, 0.94	1.21	0.97, 1.48	0.76, 1.69	0.75
		6	1:6	0.71	0.41, 0.94	0.08, 0.97	1.14	0.71, 1.53	0.32, 2.00	0.57

Table A.2: continued

Y	X	q	q_1:q_2	$\rho_{q_1:q_2}$			$\beta_{q_1:q_2}$			$\hat{\sigma}_{q_1:q_2,y\|x}$
				$\hat{\rho}$	67% CI	90% CI	$\hat{\beta}$	67% CI	90% CI	
Cons. (Nondur)	GDP	12	1:12	0.41	0.08, 0.61	-0.09, 0.72	0.35	0.13, 0.57	-0.07, 0.76	0.88
		12	1:6	0.42	0.07, 0.62	-0.54, 0.86	0.32	0.08, 0.57	-0.41, 0.76	0.64
		12	7:12	0.42	0.08, 0.65	-0.14, 0.73	0.40	0.13, 0.64	-0.07, 0.96	0.56
		18	1:12	0.41	0.14, 0.56	-0.20, 0.76	0.33	0.15, 0.51	-0.17, 0.65	0.83
		6	1:6	0.32	-0.03, 0.60	-0.30, 0.86	0.37	-0.03, 0.74	-0.37, 1.11	0.76
Cons. (Serv.)	GDP	12	1:12	0.84	0.71, 0.90	0.57, 0.96	0.83	0.66, 0.99	0.54, 1.24	0.60
		12	1:6	0.87	0.74, 0.95	0.58, 0.98	0.88	0.70, 1.18	0.54, 1.45	0.47
		12	7:12	0.82	0.67, 0.88	0.45, 0.94	0.70	0.51, 0.87	0.35, 0.99	0.32
		18	1:12	0.84	0.72, 0.92	0.63, 0.96	0.81	0.66, 0.97	0.54, 1.25	0.62
		6	1:6	0.83	0.60, 0.95	0.34, 0.98	0.93	0.66, 1.22	0.42, 1.51	0.47
Cons. (Dur.)	GDP	12	1:12	0.78	0.68, 0.90	0.53, 0.92	1.86	1.46, 2.25	1.15, 2.56	1.52
		12	1:6	0.82	0.67, 0.93	0.48, 0.96	1.72	1.33, 2.12	0.93, 2.47	1.05
		12	7:12	0.77	0.67, 0.89	0.56, 0.94	2.12	1.63, 2.78	1.28, 3.36	1.02
		18	1:12	0.75	0.63, 0.83	0.52, 0.91	1.80	1.41, 2.20	1.11, 2.50	1.82
		6	1:6	0.87	0.70, 0.94	0.47, 0.98	1.45	1.13, 1.84	0.83, 2.23	0.67
Inv. (Nonres.)	GDP	12	1:12	0.42	0.13, 0.65	-0.05, 0.75	0.97	0.41, 1.46	-0.09, 1.90	2.18
		12	1:6	0.47	0.15, 0.72	-0.09, 0.89	0.97	0.41, 1.59	-0.09, 2.02	1.51
		12	7:12	0.38	0.07, 0.62	-0.20, 0.73	0.90	0.22, 1.52	-0.59, 2.14	1.47
		18	1:12	0.55	0.38, 0.73	0.14, 0.78	1.32	0.81, 1.78	0.36, 2.18	2.23
		6	1:6	0.53	0.20, 0.79	-0.05, 0.94	0.91	0.43, 1.40	-0.16, 1.94	1.00

Table A.2: continued

Y	X	q	q_1:q_2	$\rho_{q_1:q_2}$			$\beta_{q_1:q_2}$			$\hat{\sigma}_{q_1:q_2,y\|x}$
				$\hat{\rho}$	67% CI	90% CI	$\hat{\beta}$	67% CI	90% CI	
Inv. (Res.)	GDP	12	1:12	0.40	0.10, 0.64	-0.10, 0.72	2.15	0.77, 3.53	-0.27, 4.56	5.63
		12	1:6	0.44	0.08, 0.63	-0.16, 0.86	1.97	0.60, 3.35	-0.61, 4.56	3.67
		12	7:12	0.40	0.07, 0.65	-0.12, 0.72	2.32	0.77, 4.56	-1.30, 6.80	4.08
		18	1:12	0.46	0.29, 0.63	0.07, 0.74	2.49	1.27, 3.86	0.35, 4.78	5.68
		6	1:6	0.45	0.09, 0.72	-0.13, 0.84	1.61	0.54, 2.78	-0.31, 4.06	2.18
Inv. (Equip.)	GDP	12	1:12	0.33	0.00, 0.54	-0.20, 0.70	0.81	0.12, 1.57	-0.41, 2.11	2.75
		12	1:6	0.36	-0.00, 0.58	-0.38, 0.83	0.89	0.05, 1.65	-0.64, 2.18	1.99
		12	7:12	0.29	-0.00, 0.56	-0.20, 0.70	0.89	0.05, 1.73	-0.87, 2.56	1.76
		18	1:12	0.50	0.29, 0.69	0.07, 0.75	1.33	0.61, 2.04	-0.10, 2.53	2.88
		6	1:6	0.32	-0.03, 0.64	-0.46, 0.77	0.89	-0.07, 1.94	-1.21, 2.99	1.95
10Y nom. rates	3M nom.rates	12	1:12	0.96	0.92, 0.98	0.89, 0.98	0.92	0.83, 1.06	0.75, 1.15	0.70
		12	1:6	0.96	0.91, 0.98	0.88, 0.98	0.93	0.83, 1.09	0.75, 1.18	0.61
		12	7:12	0.94	0.88, 0.97	0.76, 0.98	0.87	0.67, 0.97	0.60, 1.05	0.28
		18	1:12	0.95	0.91, 0.98	0.83, 0.98	0.87	0.75, 1.02	0.67, 1.12	0.80
		6	1:6	0.96	0.92, 0.98	0.83, 0.99	0.97	0.86, 1.09	0.75, 1.20	0.61
10Y real rates	3M real rates	12	1:12	0.95	0.89, 0.96	0.80, 0.98	0.98	0.86, 1.09	0.71, 1.26	0.68
		12	1:6	0.95	0.87, 0.97	0.72, 0.98	0.98	0.85, 1.17	0.65, 1.34	0.57
		12	7:12	0.95	0.88, 0.97	0.83, 0.98	0.97	0.85, 1.08	0.76, 1.17	0.31
		18	1:12	0.89	0.79, 0.94	0.71, 0.98	0.91	0.77, 1.06	0.66, 1.24	0.83
		6	1:6	0.94	0.86, 0.97	0.70, 0.98	1.02	0.87, 1.17	0.64, 1.42	0.61

Table A.2: continued

Y	X	q	q_1:q_2	$\rho_{q_1:q_2}$			$\beta_{q_1:q_2}$			$\hat{\sigma}_{q_1:q_2,y\|x}$
				$\hat{\rho}$	67% CI	90% CI	$\hat{\beta}$	67% CI	90% CI	
CPI Infl.	PCE Infl.	12	1:12	0.98	0.96, 0.99	0.95, 0.99	1.13	1.07, 1.19	0.98, 1.24	0.36
		12	1:6	0.98	0.95, 0.99	0.94, 0.99	1.12	1.06, 1.18	0.96, 1.24	0.29
		12	7:12	0.98	0.95, 0.99	0.92, 0.99	1.17	1.10, 1.33	1.04, 1.41	0.17
		18	1:12	0.97	0.95, 0.98	0.94, 0.99	1.14	1.08, 1.20	0.99, 1.26	0.37
		6	1:6	0.98	0.96, 0.99	0.93, 0.99	1.11	1.04, 1.17	0.97, 1.22	0.28
3M rates	PCE Infl.	12	1:12	0.47	0.21, 0.83	-0.00, 0.91	0.73	0.34, 1.49	-0.09, 1.91	2.20
		12	1:6	0.52	0.24, 0.89	-0.02, 0.95	0.84	0.38, 1.59	0.05, 2.02	1.92
		12	7:12	0.23	-0.05, 0.56	-0.38, 0.70	0.36	-0.14, 0.72	-0.64, 1.07	0.88
		18	1:12	0.54	0.28, 0.83	0.00, 0.91	0.82	0.47, 1.47	0.03, 1.91	2.20
		6	1:6	0.65	0.30, 0.89	0.00, 0.96	1.02	0.55, 1.59	0.13, 2.20	1.98
10Y rates	PCE Infl.	12	1:12	0.47	0.23, 0.83	-0.00, 0.91	0.66	0.30, 1.39	-0.06, 1.73	2.12
		12	1:6	0.52	0.24, 0.89	-0.02, 0.94	0.73	0.33, 1.43	0.03, 1.86	1.89
		12	7:12	0.23	0.03, 0.58	-0.29, 0.71	0.33	0.00, 0.70	-0.36, 0.96	0.79
		18	1:12	0.49	0.22, 0.83	0.03, 0.91	0.64	0.29, 1.28	0.08, 1.63	2.03
		6	1:6	0.57	0.23, 0.89	-0.03, 0.95	0.88	0.47, 1.43	0.01, 2.02	2.02
CPI Infl.	Mon. Supply	12	1:12	0.12	-0.17, 0.54	-0.60, 0.76	0.11	-0.17, 0.52	-0.58, 0.90	2.45
		12	1:6	0.16	-0.15, 0.54	-0.60, 0.83	0.13	-0.17, 0.52	-0.63, 1.04	2.15
		12	7:12	0.01	-0.35, 0.31	-0.66, 0.47	-0.00	-0.33, 0.30	-0.77, 0.54	0.96
		18	1:12	0.13	-0.13, 0.42	-0.36, 0.76	0.11	-0.10, 0.56	-0.28, 0.91	2.24
		6	1:6	0.27	-0.08, 0.59	-0.37, 0.86	0.27	-0.08, 0.66	-0.39, 1.09	2.34

Table A.2: continued

Y	X	q	q_1:q_2	$\rho_{q_1:q_2}$			$\beta_{q_1:q_2}$			$\hat{\sigma}_{q_1:q_2,y\|x}$
				$\hat{\rho}$	67% CI	90% CI	$\hat{\beta}$	67% CI	90% CI	
Un. Rate	PCE Infl.	12	1:12	0.26	-0.03, 0.60	-0.27, 0.80	0.21	-0.04, 0.45	-0.24, 0.76	1.44
		12	1:6	0.27	-0.03, 0.55	-0.28, 0.83	0.21	-0.04, 0.47	-0.26, 0.83	1.24
		12	7:12	0.09	-0.19, 0.43	-0.43, 0.56	0.12	-0.17, 0.43	-0.63, 0.67	0.66
		18	1:12	0.21	-0.04, 0.54	-0.46, 0.80	0.18	-0.06, 0.42	-0.24, 0.78	1.47
		6	1:6	0.27	-0.08, 0.57	-0.38, 0.86	0.23	-0.07, 0.49	-0.36, 0.85	1.34
Un. Rate	TFP	12	1:12	-0.65	-0.75,-0.34	-0.91,-0.13	-1.00	-1.39,-0.62	-1.65,-0.27	1.06
		12	1:6	-0.78	-0.92,-0.47	-0.97,-0.22	-1.22	-1.61,-0.79	-1.96,-0.44	0.75
		12	7:12	-0.31	-0.56, 0.10	-0.69, 0.38	-0.53	-1.09, 0.21	-1.39, 0.64	0.66
		18	1:12	-0.42	-0.76,-0.26	-0.91,-0.08	-0.92	-1.31,-0.49	-1.70,-0.15	1.17
		6	1:6	0.27	-0.08, 0.57	-0.38, 0.86	0.23	-0.07, 0.49	-0.36, 0.85	1.34
3M real rates	Consumption	12	1:12	0.42	0.08, 0.60	-0.06, 0.80	0.88	0.32, 1.45	-0.11, 2.57	1.84
		12	1:6	0.42	0.06, 0.64	-0.38, 0.80	0.88	0.26, 1.57	-1.11, 3.26	1.48
		12	7:12	0.43	0.14, 0.68	-0.13, 0.76	0.82	0.32, 1.38	-0.11, 1.76	0.88
		18	1:12	0.35	0.03, 0.54	-0.13, 0.80	0.63	0.11, 1.16	-0.28, 2.41	1.76
		6	1:6	0.30	-0.05, 0.59	-0.46, 0.86	0.97	-0.19, 2.03	-1.48, 3.43	1.71
Stock returns	Consumption	12	1:12	0.40	0.07, 0.60	-0.08, 0.80	2.85	0.98, 4.94	-0.48, 8.48	6.15
		12	1:6	0.43	0.08, 0.65	-0.16, 0.86	3.27	0.98, 5.57	-0.90,10.78	4.86
		12	7:12	0.35	0.04, 0.62	-0.16, 0.73	2.44	0.56, 4.31	-0.90, 5.77	3.17
		18	1:12	0.42	0.16, 0.60	-0.02, 0.76	3.05	1.47, 4.80	0.24, 7.43	6.12
		6	1:6	0.41	0.00, 0.70	-0.23, 0.89	4.23	0.62, 7.84	-2.63,11.46	5.65

Table A.2: continued

Y	X	q	q_1:q_2	$\rho_{q_1:q_2}$			$\beta_{q_1:q_2}$			$\hat{\sigma}_{q_1:q_2,y\mid x}$
				$\hat{\rho}$	67% CI	90% CI	$\hat{\beta}$	67% CI	90% CI	
Stock prices	Dividends	12	1:12	0.20	-0.05, 0.43	-0.30, 0.72	0.45	-0.17, 1.06	-0.60, 1.68	7.27
		12	1:6	0.36	-0.00, 0.66	-0.25, 0.91	0.76	-0.05, 1.56	-0.60, 3.16	5.26
		12	7:12	0.03	-0.46, 0.29	-0.72, 0.46	-0.11	-0.72, 0.69	-1.22, 1.31	4.03
		18	1:12	0.39	0.14, 0.55	-0.06, 0.72	0.75	0.31, 1.24	-0.08, 1.63	6.34
		6	1:6	0.42	0.02, 0.71	-0.20, 0.89	1.15	0.28, 1.94	-0.50, 2.72	5.56
Stock prices	Earnings	12	1:12	0.21	-0.04, 0.42	-0.27, 0.57	0.38	-0.15, 0.92	-0.53, 1.35	7.23
		12	1:6	0.29	-0.06, 0.52	-0.29, 0.76	0.49	-0.10, 1.19	-0.64, 2.16	5.23
		12	7:12	0.11	-0.15, 0.35	-0.46, 0.50	0.17	-0.26, 0.65	-0.64, 1.14	4.40
		18	1:12	0.38	0.09, 0.44	-0.09, 0.56	0.24	0.05, 0.44	-0.10, 0.60	6.62
		6	1:6	0.23	-0.08, 0.57	-0.37, 0.73	0.64	-0.22, 1.50	-0.99, 2.18	5.92
Exchange rates	Rel. price ind.	12	1:12	0.42	0.13, 0.57	-0.06, 0.72	1.19	0.51, 1.95	0.00, 2.54	6.10
		12	1:6	0.42	0.11, 0.56	-0.16, 0.80	0.93	0.26, 1.61	-0.33, 2.29	4.10
		12	7:12	0.51	0.14, 0.67	0.03, 0.80	1.87	0.85, 3.43	0.17, 4.70	4.10
		18	1:12	0.38	0.16, 0.54	-0.06, 0.63	1.01	0.43, 1.68	-0.02, 2.19	5.57
		6	1:6	0.41	0.00, 0.70	-0.27, 0.91	0.54	0.04, 0.95	-0.38, 1.29	2.08

Notes: Results are based on $\Omega_{q_1:q_2}$ (col. 4 lists q_1 and q_2) and sample information in q cosine transforms (col. 3 shows q).

Additional References

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