A.1 Data Appendix

The data used in the paper are generally growth rates computed as differences in logarithms in percentage points at annual rate ($400 \times \ln(X_t/X_{t-1})$ if $X_t$ is measured quarterly and $100 \times \ln(X_t/X_{t-1})$ if $X_t$ is measured annually). The exceptions are stock returns which are $400 \times X_t$, where $X_t$ is the gross quarterly real return (that is, $X_t = 1 + R_t$, where $R_t$ is the net return), and TFP which is reported in percentage points at annual rate in the source data. The data are described in the following table.
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A.2 Proof of Theorem 1

We now introduce some notation that is used in the proof of Theorem 1, and in auxiliary Lemmas. We first state these Lemmas and then the proof of Theorem 1.

Notation: Define $\sigma^2_T = \text{Var}[T^{-1/2} \int_0^T g(s) x_{T:sT} ds]$. Note that with $\bar{g}_{T:t} = T^{-1/2} \int_{(t-1)/T}^t g(s) ds$ and $\tilde{G}_{T:t} = T^{-1} \sum_{s=1}^{t-1} \bar{g}_{T:t} = \sum_{s=1}^{t-1} \int_{(s-1)/T}^{s/T} g(s) ds = \int_{(t-1)/T}^{(t-1)/T} g(s) ds = G(t^{-1})$, we find using summation by parts

$$T^{-1/2} \int_0^H g(s) x_{T:sT} ds = T^{-3/2} \sum_{t=1}^{HT} \bar{g}_{T:t} x_{T:t}$$

$$= T^{-1/2} \tilde{G}_{T,HT+1} x_{T,HT} - T^{-1/2} \sum_{t=1}^{HT} \tilde{G}_T(t/T) \Delta x_{T:t}$$

$$= -T^{-1/2} \sum_{t=1}^{HT} G(t^{-1}) \Delta x_{T:t}$$

since $\tilde{G}_{T,HT+1} = G(H) = 0$.

Lemma A.1 Under assumption (iv) of Theorem 1, there exists $C < \infty$ such that $\sup_{0 \leq \lambda \leq \pi, T} \sum_{t=1}^{HT} e^{-i\lambda t} G(t^{-1}) \leq C \lambda^{-2} T^{-1}$.

Proof. Let $r_t = \sum_{t=1}^T e^{i\lambda} = e^{i\lambda}(1 - e^{i\lambda})^{-1}(e^{i\lambda} - 1)$. By the above summation by parts argument,

$$\sum_{t=1}^{HT} e^{-i\lambda t} G(t^{-1}) = -\int_0^H g(s) r_{[sT]+1} ds$$

$$= -\frac{e^{i\lambda}}{1 - e^{i\lambda}} \int_0^H g(s) e^{i\lambda \Delta s} ds$$

$$= -\frac{e^{i\lambda}}{1 - e^{i\lambda}} T^{-1} \sum_{t=1}^{HT} \bar{g}_{T:t} e^{i\lambda t}$$

where the second equality uses $\int_0^H g(s) ds = 0$. Furthermore, by summation by parts,

$$\sum_{t=1}^{HT} \bar{g}_{T:t} e^{i\lambda t} = \bar{g}_{T,HT+1} - \sum_{t=1}^{HT} (\bar{g}_{T:t} - \bar{g}_{T:t-1}) r_{t-1}$$

$$= \frac{e^{i\lambda}}{1 - e^{i\lambda}} \left( \bar{g}_{T,HT} (e^{i\lambda HT} - 1) - \sum_{t=1}^{HT} (\bar{g}_{T:t} - \bar{g}_{T,t-1})(e^{i\lambda(t-1)} - 1) \right).$$

Thus

$$\left| \sum_{t=1}^{HT} e^{-i\lambda t} G(t^{-1}) \right| \leq 2T^{-1} \left| e^{i\lambda}(1 - e^{i\lambda})^{-1} \right|^2 (\bar{g}_{T,HT} + \sum_{t=1}^{HT} |\bar{g}_{T:t} - \bar{g}_{T:t-1}|).$$
For $0 \leq \lambda \leq \pi$, $|e^{i\lambda}(1 - e^{i\lambda})^{-1}| = (2\sin(\lambda/2))^{-1} \leq 2\lambda$. Furthermore, since $g$ is of bounded variation, it can be written as $g(s) = g^+(s) - g^-(s)$, with both $g^+$ and $g^-$ positive and monotonically increasing. With $\bar{g}_{T,t}^+ = T \int_{(t-1)/T}^{t/T} g^+(s)ds$ and $\bar{g}_{T,t}^- = T \int_{(t-1)/T}^{t/T} g^-(s)ds$, this implies the bound

$$\sum_{t=1}^{HT} |\bar{g}_{T,t} - \bar{g}_{T,t-1}| = \sum_{t=1}^{HT} |\bar{g}_{T,t}^+ - \bar{g}_{T,t}^- - \bar{g}_{T,t-1}^+ + \bar{g}_{T,t-1}^-| \leq \sum_{t=1}^{HT} (|\bar{g}_{T,t}^+ - \bar{g}_{T,t-1}^+| + |\bar{g}_{T,t}^- - \bar{g}_{T,t-1}^-|) \leq g^+(HT) + g^-(HT) < \infty,$$

and the result follows. □

**Lemma A.2** Under the assumptions Theorem 1, $\sigma_T^2 \rightarrow \sigma^2 = \int_{-\infty}^{\infty} S(\omega) \left| \int_{0}^{H} e^{-i\omega s} g(s) ds \right|^2 d\omega$.

**Proof.** Let $\gamma_T$ be the autocovariances of $\Delta x_{T,t}$. We find

$$\text{Var}[T^{-1/2} \sum_{t=1}^{HT} G(t-1/T) \Delta x_{T,t}] = T^{-1} \sum_{j,k=1}^{HT} \gamma_T(k-j) G\left(\frac{k-1}{T}\right) G\left(\frac{j-1}{T}\right)$$

$$= T^{-1} \sum_{j,k=1}^{HT} \left( \int_{-\pi}^{\pi} e^{i\lambda(k-j)} F_T(\lambda) d\lambda \right) G\left(\frac{k-1}{T}\right) G\left(\frac{j-1}{T}\right)$$

$$= T^{-1} \int_{-\pi}^{\pi} F_T(\lambda) \left| \sum_{t=1}^{HT} e^{i\lambda t} G\left(\frac{t-1}{T}\right) \right|^2 d\lambda.$$

Now for any fixed $K$, we will show

$$T^{-1} \int_{-K/T}^{K/T} F_T(\lambda) \left| \sum_{t=1}^{HT} e^{i\lambda t} G\left(\frac{t-1}{T}\right) \right|^2 d\lambda = \int_{-K}^{K} F_T(\omega) \left| T^{-1} \sum_{t=1}^{HT} e^{i\omega t/T} G\left(\frac{t-1}{T}\right) \right|^2 d\omega$$

$$\rightarrow \int_{-K}^{K} S(\omega) \omega^2 \left| \int_{0}^{H} e^{i\omega s} G(s) ds \right|^2 d\omega. \quad (A.1)$$

First note that for any two complex numbers $a, b$, $|a - b| \geq ||a| - |b||$, so that $||a|^2 - |b|^2| = |(|a| + |b|)(|a| - |b|)| \leq (|a| + |b|)|a - b|$. Thus,

$$\left| \int_{0}^{H} e^{i\omega s} G(s) ds \right|^2 - T^{-1} \sum_{t=1}^{HT} e^{i\omega t/T} G\left(\frac{t-1}{T}\right) \right|^2 \leq 2H \sup_{s} |G(s)| \cdot \left| T^{-1} \sum_{t=1}^{HT} e^{i\omega t/T} G\left(\frac{t-1}{T}\right) - \int_{0}^{H} e^{i\omega s} G(s) ds \right|$$

and

$$\left| T^{-1} \sum_{t=1}^{HT} e^{i\omega t/T} G\left(\frac{t-1}{T}\right) - \int_{0}^{H} e^{i\omega s} G(s) ds \right| \leq \int_{0}^{H} |e^{i\omega s} G(s) - e^{i\omega |sT|/T} G(|sT|/T)| ds$$

$$= \int_{0}^{H} |e^{i\omega s} G(s) - G(|sT|/T)) + G(|sT|/T)(e^{i\omega s} - e^{i\omega |sT|/T})| ds$$
\[ \leq \int_0^H |G(s) - G([sT]/T)|ds + \sup_s |G(s)| \int_0^H |1 - e^{i\omega([sT]/T-s)}|ds. \]

Since for any real \(a, |1-e^{ia}| < |a|, \sup_{|\omega| \leq K} |1-e^{i\omega([sT]/T-s)}| \leq K/T, \text{ and also } |G(s) - G([sT]/T)| \leq T^{-1} \sup_s |g(s)|. \] Thus,

\[ \sup_{|\omega| \leq K} \left| T^{-1} \sum_{t=1}^{HT} e^{i\omega t/T}G\left(\frac{t-1}{T}\right)^2 - \int_0^H e^{i\omega s}G(s)ds \right| \rightarrow 0 \]

and (A.1) follows from assumption (iii.a) by straightforward arguments.

Further, since \( \int_0^H e^{i\omega s}g(s)ds = -i\omega \int_0^H e^{i\omega s}G(s)ds, \)

\[ \int_{-K}^K S(\omega)\omega^2 \left| \int_0^H e^{i\omega s}G(s)ds \right|^2 d\omega = \int_{-K}^K S(\omega) \int_0^H e^{i\omega s}g(s)ds \left| \int_0^H e^{i\omega s}G(s)ds \right|^2 d\omega. \]

Thus, for any fixed \(K, \)

\[ \delta_K(T) = T^{-1} \int_{-K/T}^{K/T} F_T(\lambda) \left| \sum_{t=1}^{HT} e^{i\lambda t}G\left(\frac{t-1}{T}\right)^2 \right| d\lambda - \int_{-K}^K S(\omega) \int_0^H e^{i\omega s}g(s)ds \left| \int_0^H e^{i\omega s}G(s)ds \right|^2 d\omega \rightarrow 0. \]

Now for each \(T, \) define \(K_T \) as the largest integer \(K \leq T\) for which \(\sup_{T' \geq T} |\delta_K(T')| \leq 1/K \) (and zero if no such \(K \) exists). Note that \(\delta_K(T) \rightarrow 0\) for all fixed \(K \) implies that \(K_T \rightarrow \infty, \) and by construction, also \(\delta_{K_T}(T) \rightarrow 0. \) The result now follows from

\[ T^{-1} \int_{K_T/T}^{\pi} F_T(\lambda) \left| \sum_{t=1}^{HT} e^{i\lambda t}G\left(\frac{t-1}{T}\right)^2 \right| d\lambda \leq C^2 T^{-3} \int_{K_T/T}^{\pi} F_T(\lambda) \frac{1}{\lambda} d\lambda \rightarrow 0 \]

by assumption (iii.b) and Lemma A.1. \(\blacksquare\)

**Lemma A.3** Under the assumptions Theorem 1, \(T^{-1/2} \sup_{t,T} |\sum_{j=1}^{HT} G(t-j-1)ct_{j-1}| \rightarrow 0.\)

**Proof.** Recall that for any two real, square integrable sequences \(\{a_j\}_{j=-\infty}^{\infty} \) and \(\{b_j\}_{j=-\infty}^{\infty}, \)

\(\sum_{j=-\infty}^{\infty} a_jb_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{A}(\lambda)\hat{B}^*(\lambda)d\lambda, \) where \(\hat{A}(\lambda) = \sum_{j=-\infty}^{\infty} a_je^{-i\lambda j}, \) and \(\hat{B}^*(\lambda) = \sum_{j=-\infty}^{\infty} b_je^{i\lambda j}. \) Thus

\[ \sum_{j=1}^{HT} G\left(\frac{j-1}{T}\right)ct_{j-1-t} = \sum_{t=1}^{HT-t} G\left(\frac{t+j-1}{T}\right)ct_{t-j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda} \hat{G}_T(\lambda)\hat{C}_T^*(\lambda)d\lambda \]

where \(\hat{C}_T(\lambda) = \sum_{j=-\infty}^{\infty} c_T e^{-i\lambda j} \) and \(\hat{G}_T(\lambda) = \sum_{j=1}^{HT} e^{-i\lambda j}G\left(\frac{t-1}{T}\right), \) so that \(e^{i\lambda t}\hat{G}_T(\lambda) = \sum_{j=1}^{HT} e^{-i\lambda(j-1)}G\left(\frac{t+j-1}{T}\right) = \sum_{j=1-t}^{HT-t} e^{-i\lambda(j-1)}G\left(\frac{t+j-1}{T}\right), \) and \(\hat{C}_T^* \) is the complex conjugate of \(\hat{C}_T. \) Now since \(F_T(\lambda) = \frac{1}{2\pi} |\hat{C}_T(\lambda)|^2, \) we find

\[ 2\pi |\sum_{j=1}^{HT} G\left(\frac{j-1}{T}\right)ct_{j-1}| = |\int_{-\pi}^{\pi} e^{i\lambda} \hat{G}_T(\lambda)\hat{C}_T^*(\lambda)d\lambda| \leq \sqrt{2\pi} \int_{-\pi}^{\pi} |\hat{G}_T(\lambda)|F_T(\lambda)^{1/2}d\lambda. \]
Also, using $|\hat{G}_T(\lambda)| \leq \sum_{j=1}^{HT} |G(\frac{j-1}{T})|$, we have
\[
\int_0^{1/T} |\hat{G}_T(\lambda)| F_T(\lambda)^{1/2} d\lambda \leq T^{-1} \sum_{j=1}^{HT} |G(\frac{j-1}{T})| \cdot \int_0^{1} F_T(\omega/T)^{1/2} d\omega
\]
\[
\rightarrow \int_0^{H} |G(s)| ds \cdot \int_0^{1} \omega S(\omega)^{1/2} d\omega < \infty
\]
where the convergence follows from assumption (iii.a). Furthermore, by Lemma A.1,
\[
\int_0^{1/T} |\hat{G}_T(\lambda)| F_T(\lambda)^{1/2} d\lambda \leq CT^{-1} \int_0^{1/T} F_T(\lambda)^{1/2} \lambda^{-2} d\lambda.
\]
The result now follows from assumption (iii.c). ■

**Lemma A.4** Under the assumptions Theorem 1, for every $\epsilon > 0$ there exists a $M > 0$ such that
\[
\limsup_{T \to \infty} \text{Var}[T^{-1/2} \int_0^{H} g(s)x_{T,[st]+1} ds + T^{-1/2} \sum_{t=-MT}^{MT} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T})c_{T,j-t} \right) \varepsilon_t] < \epsilon.
\]
For this $M$, $\sigma_{T,M}^2 = \text{Var}[T^{-1/2} \sum_{t=-MT}^{MT} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T})c_{T,j-t} \right) \varepsilon_t]$ satisfies $\limsup_{T \to \infty} |\sigma_{M,T}^2 - \sigma^2| < \epsilon$.

**Proof.** We have
\[
- \int_0^{H} g(s)x_{T,[st]+1} ds = \sum_{t=1}^{HT} G(\frac{t-1}{T}) \Delta x_{T,t}
\]
\[
\quad = \sum_{t=1}^{HT} G(\frac{t-1}{T}) \sum_{s=-\infty}^{\infty} c_{T,s} \varepsilon_{t-s}
\]
\[
\quad = \sum_{t=-\infty}^{\infty} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t} \right) \varepsilon_t
\]
so that, with $\hat{G} = \sup_{0 \leq s < H} |G(s)|$
\[
\text{Var}[T^{-1/2} \int_0^{H} g(s)x_{T,[st]+1} ds + T^{-1/2} \sum_{t=-MT}^{MT} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T})c_{T,j-t} \right) \varepsilon_t]
\]
\[
= T^{-1} \sum_{t=-\infty}^{MT-1} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T})c_{T,j-t} \right)^2 + T^{-1} \sum_{t=MT+1}^{\infty} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T})c_{T,j-t} \right)^2
\]
\[
\leq \hat{G}^2 T^{-1} \sum_{t=MT+1}^{\infty} \left( \sum_{j=1}^{j} (|c_{T,j-t}| + |c_{T,j+t}|) \right)^2
\]
Thus, it suffices to show that
\[
\sum_{t=MT+1}^{\infty} \left( T \sup_{|s| \geq t-HT} |c_{T,s}| \right)^2
\]
which can be made arbitrarily small by choosing $M$ large enough via assumption (ii).

The second claim follows directly from Lemma A.2.

**Lemma A.5** Under the assumptions Theorem 1, for any large enough integer $M > 0$, $\sigma_{M,T}^{-1} T^{-1/2} \sum_{t=-MT}^{MT} (\sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t}) \varepsilon_t \Rightarrow N(0,1)$.

**Proof.** By the second claim in Lemma A.4 and Lemma A.2, $\sigma_{M,T} = O(1)$ and $\sigma_{M,T}^{-1} = O(1)$. By Theorem 24.3 in Davidson (1994), it thus suffices to show (a) $T^{-1/2} \sup_{1 \leq t \leq HT} |\sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t} \varepsilon_t| \overset{p}{\rightarrow} 0$ and (b) $T^{-1} \sum_{t=-MT}^{MT} (\sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t})^2 (\varepsilon_t^2 - 1) \overset{p}{\rightarrow} 0$.

(a) is implied by the Lyapunov condition via Davidson’s (1994) Theorems 23.16 and 23.11. Thus, it suffices to show that
\[
\sum_{t=-MT}^{MT} E[|T^{-1/2} (\sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t}) \varepsilon_t|^{2+\delta}] \rightarrow 0.
\]

Now
\[
\sum_{t=-MT}^{MT} E \left[ \left| T^{-1/2} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t} \right) \varepsilon_t \right|^{2+\delta} \right]
\]
\[
\leq T^{-1-\delta/2} \sum_{t=-MT}^{MT} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t} \right)^{2+\delta} E(|\varepsilon_t|^{2+\delta})
\]
\[
\leq (\sup_t E(|\varepsilon_t|^{2+\delta})) \cdot T^{-\delta/2} \sup_t \left( \sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t} \right)^{\delta} \cdot T^{-1} \sum_{t=-MT}^{MT} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t} \right)^{2}
\]
\[
= (\sup_t E(|\varepsilon_t|^{2+\delta})) \cdot \left( T^{-1/2} \sup_t \left( \sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t} \right) \right)^{\delta} \cdot \sigma_{M,T}^2 \rightarrow 0
\]
where the convergence follows from Lemma A.3.

For (b), we apply Theorem 19.11 of Davidson (1994) with Davidson’s $X$ and $c$ chosen as $X_{P,t} = c_{P,t}^2 (\varepsilon_t^2 - 1)$ and $c_{P,t}^2 = T^{-1} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t} \right)^2$. Then $X_{P,t}^2 / c_{P,t}^2 = \varepsilon_t^2 - 1$ is uniformly integrable, since $\sup_t E(|\varepsilon_t|^{2+\delta}) < \infty$. Further,
\[
\sum_{t=-MT}^{MT} c_{P,t}^2 = T^{-1} \sum_{t=-MT}^{MT} \left( \sum_{j=1}^{HT} G(\frac{j-1}{T}) c_{T,j-t} \right)^2 = \sigma_{M,T}^2 = O(1)
\]
and

$$\sum_{t=-MT}^{MT} (e_{T,t}^D)^2 = T^{-2} \sum_{t=-MT}^{MT} \left( \sum_{j=1}^{HT} G\left(\frac{j-1}{T}\right) c_{T,j-t} \right)^4 \leq \sup_{-MT \leq t \leq MT} \left| T^{-1/2} \sum_{j=1}^{HT} G\left(\frac{j-1}{T}\right) c_{T,j-t} \right|^2 \cdot T^{-1} \sum_{t=-MT}^{MT} \left( \sum_{j=1}^{HT} G\left(\frac{j-1}{T}\right) c_{T,j-t} \right)^2 \cdot \sigma_{M,T}^2 \rightarrow 0$$

where the convergence follows from Lemma A.3.

**Proof of Theorem 1:**

By Lemmas A.2, A.4 and A.5, for large enough $M$,

$$\sigma^{-1} T^{-1/2} \int_0^H g(s) x_{T, [sT] + 1} ds = \frac{\sigma_{M,T}}{\sigma} A_T + \frac{\sigma_{M,T}}{\sigma} B_T$$

where $A_T \Rightarrow N(0,1)$, $\limsup_{T \rightarrow \infty} E(B_T^2) < \epsilon$ and $\limsup_{T \rightarrow \infty} |\sigma_{M,T}^2 - \sigma^2| < \epsilon$. Thus, by Slutsky’s Theorem, as $\epsilon \rightarrow 0$, the desired convergence in distribution follows. But $\epsilon$ was arbitrary, which proves the Theorem.

**A.3 $\Sigma$ as a Function of the local-to-zero Spectrum**

Letting $\eta_{j,T} = T^{-1/2} \int_0^{1+r} g_j(s) x_{[sT] + 1} ds$ and $\eta_{k,T} = T^{-1/2} \int_0^{1+r} g_k(s) x_{[sT] + 1} ds$, Corollary 1 shows that the $j$th element of $\Sigma$ is given by

$$\Sigma_{j,k} = \int_{-\infty}^{\infty} S(\omega) \left( \int_0^{1+r} g_k(s) e^{i\omega s} ds \right) \left( \int_0^{1+r} g_j(s) e^{-i\omega s} ds \right). \quad (A.2)$$

Since $S$ is an even function and $g_j$ and $g_k$ are real valued, (A.2) can be rewritten as $\Sigma_{j,k} = \int_0^{\infty} S(\omega) w_{jk}(\omega) d\omega$, where $w_{jk}(\omega) = \text{Re} \left( \int_0^{1+r} g_j(s) e^{-i\omega s} ds \right) \left( \int_0^{1+r} g_k(s) e^{i\omega s} ds \right)$, as noted in the main text. With $g_j = \sqrt{2} \cos(\pi j s)$, a calculation shows that $w_{jk}(\omega) = 0$ for $1 \leq j, k \leq q$ and $j + k$ odd, so that $E(X_j X_k) = 0$ for all odd $j + k$, independent of the local-to-zero spectrum $S$. Figure A.1 plots $w_{jk}(\cdot)$ for some selected values of $j$ and $k$ and $r = 1/2$. The figure displays the weights $w_{jk}$ corresponding to the covariance matrix of the vector $(X_1, X_2, X_3, X_{10}, X_{11}, X_{12}, Y)'$, organized into a symmetric matrix of 9 panels. The first panel plots the weights for the covariance matrix of $(X_1, X_2, X_3)'$, where the weights for the variances are shown in bold and the weights for the covariances are shown as thin curves. The second panel plots the weights associated with covariances between $(X_1, X_2, X_3)'$ and $(X_{10}, X_{11}, X_{12})'$, and so forth.
Figure A.1: Integral weights on local-to-zero spectrum for selected elements of $\Sigma$

Notes: The curves denote weights for variances (bold) and covariances (thin) of $(X_1, X_2, X_3, X_{10}, X_{11}, X_{12}, Y)$ with $w_{q, q+1}$ corresponding to $E(X_i Y)$, and $w_{q, q+1}$ corresponding to $E(Y^2)$. The nine panels show weights for the covariances of the three sets of variables $(X_1, X_2, X_3, X_{10}, X_{11}, X_{12})$, and $Y$, respectively. All weights $w_{j,k}$ on covariance terms $\Sigma_{jk} = E(X_i X_k)$ with $j,k \leq q$ and $j+k$ odd are equal to zero.
A.4 Computation of $\Sigma$ in bcd-model

One possibility to compute $\Sigma$ is to notice that for all $g_j$ under consideration, $\int_0^{1+r} g_j(s)e^{i\omega s}ds$ can be obtained in closed form, so that one can apply one-dimensional numerical integration to (9) for $S(\omega) = (2\pi)^{-1}(c^2 + \omega^2)^{-d} + b^2$. It turns out, however, that the integrand is highly oscillatory and, for $d$ small ($d = -0.4$), very slowly decaying.

We therefore now derive a numerically more stable expression for $\Sigma_{j,k}$ in the bcd-model. Note that $\Sigma$ in (9) is linear in $S$, and for and $S(\omega) = (2\pi)^{-1}b^2$, $\Sigma$ can be computed straightforwardly from (9). It thus suffices to obtain a more convenient expression for $S(\omega) = (2\pi)^{-1}(c^2 + \omega^2)^{-d}$, and to add the component stemming from $b > 0$ at the end.

Now for $c > 0$ and $1/2 < d < 1/2$, from (4.58) in Lindgren (2013)

$$\gamma_{c,d}(s) = (2\pi)^{-1}\int_{-\infty}^{\infty} (c^2 + \omega^2)^{-d}e^{i\omega s}d\omega = \frac{2^{1/2-d}\sqrt{\pi}|s|/c)^{d-1/2}I_{d-1/2}(|s|)}{\pi \Gamma(d)} \quad (A.3)$$

where $I_\nu(x)$ is the modified Bessel function of the second kind, so that under $b = 0$,

$$\Sigma_{j,k} = \int_0^{H_L} \int_0^{H_U} g_j(s)e^{i\omega s}g_k(u)e^{-i\omega u}(c^2 + \omega^2)^{-d}e^{i\omega s}d\omega dsdu \quad (A.4)$$

We approximate (A.5) by a double sum with a step length of 0.0002. For $c = 0$, we use the implied value for $c = 0.01$.

It remains to obtain a suitable expression for $-1/2 < d < 1/2$. Suppose $g$ is differentiable on $(H_L, H_U)$ with derivative $g'$. Then by integration by parts,

$$\int_{H_L}^{H_U} g(s)e^{i\omega s}ds = \int_{H_L}^{H_U} g(s)e^{-cs}e^{(c+i\omega)s}ds$$

$$= g(H_U)e^{-H_Ue^{(c+i\omega)H_U}} - g(H_L)e^{-cH_L} - \int_{H_L}^{H_U} (g'(s)e^{-cs} - cg(s)e^{-cs})e^{(c+i\omega)s}ds$$

$$= g(H_U)e^{i\omega H_U} - g(H_L)e^{i\omega H_L} - \int_{H_L}^{H_U} (g'(s) - cg(s))e^{i\omega s}ds.$$  

All $g$ functions entering $\Sigma_{j,k}$ are differentiable on $(0, 1)$ and on $(1, 1+r)$, so we obtain

$$\int_0^{1+r} g(s)e^{i\omega s}ds = \frac{g(1+r)e^{i\omega(1+r)} - (g^+(1) - g(1))e^{i\omega} - g(0) - \int_0^{1+r} g'(s) - cg(s)e^{i\omega s}ds}{c + i\omega} \quad (A.6)$$

where $g^+(1)$ the right limit of $g(s)$ at $s = 1$, and $g'(s)$ for $s \in \{0, 1, 1+r\}$ may be defined arbitrarily. Now applying (A.6) in (A.4) twice, and noting that $(c + i\omega)(c - i\omega) = c^2 + \omega^2$, all integrals with respect to $d\omega$ in (A.4) are of the form $C_1 \int_{-\infty}^{\infty} (c^2 + \omega^2)^{-(d+1)}e^{i\omega(s-C_2)}d\omega$ for $C_1$ and $C_2$ not depending on $\omega$, which can be computed in closed form using (A.3). We obtain

$$\Sigma_{j,k} = \begin{pmatrix} g_j(1+r) \\ g_j(1) - g_j^+(1) \\ -g_j(0) \end{pmatrix}' \begin{pmatrix} \gamma_{c,d+1}(0) & \gamma_{c,d+1}(1+r) & \gamma_{c,d+1}(1+r) \\ \gamma_{c,d+1}(r) & \gamma_{c,d+1}(0) & \gamma_{c,d+1}(1) \\ \gamma_{c,d+1}(1+r) & \gamma_{c,d+1}(1) & \gamma_{c,d+1}(0) \end{pmatrix} \begin{pmatrix} g_k(1+r) \\ g_k(1) - g_k^+(1) \\ -g_k(0) \end{pmatrix}$$
Thus, the expression for

\[
- \left( \begin{array}{c}
g_j(1 + r) \\
g_j(1) - g_j^+(1) \\
-g_j(0)
\end{array} \right) \left( \begin{array}{c}
f_0^{1+r}(g_k(s) - cg_k(s))\gamma_{c,d+1}(1 + r - s)ds \\
f_0^{1+r}(g_k(s) - cg_k(s))\gamma_{c,d+1}(1 - s)ds \\
f_0^{1+r}(g_k(s) - cg_k(s))\gamma_{c,d+1}(s - u)dsu.
\end{array} \right)
\]

where

\[
\gamma_k = \int_0^\pi \cos(k\lambda)R(\lambda)d\lambda
\]

and the \(k\)th autocovariance is therefore

\[
\gamma_k = \frac{\sigma^2}{2\pi} \int_0^\pi \cos(k\lambda)(1 + \rho)^{-d}d\lambda
\]

\[
= \frac{\sigma^2}{2\pi} \left( \frac{\rho}{1 + \rho^2} \right)^k (1 + \rho)^{-2d} \left( \frac{\Gamma(d + k)}{\Gamma(d)\Gamma(1 + k)} \right) \quad 2F_1 \left( \frac{0.5 + k, d + k; 1 + 2k; 4\rho}{(1 + \rho)^2} \right)
\]

where \(2F_1(a,b;c,z)\) is the hypergeometric function.

From Luke (1975, page 271 equation (7)):

\[
2F_1(a,b;2a;z) = \left( \frac{2}{1 + y} \right)^{2b} 2F_1 \left( b, b + 0.5 - a; a + 0.5; \left( \frac{1 - y}{1 + y} \right)^2 \right), \text{ where } y = (1 - z)^{1/2}.
\]

Thus, the expression for \(\gamma_k\) simplifies:

\[
\gamma_k = \frac{\sigma^2}{2\pi} \left( \frac{\rho}{1 + \rho^2} \right)^k (1 + \rho)^{2(d+k)}(1 + \rho)^{-2d} \left( \frac{\Gamma(d + k)}{\Gamma(d)\Gamma(1 + k)} \right) 2F_1 \left( d + k, d; 1 + k; \rho \right)
\]

\[
= \frac{\sigma^2}{2\pi} \rho^k \left( \frac{\Gamma(d + k)}{\Gamma(d)\Gamma(1 + k)} \right) 2F_1 \left( d + k, d; 1 + k; \rho^2 \right).
\]
A.6 Analysis of the Weighting Function $W$

As discussed in the text, construction of the $A_{MN}$ sets requires specification of the weighting function $W$ which determines the trade-off between expected length for various of $\theta$. Our choice of $W$ uses the minimized values of $V_\theta(A)$. Denote the scaled version of $V_\theta(A)$ by $R_\theta(A) = V_\theta(A)/V_\theta^{\text{known}}$, where $V_\theta^{\text{known}}$ is the expected length of the prediction set for known value of $\theta$ implied by equation (8). This relative measure of length is readily interpretable: $R_\theta(A) > 1$ represents the "regret" about not knowing $\theta$ when using the set $A$. In terms of $R_\theta(A)$ we use a weighting function that coincides with the Bayes prior: uniform on $d \in [-0.4,1.0]$ and with $b = c = 0$ (so in terms of $V_\theta(A)$, the weighting function $W$ is proportional to $1/V_\theta^{\text{known}}$). Thus, we robustify the Bayes sets in a way that minimizes regret using the Bayes prior while achieving frequentist coverage.

Table A.2 examines the expected length of the $A_{MN} = A_{MN}^{(b,c,d)}$ sets and answers questions regarding the choice of $W$ and $\Gamma$. Because $A_{MN}$ minimizes a weighted average of the values of the expected (normalized) length $R_\theta(A)$, it is impossible to choose an alternative weighting function with uniformly lower values of $R_\theta(A)$. However, it is possible to reduce $R_\theta(A_{MN})$ for any particular value of $\theta$. Thus, Table A.2 compares the value $R_\theta(A_{MN})$ for the benchmark choice of $W$ to the value obtained by putting all $W$ mass on $\theta$, for selected values of $\theta$. Said differently, Table A.2 compares $R_\theta(A_{MN})$ to the envelope obtained by minimizing $R_\theta(A)$ over all equivariant sets $A$ that have uniform coverage in the $bcd$-model. Values of $R_\theta(A)$ are shown as a function of $d$ in the $I(d)$ model, and for selected values in the $b$ and $c$ models, respectively.

The table shows results for 90% prediction sets; conclusions for 67% prediction sets are similar. Panel (a) considers sets that are also restricted to contain the Bayes prediction set, as in (13). For

<table>
<thead>
<tr>
<th>Parameter values ($\theta$)</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{\text{Bench}}$</td>
<td>1.85</td>
<td>1.88</td>
<td>1.80</td>
</tr>
<tr>
<td>$V_\theta^{\text{known}}$</td>
<td>0.54</td>
<td>1.15</td>
<td>1.15</td>
</tr>
<tr>
<td>$V_\theta^{\text{Env}}$</td>
<td>1.78</td>
<td>1.38</td>
<td>1.22</td>
</tr>
<tr>
<td>$V_\theta^{\text{Env}}$</td>
<td>1.89</td>
<td>1.19</td>
<td>1.40</td>
</tr>
<tr>
<td>$V_\theta^{\text{En}}$</td>
<td>1.88</td>
<td>1.19</td>
<td>1.40</td>
</tr>
</tbody>
</table>

Notes: The table shows the values of $R_\theta(A)$ for the value of $\theta = (b,c,d)$ shown in the first three rows. $W_{\text{Bench}}$ denotes the benchmark weighting function described in the text, and in the rows labeled "Env," $A$ is a $\theta$-specific set that minimizes expected length at that value of $\theta$. The non-zero values of $b$ and $c$ are chosen to lie between the I(0) $(b \rightarrow \infty, c \rightarrow \infty)$ and I(1) $(b=0, c=0)$ extremes.
Table A.3: Average length of 90% $A^{MN}$ prediction sets, $r = 0.5$

<table>
<thead>
<tr>
<th></th>
<th>$q = 12$</th>
<th>$q = 6$</th>
<th>$q = 24$</th>
<th>$q = 48$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$ unknown</td>
<td>1.57</td>
<td>1.69</td>
<td>1.45</td>
<td>1.38</td>
</tr>
<tr>
<td>$\theta$ known</td>
<td>1.00</td>
<td>1.08</td>
<td>0.97</td>
<td>0.95</td>
</tr>
</tbody>
</table>

Notes: Entries are weighted average regret (relative to knowing $\theta$ in the $q=12$ case) of $A^{MN}$ prediction sets ($\theta$ unknown) and minimum length sets $A^{\text{known}}_{\theta}$ for $\theta$ known.

the benchmark choice of $W$, $R_\theta(A) = R_\theta(A^{MN})$ ranges from 1.15 (so that $V_\theta(A^{MN})$ is 15% larger than the attainable expected length with $\theta$ known) when $b = c = 0$ and $d = 1.0$, to 1.89 when $b = 0$, $c = 27$, and $d = 1.0$. These numbers demonstrate that parameter uncertainty is a significant component of $A^{MN}$. The envelope numbers are smaller, of course, but are typically within 10% of the $A^{MN}$ values even for $\theta$ with non-zero $b$ or $c$ (where the benchmark weighting function puts zero mass).

Panel (b) of Table A.2 takes up the question of the cost of imposing the Bayes-superset constraint (13). The two rows in panel (b) are computed exactly as those in panel (a), except that the Bayes-superset constraint is no longer enforced. Evidently, for most values of $\theta$, the Bayes-superset constraint has a negligible impact on the expected length of the prediction set. What is more, the envelope numbers in panel (b) are typically not much smaller than those for $A^{MN}$ in panel (a), which means that the substantial regret of not knowing $\theta$ is mostly driven by the coverage constraint, and is not an artifact of our benchmark choices for $\Gamma$ or $W$. (And, as a corollary, it also implies that alternative choices for $\Gamma$ for $A^{MN}$ can not lead to substantially shorter sets on average).

### A.7 Choice of $q$

As discussed in Section 4.3, the sample cosine transforms contain information on both the scale and shape of the local-to-zero spectrum. Table A.3 quantifies the combined scale and shape effects by reporting the value of the objective $\int V_\theta(A) dW(\theta)$ in the program (10) for $q \in \{6, 12, 24, 48\}$. To be able to make meaningful comparisons across different values of $q$, we choose $W$ as the benchmark weighting function from the $q = 12$ case for all values of $q$, normalized to integrate to one (so that the values in Table 3 are the weighted average regret relative to knowing $\theta$ in the $q = 12$ case).

In this $\theta$ unknown case, there is an 8% decrease in average length as $q$ increases from $q = 12$ to $q = 24$ and a further reduction of 5% for $q = 48$. As a point of reference, the table also reports the same average length for $\theta$ known.\textsuperscript{16} The reductions in lengths for $q > 12$ are only marginally larger.

\textsuperscript{16} These are for comparison only; the parameters of the $bcd$-model cannot be consistently estimated even under $q \to \infty$ asymptotics.
than the pure scale effect in the $I(0)$ model, corroborating the intuition above about the relative unimportance of the spectral shape outside the $[-12\pi/T, 12\pi/T]$ interval for long-run forecasting.

A.8 Computation of Prediction Sets

A.8.1 Numerical Determination of $C_\theta(A)$ and $V_W(A)$

The algorithm for determining $A_{MN}$ below requires repeated evaluation of $V_\theta(A)$ and $C_\theta(A)$, with $A$ some set valued function of $x^s$. We employ an importance sampling Monte Carlo scheme: Write $f_\theta$ for the density of $Z^s = (X^s, Y^s)$ under $\theta$, i.e. $f_\theta(z^s) = f_{(X^s, Y^s)|\theta}(x^s, y^s)$, and let $f_p$ be a proposal density for $Z^s$ such that $f_\theta$ is absolutely continuous with respect to $f_p$ for all $\theta$. Then

$$C_\theta(A) = E_p[\frac{f_\theta(Z^s)}{f_p(Z^s)} 1[Y^s \in A(X^s)]]$$

with $E_p$ denoting integration with respect to $f_p$. Furthermore, from

$$V_\theta(A) = E_\theta [g_\theta(X^s) \text{vol}(A(X^s))]$$

$$= E_\theta \left[ g_\theta(X^s) \int 1[y^s \in A(X^s)] dy^s \right]$$

$$= \int \int g_\theta(x^s)f_{X^s|\theta}(x^s)1[y^s \in A(x^s)] dx^s dy^s$$

we obtain

$$V_\theta(A) = E_p[\frac{g_\theta(X^s)f_{X^s|\theta}(X^s)}{f_p(Z^s)} 1[Y^s \in A(X^s)]]$$

With $N$ i.i.d. draws $Z_1^s, \ldots, Z_n^s$ from $f_p$, we thus obtain the importance sampling approximations

$$\hat{C}_\theta(A) = N^{-1} \sum_{i=1}^N \frac{f_\theta(Z_i^s)}{f_p(Z_i^s)} 1[Y_i^s \in A(X_i^s)] \quad (A.7)$$

$$\hat{V}_\theta(A) = N^{-1} \sum_{i=1}^N \frac{g_\theta(X_i^s)f_{X^s|\theta}(X_i^s)}{f_p(Z_i^s)} 1[Y_i^s \in A(X_i^s)]. \quad (A.8)$$

The expression (A.8) is numerically advantageous, as it does not require any explicit numerical determination of the length of a set $A(x^s)$.

A.8.2 Numerical Determination of an ALFD

Our algorithm is analogous to what is suggested in Elliott, Müller, and Watson (2015) (EMW). It consists of two main parts: (I) determination of a candidate $e$-ALFD $A^*$; (II) numerical check of $\inf_{\theta \in \Theta} P_\theta(Y^s \in A_{\Lambda^*, cv^*}(X^s)) \geq 1 - \alpha$. Both parts are based on a discrete grid of values for $\theta$, with the grid $\Theta_{II}$ for part (II) finer and with a wider range than the grid $\Theta_I$ for part (I). In particular, with $\Delta_I = \{-0.4, -0.2, \ldots, 1.0\}$, $\Delta_{II} = \{-0.4, -0.3, \ldots, 1.0\}$, $\tilde{B}_I = \{0\} \cup \{e^{-5}, e^{-4}, \ldots, e^5\}$, $\tilde{B}_{II} =$
\[ \{0\} \cup \{e^{-5.5}, e^{-5.0}, \ldots, e^{7.5}\}, \quad C_I = \{0\} \cup \{e^{-3.0}, e^{-2.3}, \ldots, e^{4.0}\}, \quad C_{II} = \{0\} \cup \{e^{-3.35}, e^{-3.0}, \ldots, e^{7.5}\}, \]
\[ \Theta_I = \{(b,0,d) : d \in \Delta_I, ((\pi)^2)^{b^2} \in B_I\} \cup \{(0,c,d) : d \in \Delta_I, c \in C_I\} \cup \{(b,c,1.0) : c \in C_I, ((\pi)^2 + a^2)^{b^2} \in B_{II}\} \quad \text{and} \quad \Theta_{II} = \{(b,c,d) : c \in C_{II}, d \in \Delta_{II}, ((\pi)^2 + a^2)^{b^2} \in B_{II}\} \]
Also let \( \Theta_T = \{\theta : \theta = (0,0,d), d \in \{-0.4, -0.2, \ldots, 1.0\}\} \) denote the support of the prior \( \Gamma \).

The range of values for \( b \) and \( c \) in grid \( \Theta_I \) are chosen to approximately cover all possible values of \( \Sigma \) in the bcdf-model (as \( b \to \infty \) or \( c \to \infty \), \( \Sigma \) converges to its value in the I(0) model, for any value of \( d \in \Delta_{II} \)). Preliminary results suggested that the ALFD puts no mass on \( \theta \) with \( b > 0 \), \( c > 0 \) and \( d < 1 \), motivating our smaller choice for \( \Theta_I \).

With \( \Theta_I \) the support of candidate ALFDs \( \Lambda \), and \( W \) putting weight \( w_\theta \) on \( \theta \in \Theta_T \), (28) becomes
\[ A_{\Lambda,cv}(x^s) = \left\{ y^s : \sum_{\theta \in \Theta_I} \frac{\lambda_\theta f(y^s, x^s)\theta(y^s, x^s)}{\sum_{\theta \in \Theta_I} w_\theta\theta(y^s, x^s)f(x^s|\theta)} > cv \right\} \]

where \( \lambda_\theta \geq 0 \) and \( \sum_{\theta \in \Theta} \lambda_\theta = 1 \). Note that \( A_{\Lambda,cv} \) is fully determined by the ratios \( \kappa_\theta = \lambda_\theta/cv \), \( \theta \in \Theta_I \), so introduce the notation \( A_{\Lambda,cv} = A_\kappa \).

Part (I) of the algorithm seeks to determine appropriate values of \( \lambda_\theta \), \( \theta \in \Theta_I \), for an \( \epsilon \)-ALFD. The following steps are for the \( A^{MN} \) set that also enforces the Bayes superset constraint (13).

1. Generate i.i.d. draws \( Z^s_l \), \( l = 1, \ldots, N \) from \( f_p(z^s) = |\Theta_I|^{-1} \sum_{\theta \in \Theta_I} f_\theta(z^s) \) (that is, the proposal density \( f_p \) is the equal probability mixture of \( f_\theta \) with \( \theta \) drawn uniformly from \( \Theta_I \)).

2. Compute and store \( f_\theta(Z^s_l) \) and \( f_p(Z^s_l) \) for \( \theta \in \Theta_I \), and \( g_\theta(X^s_l)f_{X^s|\theta}(X^s_l) \) for \( \theta \in \Theta_T \), \( l = 1, \ldots, N \).

3. Compute \( V^{\text{known}}_\theta = \hat{V}_\theta(A^{\text{known}}_\theta) \), \( \theta \in \Theta_T \) where \( A^{\text{known}}_\theta \) is the prediction set for known \( \theta \). To be specific, \( A^{\text{known}}_\theta = \left[ \mu_\theta(x^s) - f_{\alpha/2}(1/\alpha)\sigma_\theta(x^s), \mu_\theta(x^s) + f_{\alpha/2}(1/\alpha)\sigma_\theta(x^s) \right] \) with \( \mu_\theta(x^s) = \Sigma_{XY} - \Sigma_{X\hat{X}}x^s \) and \( \sigma^2_\theta(x^s) = (\Sigma_{YY} - \Sigma_{Y\hat{X}}\Sigma_{X\hat{Y}}x^s)\Sigma_{XX}^{-1}x^s/q \), with \( \Sigma_{XX}, \Sigma_{YY}, \Sigma_{X\hat{Y}} \) and \( \Sigma_{YY} \) as implied by \( \theta \). Set \( w_\theta = (V^{\text{known}}_\theta)^{-1}/\sum_{\theta \in \Gamma_\theta}(V^{\text{known}}_\theta)^{-1}, \theta \in \Gamma_\theta \).

4. For each \( Z^s_l \), determine \( 1[Y^s_l \in A^{\text{Bayes}}(X^s_l)] = 1[F^{\text{Bayes}}(Y^s_l|X^s_l) \in (\alpha/2, 1 - \alpha/2)] \), where \( F^{\text{Bayes}} \) is the c.d.f. of the Bayes predictive distribution, that is
\[ F^{\text{Bayes}}(Y^s|X^s) = \frac{\sum_{\theta \in \Theta_T} \gamma_\theta f_{X^s|\theta}(X^s)F^q_t \left( \frac{Y^s - \mu_\theta(X^s)}{\sigma_\theta(X^s)} \right)}{\sum_{\theta \in \Theta_T} \gamma_\theta f_{X^s|\theta}(X^s)} \]
with \( F^q_t \) the c.d.f. of a student-t distribution with \( q \) degrees of freedom and \( \gamma_\theta = 1/|\Theta_T| \).

5. Set \( \kappa^{(0)} = (1, \ldots, 1) \in \mathbb{R}^{\Theta_I} \), \( \omega^{(0)} = (4, \ldots, 4) \in \mathbb{R}^{\Theta_I} \), \( \text{RP}^{(0)} = (\alpha, \ldots, \alpha) \in \mathbb{R}^{\Theta_I} \)

6. Iterate over \( i = 1, \ldots, 500 \)

(a) Compute \( \text{RP}^{(i)} = 1 - \hat{C}_\theta(A^{\kappa^{(i)}} \cup A^{\text{Bayes}}), \theta \in \Theta_I \).

(b) Set \( \kappa^{(i+1)} = \kappa^{(i)} \exp(\omega^{(i)}(\text{RP}^{(i)} - \alpha)), \theta \in \Theta_I \).
(c) If \( i \geq 400 \), set \( \omega^{(i+1)} = 0.1 \). Otherwise, set \( \omega^{(i+1)} = 1.03 \omega^{(i)} \) if \((R\!P_\theta^{(i)} - \alpha)(R\!P_\theta^{(i)} - \alpha) > 0\) and \( \omega^{(i+1)} = 0.5 \omega^{(i)} \) otherwise, \( \theta \in \Theta_I \).

7. Set \( \lambda_\theta = \kappa_\theta^{(500)} / \sum_{i \in \Theta_I} \kappa_i^{(500)} \), \( \theta \in \Theta_I \).

As in EMW, the simple idea underlying Step 6.b is a fixed-point iteration that decreases \( \kappa_\theta \) under overcoverage, and increases it otherwise, with the change (in logarithms) proportional to the degree of over- or undercoverage. The idea of Step 6.c is to half the step length if the sign of the coverage violation switched in the last iteration, and to gradually increase it otherwise. In the last 100 of the 500 iterations, a small step length is enforced regardless.

One would expect \( \lambda_\theta \) of Step 7 to be a rough approximation to the least favorable distribution with the parameter space restricted to \( \Theta_I \). With \( \Theta_I \) a sufficiently fine grid, this in turn is a candidate for an \( \epsilon \)-ALFD \( \Lambda^* \) in the sense of Definition 2. Part (II) of the algorithm checks whether \( \Lambda^* \) constructed from \( \lambda_\theta \) does indeed satisfy the properties of an \( \epsilon \)-ALFD \( \Lambda^* \).

1. Compute the number \( cv^* \) such that \( \sum_{\theta \in \Theta_I} w_\theta C_\theta(A_{\Lambda^*,cv^*} \cup A^{Bayes}) = 1 - \alpha \).

2. Compute the number \( cv^{**} \) such that \( \sum_{\theta \in \Theta_I} w_\theta \hat{V}_\theta(A_{\Lambda^*,cv^{**}} \cup A^{Bayes}) = \epsilon + \sum_{\theta \in \Theta_I} w_\theta \hat{V}_\theta(A_{\Lambda^*,cv^{**}} \cup A^{Bayes}) \).

3. Check whether \( \inf_{\theta \in \Theta_I} C_\theta(A_{\Lambda^*,cv^{**}} \cup A^{Bayes}) \geq 1 - \alpha \).

We set \( \epsilon = 0.01 \) throughout, so that in terms of the \( W \)-average expected length, the set \( A^{MN} = A_{\Lambda^*,cv^{**}} \cup A^{Bayes} \) is at most 1% longer than any set satisfying (11)-(13). We use \( N = 250,000 \), leading to Monte Carlo errors of \( \hat{C}_\theta \) of approximately 0.1-0.3%. For a given value of \( q \), \( \alpha \) and \( r \), these computations take approximately one minute on a modern PC using Fortran. We compute \( A^{MN} \) sets for \( q \in \{ 6, 12, 24, 48 \} \) and \( r \in \{ 0.075, 0.1, 0.15 \} \cup \{ 0.2, 0.3, \ldots, 1.0 \} \cup \{ 1.2, 1.4, 1.6, 1.8 \} \). In the empirical work, prediction intervals for other values of \( r \) are computed via linear interpolation.

Figure A.2 displays the approximate least favorable distributions (ALFDs) that underlie the \( A^{MN}_{(b,c,d)} \) sets. The ALFDs are displayed in terms of the local-to-zero spectral shapes from Figure 3 in the text, with thicker curves representing more mass in the ALFD. The ALFD is non-degenerate and has most of its mass on spectra that are relatively flat for larger \( \omega \), but with a pronounced pole at zero (these spectra arise, for instance, in the local-level with moderate \( b \)). Intuitively, in the local-level model, the strong mean reversion of the \( I(0) \) component masks the pronounced long-run uncertainty, making it relatively hardest to control coverage.

### A.9 Parameter values in the Monte Carlo Simulations

Models 1-4 are calibrated to quarterly data and set \( T = 260 \) quarters. Model 5 is calibrated to monthly data and sets \( T = 780 \) months.
Figure A.2: ALFDs in terms of implied local-to-zero spectra

(a) 67% $A^{MN}$ Sets

(b) 90% $A^{MN}$ Sets

Notes: The panels represent the logarithm of the local-to-zero spectra that receive more than 1% mass in the approximate least favorable distribution underlying the $A^{MN}$ predictive sets for $r=0.5$ and $q=12$. Line widths are proportional to mass in the ALFD. Same scale normalization as in Figure 3.

Model 1: $x_t = \mu_t + u_t$ with $u_t \sim iid\mathcal{N}(0, 3.7^2)$, $\Delta\mu_t = s_t\delta_t$, where $s_t \sim iid\text{Bernoulli}(p)$, and $\delta_t \sim iid(-\delta)$ with probability = 0.5, $+\delta$ with probability = 0.5. The experiments use $(p, \delta) = (0, 0)$, $(1/40, 0.125)$, $(1/40, 0.375)$, $(1/260, 0.250)$, and $(1/260, 0.750)$.

Model 2: $x_t = \mu_t + u_t$ with $(1 - 0.98L)u_t = \varepsilon_t \sim iid\mathcal{N}(0, 0.46^2)$, $\Delta\mu_t = s_t\delta_t$, where $s_t \sim iid\text{Bernoulli}(p)$, and $\delta_t \sim iid(+\delta)$ with probability = 0.5, $+\delta$ with probability = 0.5. The experiments use $(p, \delta) = (0, 0)$, $(1/40, 0.5)$, $(1/40, 1.0)$, $(1/260, 1.0)$, and $(1/260, 2.0)$.

Model 3: $x_t = \sigma_t u_t$ with $u_t \sim iid\mathcal{N}(0, 1)$, $\sigma_0 = 1$, $\Delta \ln(\sigma_t) = s_t\delta_t$, where $s_t \sim iid\text{Bernoulli}(p)$, and $\delta_t \sim iid(+\delta)$ with probability = 0.5, $+\delta$ with probability = 0.5. The experiments use $(p, \delta) = (0, 0)$, $(1/40, 0.0625)$, $(1/40, 0.1875)$, $(1/260, 0.125)$, and $(1/260, 0.375)$.

Model 4: $x_t = u_{3t} + \sigma_t u_{2t}$ with $u_{3t} \sim iid\mathcal{N}(0, 1)$, $(1 - L)u_{2t} = \varepsilon_t \sim iid\mathcal{N}(0, 0.707^2)$, $\sigma_0 = 1$, $\Delta \ln(\sigma_t) = s_t\delta_t$, where $s_t \sim iid\text{Bernoulli}(p)$, and $\delta_t \sim iid(+\delta)$ with probability = 0.5, $+\delta$ with probability = 0.5. The experiments use $(p, \delta) = (0, 0)$, $(1/40, 0.407)$, $(1/40, 0.806)$, $(1/260, 0.814)$, and $(1/260, 1.612)$.

Model 5: Within regime $k$, $\Delta x_t = \alpha_k\phi_k - \phi_k x_{t-1} + \varepsilon_t$, with $\varepsilon_t \sim iid\mathcal{N}(0, \sigma_k^2)$, where $\alpha_k \sim \max(0, \mathcal{N}(\mu_\alpha, \sigma_\alpha^2))$, $\phi_k \sim \mathcal{N}(\mu_\phi, \sigma_\phi^2)$ truncated on (0, 1), and $\sigma_k^{-1} \sim \text{Gamma}(v, d)$. Regime $k$ ends with probability $1 - p_k$, where $p_k \sim \text{Beta}(a, b)$. Values of $\mu_\alpha, \sigma_\alpha, \mu_\phi, \sigma_\phi, v, d$ were taken from Pesaran, Pettenzuzzo, and Timmermann (2006), Table 5: $\mu_\alpha = 1.96, \sigma_\alpha^2 = 125.3, \mu_\phi = 0.084, \sigma_\phi^2 = .179, v = .884, d = .050$. Values of $a, b$ are not given in Pesaran, Pettenzuzzo, and Timmermann (2006), and we take them from Pettenzuzzo and Timmermann (2011), Table 4: $a = 34.77$, and $b = 0.81$. Sample sizes of $2T$ are generated and the first $T$ observations are discarded.
Additional References


B.1 Growth rate of real per-capita GDP
Sample period: 1947:Q2 – 2014:Q4, $\bar{x}_{1,T} = 1.94$, $s_{LR} = 4.82$

Notes: The cosine transformations shown in (ii) are the standardized values, $X_{1,q}^\tau$. The low-frequency $l(d)$ likelihood is computed using $X_{1,q}^\tau$ and its asymptotic distribution given in the text; values are relative to the $l(0)$ model. The low-frequency components in (iv) are the projection of the series onto $\cos[(t-0.5)\pi j/T]$ for $j = 0, \ldots, 12$. The predictions sets in panels (v) and (vi) are $A^{\text{Bayes}}$ (thick line), $A^{\text{MN}}$ (thin line), and $A^{(0)}$ (dashed line), and are computed separately for each horizon, so coverage levels hold pointwise, not jointly across horizons.
B.2 Growth rate of real per-capita consumption expenditures
Sample period: 1947:Q2 – 2014:Q4, $\bar{x}_{1,T} = 2.09$, $s_{LR} = 4.53$

See notes to B.1.
B.3 Growth rate of total factor productivity
Sample period: 1947:Q2 – 2014:Q4, \( \bar{x}_{1R} = 1.29 \), \( s_{LR} = 3.70 \)

See notes to B.1.
B.4 Growth rate of labor productivity

Sample period: 1947:Q2 - 2014:Q4, $\bar{Y}_T = 2.15$, $s_{LR} = 3.84$

See notes to B.1.
B.5 Growth rate of population
Sample period: 1947:Q2 – 2014:Q4, $\bar{\lambda}_{1T} = 1.19$, $s_{LR} = 1.53$

See notes to B.1.
B.6 Inflation (PCE)

Sample period: 1947:Q2 – 2014:Q4, $\bar{\lambda}_{1T} = 3.14$, $s_{LR} = 9.22$

See notes to B.1.
B.7 Inflation (CPI)
Sample period: 1947:Q2 – 2014:Q4, $\bar{x}_{1T} = 3.53$, $s_{LR} = 10.38$

See notes to B.1.
B.8 Inflation (CPI, Japan)

Sample period: 1955:Q2 – 2014:Q4, $\bar{X}_{LR} = 2.96$, $s_{LR} = 15.02$

See notes to B.1.
B.9 Stock Returns
Sample period: 1947:Q2 – 2014:Q4, $\bar{x}_{LR} = 7.30$, $s_{LR} = 33.11$

See notes to B.1.
B.10 Growth rate of real per-capita GDP (annual data)
Sample period: 1901 – 2014, $\overline{\lambda}_T = 1.95, s_{LR} = 5.99$

See notes to B.1.
B.11 Growth rate of real per-capita consumption expenditures (annual data)
Sample period: 1901 – 2014, \( \bar{x}_{1.7} = 1.76 \), \( s_{LR} = 3.27 \)

See notes to B.1.
B.12 Growth rate of population (annual data)
Sample period: 1901 - 2014, $\bar{\gamma}_{1T} = 1.26$, $s_{LR} = 1.13$

See notes to B.1.
B.13 Inflation (CPI, annual data)
Sample period: 1914 - 2014, $\bar{\mu}_T = 3.15$, $s_{LR} = 8.56$

See notes to B.1.
B.14 Stock Returns

Sample period: 1926:Q2 – 2014:Q4, $\bar{\alpha}_{1r} = 6.72$, $s_{lr} = 33.56$

See notes to B.1.