Nearly Optimal Tests when a Nuisance Parameter is Present Under the Null Hypothesis*

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January 2012 (Revised June 2014)

Abstract

This paper considers nonstandard hypothesis testing problems that involve a nuisance parameter. We establish an upper bound on the weighted average power of all valid tests, and develop a numerical algorithm that determines a feasible test with power close to the bound. The approach is illustrated in six applications: inference about a linear regression coefficient when the sign of a control coefficient is known; small sample inference about the difference in means from two independent Gaussian samples from populations with potentially different variances; inference about the break date in structural break models with moderate break magnitude; predictability tests when the regressor is highly persistent; inference about an interval identified parameter; and inference about a linear regression coefficient when the necessity of a control is in doubt.

JEL classification: C12; C21; C22

Keywords: Least favorable distribution, composite hypothesis, maximin tests

*This research was funded in part by NSF grant SES-0751056 (Müller). The paper supersedes the corresponding sections of the previous working papers "Low-Frequency Robust Cointegration Testing" and "Pre and Post Break Parameter Inference" by the same set of authors. We thank Lars Hansen, Andriy Norets, Andres Santos, two referees, the participants of the AMES 2011 meeting at Korea University, of the 2011 Greater New York Metropolitan Econometrics Colloquium, and of a workshop at USC for helpful comments.

1 Introduction

Consider a statistical hypothesis test concerning a parameter $\theta = (\beta', \delta')'$ where β is the parameter of interest and δ is a nuisance parameter. Both the null and alternative are composite

$$H_0 : \beta = \beta_0, \delta \in D \quad \text{against} \quad H_1 : \beta \in B, \delta \in D \tag{1}$$

so that the null specifies the value of β, but not δ.

A key example of a hypothesis testing problem with a nuisance parameter is the Gaussian shift experiment, where the single observation Y is drawn from

$$Y = \begin{pmatrix} Y_\beta \\ Y_\delta \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \beta \\ \delta \end{pmatrix}, \Sigma \right) \tag{2}$$

and the positive definite covariance matrix Σ is known. With an unrestricted nuisance parameter space D, there are good reasons for simply ignoring Y_δ, even if Σ is not block-diagonal: For scalar β, the one-sided test of (1) based on Y_β is uniformly most powerful. In the two-sided problem, rejecting for large values of $|Y_\beta - \beta_0|$ yields the uniformly most powerful unbiased test. These arguments can be generalized to vector valued β_0 and unrestricted B by either imposing an appropriate rotational invariance for Y_β, by focussing on most stringent tests or by maximizing weighted average power on alternatives that are equally difficult to distinguish (see, for instance, Choi, Hall, and Schick (1996) and Lehmann and Romano (2005) for a comprehensive treatment and references).

These results are particularly significant because LeCam's Limits of Experiments Theory implies that inference about the parameter of a well behaved parametric model is large sample equivalent to inference in a Gaussian shift experiment. See, for instance Lehmann and Romano (2005) or van der Vaart (1998) for textbook introductions. As a consequence, the usual likelihood ratio, Wald and score tests have a well defined asymptotic optimality property also in the presence of a nuisance parameter.

These standard results only apply to the Gaussian shift experiment with unrestricted D, however. Outside this class it is sometimes possible to derive powerful tests in the presence of nuisance parameters using specific techniques. One approach is to impose invariance constraints. For example, Dufour and King's (1991) and Elliott, Rothenberg and Stock's (1996) optimal unit root tests impose translation invariance that eliminates the mean parameter. In many problems, however, invariance considerations only reduce the dimensionality of the nuisance parameter space. In the weak instrument problem with multiple instruments, for instance, rotational invariance reduces the effective nuisance parameter to the concentration parameter, a nonnegative scalar. What is more, even if an invariance transformation can be

found such that the maximal invariant is pivotal under the null hypothesis, the restriction to invariant tests might not be natural. Imposing invariance can then rule out perfectly reasonable, more powerful procedures. We provide such an example in Section 5.2 below.

A second approach is to impose similarity, unbiasedness or conditional unbiasedness. In particular, conditioning on a statistic that is sufficient for δ ensures by construction that conditional distributions no longer depend on δ. Depending on the exact problem, this allows the derivation of optimal tests in the class of all similar or conditionally unbiased tests, such as Moreira's (2003) CLR test for the weak instrument problem. The applicability of this approach, however, is quite problem specific. In addition, it is again possible that an exclusive focus on, say, similar tests rules out many reasonable and powerful tests a priori.[1]

In this paper, we adopt a well-known general solution to hypothesis tests in the presence of a nuisance parameter by integrating out the parameter θ with respect to some probability distribution Λ under the null, and some probability distribution F under the alternative. The test statistic is then simply the likelihood ratio of the resulting integrated null and alternative densities. We treat F as given, so that the problem effectively reduces to testing against the point alternative of a hyper-model where θ is drawn from F. In terms of the original composite alternative hypothesis, F represents the relative weights a researcher attaches to the power under various alternatives, so we seek tests that are optimal in the sense of maximizing weighted average power. The main concern of the paper is the probability distribution Λ under the null hypothesis, which has to be carefully matched to the problem and F. Technically, the distribution Λ that yields the optimal likelihood ratio test is known as the "least favorable distribution" (see Lehmann and Romano (2005) for details).

The least favorable approach is very general. Indeed, the standard results about the Gaussian location problem (2) reviewed above are obtained in this fashion. For nonstandard problems, however, it can be challenging to identify the least favorable distribution, and thus the efficient test. This is the problem that we address in this paper.

Our approach is based on the notion of an "approximate least favorable distribution" (ALFD), that we determine numerically. The ALFD plays two conceptually distinct roles: first, it yields an analytical upper bound on the weighted average power of *all* valid tests, and thus can be used to evaluate the optimality or near-optimality of extant tests. For example, Andrews, Moreira, and Stock (2008) show that the Moreira's (2003) CLR test essentially

[1] In the Behrens-Fisher problem Linnik (1966, 1968) and Salaevskii (1963) have shown that all similar tests have highly undesirable features, at least as long as the smaller sample has at least three observations. More recently, Andrews (2011) shows that similar tests have poor power in the context of moment inequality tests. However, it may sometimes be useful to impose similarity or other constraints on power functions, see Moreira and Moreira (2013).

achieves the power bound from an ALFD implying that the test is essentially optimal. Second, the test based on the likelihood ratio statistic with the null density integrated out with respect to the ALFD yields weighted average power close to the upper bound.

For most non-standard testing problems, much of the parameter space essentially corresponds to a standard testing problem. For instance, in the weak instrument problem, a large concentration parameter essentially turns the problem into a standard one. We extend our approach so that tests switch to the "standard" test (with high probability) in this "almost standard" part of the parameter space. The weighting function for power and ALFD thus only need to be determined in the genuinely non-standard part of the parameter space. A corresponding modification of the power bound result shows that the resulting tests are nearly weighted average power maximizing among all tests that have at least as much power as the standard test in the standard part of the parameter space. In our numerical work we determine tests whose weighted average power is within 0.5 percentage points of the bound, and this is the sense in which the tests are nearly optimal.

The algorithm may be applied to solve for nearly optimal tests in a variety of contexts: small sample and asymptotic Limit of Experiment-type problems, time series and cross section problems, nonstandard and Gaussian shift problems. Specifically, we consider six applications. First, we introduce a running example to motivate our general approach that involves the Gaussian shift problem (2) with scalar β and δ, where δ is known to be non-negative. This arises, for instance, in a regression context where the sign of the coefficient of one of the controls is known. Second, we consider the small sample problem of testing for the equality of means from two normal populations with unknown and possibly different variances, the so called "Behrens-Fisher problem". While much is known about this well-studied problem (see Kim and Cohen (1998) for a survey), small sample optimal tests have not been developed, making the application of the algorithm an interesting exercise. Third, we consider inference about the break date in a time series model with a single structural change. In this problem δ is related to the size of the parameter break, where ruling out small breaks (as, for example Bai (1994, 1997) and much of the subsequent literature) may lead to substantially over-sized tests (see Elliott and Müller (2007)). We compare our near-optimal test to the invariant tests developed in Elliott and Müller (2007), and find that the invariance restriction is costly in terms of power. The fourth example concerns inference in the predictive regression model with a highly persistent regressor. We compare our near-optimal tests to the tests derived by Campbell and Yogo (2006), and find that our tests have higher power for most alternatives. The fifth example considers nearly optimal inference about a set-identified parameter as in Imbens and Manski (2004), Woutersen (2006) and Stoye (2009). Finally, we consider a canonical model selection problem, where the parameter of

interest β is the coefficient in a linear regression, and the necessity of including a particular control variable is in doubt. It is well understood that standard model selection procedures do not yield satisfactory inference for this problem—Leeb and Pötscher (2005) provide a succinct review and references. The application of our approach here yields a power bound for the performance of any uniformly valid procedure, as well as a corresponding test with power very close to the power bound.

The remainder of the paper is organized as follows. Section 2 formally states the problem, introduces the running example, presents the analytical power bound result and uses the power bound to define the approximate least favorable distribution. This section also highlights the connection between the hypothesis testing problem and minimax decision rules as discussed, for instance, in Blackwell and Girshick (1954) and Ferguson (1967). Section 3 takes up the numerical problem of computing the ALFD, reviews the existing approaches of Kempthorne (1987), Chamberlain (2000), Srianthakumar and King (2006) and Chiburis (2009) and proposes a simple and effective algorithm that we recommend for practical use. Section 4 discusses modifications of the ALFD tests so that they (essentially) coincide with known optimal tests in the standard region of the parameter space, and the corresponding modification of the power bound. Finally, Section 5 contains the results for the additional five examples. The Appendix contains additional details on the algorithm and the applications.

2 Hypothesis Tests with Composite Null

2.1 Statement of the Problem

We observe a random element Y that takes values in the metric space $\mathcal{Y}$. The distribution of Y is parametric with parameter $\theta \in \Theta \in \mathbb{R}^k$, so that the probability density function is $f_\theta(y)$ relative to some sigma-finite measure ν. Based on this single observation, we seek to test the hypotheses

$$H_0 : \theta \in \Theta_0 \quad \text{against} \quad H_1 : \theta \in \Theta_1 \tag{3}$$

where $\Theta_0 \cap \Theta_1 = \varnothing$ and Θ_0 is not a singleton, so that the null hypothesis is composite.

Tests of (3) are measurable functions $\varphi : \mathcal{Y} \mapsto [0, 1]$, where $\varphi(y)$ indicates the rejection probability conditional on observing $Y = y$. Thus, a non-randomized test has restricted range $\{0, 1\}$, and critical region $\{y : \varphi(y) = 1\}$. If $\varphi(y) \in (0, 1)$ for some $y \in \mathcal{Y}$, then φ is a randomized test. In either case, the rejection probability of the test is equal to $\int \varphi f_\theta d\nu$ for a given $\theta \in \Theta$, so that the size of the test is $\sup_{\theta \in \Theta_0} \int \varphi f_\theta d\nu$, and by definition, a level α test has size smaller or equal to α.

In many problems, a composite null hypothesis arises due to the presence of a nuisance parameter. In a typical problem, θ can be parametrized as $\theta = (\beta', \delta')'$, where $\beta \in \mathbb{R}^{k_\beta}$ is the parameter of interest and $\delta \in \mathbb{R}^{k_\delta}$ is a nuisance parameter. The hypothesis testing problem (3) then is equivalent to

$$H_0 : \beta = \beta_0, \delta \in D \quad \text{against} \quad H_1 : \beta \in B, \delta \in D \tag{4}$$

where $\beta_0 \notin B$, $\Theta_0 = \{\theta = (\beta', \delta')' : \beta = \beta_0, \delta \in D\}$ and $\Theta_1 = \{\theta = (\beta', \delta')' : \beta \in B, \delta \in D\}$.

One motivation for the single observation problem involving Y is a small sample parametric problem, where Y simply contains the n observations (or a lower dimensional sufficient statistic). Alternatively, the single observation problem may arise as the limiting problem in some asymptotic approximation, as we now discuss.

Running example: To clarify ideas and help motivate our proposed testing procedures, we use the following example throughout the paper. (Related problems were considered by Moon and Schorfheide (2009) and Andrews and Guggenberger (2010)). Suppose we observe n observations from a parametric model with parameter $(\gamma, \eta) \in \mathbb{R}^2$. The hypothesis of interest is $H_0 : \gamma = \gamma_0$, and it is known a priori that $\eta \geq \eta_0$. For instance, γ and η may correspond to regression coefficients, and it is known that the coefficient associated with the control variable is non-negative. Let $\beta = \sqrt{n}(\gamma - \gamma_0)$ and $\delta = \sqrt{n}(\eta - \eta_0)$. If the model is locally asymptotic normal at $(\gamma, \eta) = (\gamma_0, \eta_0)$ at the usual parametric $\sqrt{n}$ rate with non-singular Fisher information matrix Σ^{-1}, then by Corollary 9.5 of van der Vaart (1998), the Limit Experiment local to (γ_0, η_0) concerns the bivariate normal observation

$$Y = \begin{pmatrix} Y_\beta \\ Y_\delta \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \beta \\ \delta \end{pmatrix}, \Sigma \right) \tag{5}$$

where Σ is known. The hypothesis testing problem concerning (5) is

$$H_0 : \beta = 0, \delta \geq 0 \quad \text{against} \quad H_1 : \beta \in B, \delta \geq 0 \tag{6}$$

where $B = (0, \infty)$ and $B = \mathbb{R}\backslash\{0\}$ correspond to one-sided and two-sided alternatives, respectively. It clear that in either case, we can normalize Σ to be unity on the diagonal without loss of generality, so that the testing problem is only indexed by the correlation $\rho \in (-1, 1)$.

By the Asymptotic Representation Theorem (van der Vaart (1998, Theorem 9.3 and Theorem 15.1)), the local asymptotic rejection profile of any test in the original n observation problem (should it exist) can be matched by a test in the single observation problem (5). What is more, for any test of (5), it is typically straightforward to construct a corresponding

test in the original parametric problem with the same asymptotic local power. Thus, the derivation of large sample tests with good local asymptotic power for the original problem reduces to the derivation of good tests for (5).

If the original parametric model concerns additional nuisance parameters, then the Limit Experiment (5) involves a larger dimensional normal variate. It is clear, however, that any valid test of the bivariate problem can still be applied, as the additional Gaussian observations in the Limit Experiment may simply be ignored (although additional arguments, such as invariance considerations, would be needed to argue for the optimality of such a procedure). A similar point applies in the presence of infinite dimensional additional nuisance parameters, that is if the underlying model is semiparametric (see Choi, Hall, and Schick (1996) for details).

Finally, one could also rely on the approach developed by Müller (2011) to argue for the asymptotic reduction to the single observation problem (5). We omit details for brevity. ▲

2.2 Weighted Average Power

The determination of a good test of (3) is difficult because both the null and the alternative are composite. A composite null requires that the test controls rejection probability over all values of $\theta \in \Theta_0$; a composite alternative leads to the consideration of how the test's power varies over $\theta \in \Theta_1$. A standard approach for composite alternatives is to consider weighted average power as the scalar criterion to choose among tests

$$\text{WAP}(\varphi) = \int \left(\int \varphi f_\theta d\nu \right) dF(\theta), \tag{7}$$

where F is a probability measure with support in (the closure of) Θ_1. The weighting function F describes the importance a researcher attaches to the ability of the test to reject under different alternatives. This approach underlies the optimality of Wald's (1943) statistics and has been employed in the influential work by Andrews and Ploberger (1994).

Since tests that maximize WAP equivalently maximize $\int \varphi \left(\int f_\theta dF(\theta) \right) d\nu$ (where the interchange of the order of integration is allowed by Fubini's Theorem), efficient tests under the WAP criterion also maximize power against the single density $g = \int f_\theta dF(\theta)$. Thus, with a WAP criterion, the hypothesis testing problem (3) becomes one of finding a powerful test for the problem

$$H_0 : \text{the density of } Y \text{ is } f_\theta,\ \theta \in \Theta_0 \qquad \text{against} \qquad H_{1,F} : \text{the density of } Y \text{ is } g = \int f_\theta dF(\theta) \tag{8}$$

where the alternative $H_{1,F}$ is simple. The power of a test under $H_{1,F}$ is synonymous to weighted average power under the composite alternative H_1 with weighting function F.

If a uniformly most powerful test exists, then it maximizes WAP for all choices of F, so that in this sense a focus on WAP is without loss of generality. In most problems, however, the choice of the weighting function F matters, as there is no uniformly most powerful test: there are many tests whose power functions cross, and one can reasonably disagree about the overall preferred test. We therefore offer no general remarks about F, but rather discuss our choices in the context of the running example and the particular testing problems analyzed in Section 5.

2.3 A Set of Power Bounds

Under the weighted average power criterion (7) the challenge is to derive a good test of a composite null against a simple alternative, that is a good test of (8). This subsection does not derive such a test directly, but rather provides a general set of bounds on the power of any level α test. These bounds are useful both for constructing an approximately efficient test and for evaluating the efficiency of *ad hoc* tests.

Suppose the composite null hypothesis in (8) is replaced by the single hypothesis

$$H_{0,\Lambda} : \text{The density of } Y \text{ is } \int f_\theta d\Lambda(\theta)$$

where Λ is a probability distribution with support on Θ_0. In general, the size α Neyman-Pearson test of $H_{0,\Lambda}$ against $H_{1,F}$ is *not* a level α test of H_0 in (8), as its null rejection probability is equal to α by definition only when Y is drawn from the mixture distribution $\int f_\theta d\Lambda(\theta)$ and does not satisfy the size constraint for the composite null H_0. Its properties are nevertheless helpful to bound the power of any level α test of (8).

Lemma 1 *Let φ_Λ be the size α test of $H_{0,\Lambda}$ against $H_{1,F}$ of the Neyman-Pearson form*

$$\varphi_\Lambda(y) = \begin{cases} 1 \text{ if } g(y) > \text{cv} \int f_\theta(y) d\Lambda(\theta) \\ \kappa \text{ if } g(y) = \text{cv} \int f_\theta(y) d\Lambda(\theta) \\ 0 \text{ if } g(y) < \text{cv} \int f_\theta(y) d\Lambda(\theta) \end{cases} \tag{9}$$

for some $\text{cv} \geq 0$ *and* $0 \leq \kappa \leq 1$. *Then for any level α test φ of H_0 against $H_{1,F}$, $\int \varphi_\Lambda g d\nu \geq \int \varphi g d\nu$.*

Proof. *Since φ is a level α test of H_0, $\int \varphi f_\theta d\nu \leq \alpha$ for all $\theta \in \Theta_0$. Therefore, $\int \int \varphi f_\theta d\nu d\Lambda(\theta) = \int \int \varphi f_\theta d\Lambda(\theta) d\nu \leq \alpha$, where the equality follows from Fubini's Theorem, so that φ is also a level α test of $H_{0,\Lambda}$ against $H_{1,F}$. The result now follows from the Neyman-Pearson Lemma.* ■

Lemma 1 formalizes the intuitive result that replacing the composite null hypothesis H_0 with the single mixture null hypothesis $H_{0,\Lambda}$ can only simplify the testing problem in the sense of allowing for more powerful tests. Its appeal lies in the fact that the power of the test φ_Λ can be easily computed. Thus, Lemma 1 provides a set of explicit power bounds on the original problem, indexed by the distribution Λ.

Running example, ctd: Suppose $\rho = \text{corr}(Y_\beta, Y_\delta) = -1/2$ in the running example, and consider maximizing weighted average power for the degenerate distribution F that puts all mass at $\theta_1 = (\beta, \delta)' = (1, 0)'$. Further, choose Λ as a degenerate distribution with all its mass at $\theta_0 = (0, 1)'$. The likelihood ratio test φ_Λ of $H_{0,\Lambda}$ against $H_{1,F}$ then rejects for large values of $Y_\beta - Y_\delta$. Since $Y_\beta - Y_\delta | H_{0,\Lambda} \sim \mathcal{N}(-1, 3)$, $\varphi_\Lambda(y) = \mathbf{1}[y_\beta - y_\delta > 1.85]$, where the critical value 1.85 is chosen to produce a rejection probability of 5% under $H_{0,\Lambda}$. Note that φ_Λ is not a valid 5% level test of $H_0 : \beta = 0$, $\delta \geq 0$, since it has a rejection probability greater than 5% when $\delta < 1$. Under the alternative, $Y_\beta - Y_\delta | H_{1,F} \sim \mathcal{N}(1, 3)$, so that the power of φ_Λ is given by $\int \varphi_\Lambda g d\nu = 0.31$. While φ_Λ may not control size under H_0, Lemma 1 implies that any 5% level test of $H_0 : \beta = 0$, $\delta \geq 0$ against $H_{1,F}$ has power that does not exceed 0.31. ▲

Lemma 1 follows directly from the arguments leading to a standard result concerning tests with a composite null hypothesis; see, for instance, Theorem 3.8.1 of Lehmann and Romano (2005): A distribution $\Lambda^\dagger$ is *least favorable* if the best level α test of $H_{0,\Lambda^\dagger}$ against the single alternative $H_{1,F}$ is also of level α in the testing problem with the composite null hypothesis H_0 against $H_{1,F}$, so that—using the same reasoning as in the proof of Lemma 1—this test is also the best test of H_0 against $H_{1,F}$. In contrast to this standard result, Lemma 1 is formulated without any restriction on the probability distribution Λ. This is useful because in many contexts, it is difficult to identify the least favorable distribution $\Lambda^\dagger$.

2.4 Using the Power Bound to Gauge Potential Efficiency of *ad hoc* Tests

It is sometimes possible to construct an *ad hoc* test φ_{ah} of (3) that is known to be of level α, even if the nuisance parameter space is high dimensional, but φ_{ah} has no optimality property by construction. The power bounds from Lemma 1 can then be used to check its efficiency: if the (weighted average) power of φ_{ah} is close to the power bound arising from some distribution Λ, then φ_{ah} is known to be close to optimal, as no substantially more powerful test exists. The check is partial, though, as a large difference between the power of φ_{ah} and the bound can arise either because φ_{ah} is inefficient, or because this specific Λ yields a bound far above the least upper bound.

For this strategy to work, one must try to guess a Λ that yields a low power bound. Intuitively, a low power bound arises if the density of Y under $H_{0,\Lambda}$ is close to the density g under the alternative $H_{1,F}$. This may suggest a suitable choice of Λ directly. Alternatively, one can parametrize Λ in some suitable fashion, and numerically minimize some convenient distance between $\int f_\theta d\Lambda(\theta)$ and g. For example, the testing problem of Müller and Watson (2013b) involves hypotheses about the covariance matrix of a mean zero multivariate normal, which under the null hypothesis is a function of a high dimensional nuisance parameter $\delta \in D$. With $\Lambda = \Lambda_\delta$ restricted to put point mass at some δ, one can use the Kullback-Leibler divergence between the null and alternative density as a convenient distance function, and use numerical methods to find Λ_δ. In that application, the resulting power bound comes close to the power of a particular *ad hoc* test, which shows that the *ad hoc* test is close to efficient, and also that the power bound computed in this fashion is close to the least power bound. As a second example, Andrews, Moreira, and Stock (2008) show that Moreira's (2003) CLR test almost achieves the power bound in a weak instrument IV testing problem, and thus is nearly optimal in that context.

2.5 Approximately Least Favorable Distributions

The least favorable distribution $\Lambda^\dagger$ has the property that the size α Neyman-Pearson test $\varphi_{\Lambda^\dagger}$ of the simple hypothesis $H_{0,\Lambda^\dagger}$ against $H_{1,F}$ also yields a level α test of the composite null hypothesis H_0 against $H_{1,F}$. As noted above, for many problems it is difficult to analytically determine $\Lambda^\dagger$. A natural reaction is then to try to numerically approximate $\Lambda^\dagger$. In many problems, however, it is non-trivial to approximate $\Lambda^\dagger$ arbitrarily well, as its definition depends on the typically unbounded number of constraints $\int \varphi_{\Lambda^\dagger} f_\theta d\nu \leq \alpha$ for all $\theta \in \Theta_0$. To ease the computational burden, it would be useful to be able to determine whether a potentially coarse approximation to $\Lambda^\dagger$ is good enough in terms of generating a test with near optimal power.

Lemma 1 is very helpful in this regard. Specifically, consider the following definition of an approximate least favorable distribution (ALFD).

Definition 1 *An* ε-ALFD *is a probability distribution* Λ^* *on* Θ_0 *satisfying*

(i) the Neyman-Pearson test (9) with $\Lambda = \Lambda^*$ *and* $(\mathrm{cv}, \kappa) = (\mathrm{cv}^*, \kappa^*)$, φ_{Λ^*}, *is of size* α *under* H_{0,Λ^*}, *and has power* $\bar{\pi}$ *against* $H_{1,F}$;

(ii) there exists $(\mathrm{cv}^{*\varepsilon}, \kappa^{*\varepsilon})$ *such that the test (9) with* $\Lambda = \Lambda^*$ *and* $(\mathrm{cv}, \kappa) = (\mathrm{cv}^{*\varepsilon}, \kappa^{*\varepsilon})$, $\varphi^\varepsilon_{\Lambda^*}$, *is of level* α *under* H_0, *and has power of at least* $\bar{\pi} - \varepsilon$ *against* $H_{1,F}$.

Suppose that a suitable ε-ALFD can be identified, where ε is small. By (ii), $\varphi^\varepsilon_{\Lambda^*}$ is a level

α test under H_0, and by (i), (ii) and Lemma 1, it has power that is within ε of the power bound. Thus $\varphi^{\varepsilon}_{\Lambda^*}$ is a nearly optimal test of H_0 against $H_{1,F}$.

Crucially, the demonstration of near optimality of $\varphi^{\varepsilon}_{\Lambda^*}$ only requires the rejection probability of $\varphi^{\varepsilon}_{\Lambda^*}$ under H_0 and the rejection probabilities of φ_{Λ^*} and $\varphi^{\varepsilon}_{\Lambda^*}$ under $H_{1,F}$, respectively. Thus, the argument is *not* based on the notion that Λ^* is necessarily a good approximation to the actual least favorable distribution $\Lambda^{\dagger}$ (should it exists) in some direct sense. Rather, any Λ^* that satisfies the two parts of Definition 1 yields a demonstrably nearly optimal test $\varphi^{\varepsilon}_{\Lambda^*}$ of H_0 against $H_{1,F}$.

2.6 A Decision Theoretic Interpretation of the ALFD

From a decision theoretic perspective, most powerful tests for composite hypotheses are related to minimax rules in the problem of deciding between H_0 against $H_{1,F}$ (Blackwell and Girshick (1954), Chapter 7.7), and the test based on the ALFD is an approximate minimax decision rule.

To be precise, consider the hypothesis testing problem (8) as a decision problem where a false rejection of H_0 induces a loss of 1, a false rejection of H_1 induces a loss of $\phi > 0$, and correct decision has a loss of 0. The (frequentist) risk of the decision rule φ then is $R_0(\varphi, \theta) = \int \varphi f_\theta d\nu$, $\theta \in \Theta_0$ under H_0 and $R_1(\varphi) = \phi \int (1-\varphi) g d\nu$ under $H_{1,F}$. The largest risk under H_0, $\alpha(\varphi) = \sup_{\theta \in \Theta_0} \int \varphi f_\theta d\nu$ is recognized as the size of the test φ, and the power of the test under $H_{1,F}$, $\pi(\varphi)$ equals $1 - R_1(\varphi)/\phi$. Nature's strategies consist of drawing Y from H_0 with probability $0 \leq q \leq 1$, and from H_1 with probability $1-q$ and, conditional on drawing from H_0, drawing θ from Λ, a probability distribution with support in Θ_0. An adversarial nature seeks to maximize expected risk, that is to choose (q, Λ) that maximize $r(\varphi, q, \Lambda) = q \int R_0(\varphi, \theta) d\Lambda(\theta) + (1-q) R_1(\varphi)$. Note that for any (q, Λ), $\inf_\varphi r(\varphi, q, \Lambda) \leq \inf_\varphi \max\left(\sup_{\theta \in \Theta_0} R_0(\varphi, \theta), R_1(\varphi)\right) = \bar{V}$, the econometrician's minimax risk.

For any prior (q, Λ), the posterior probabilities are $p^0_{q,\Lambda}(y) = q \int f_\theta(y) d\Lambda(\theta) / f(y)$ under H_0 and $p^1_{q,\Lambda}(y) = (1-q) g(y)/f(y)$ under $H_{1,F}$, where the marginal likelihood $f(y)$ is given by $f(y) = q \int f_\theta(y) d\Lambda(\theta) + (1-q) g(y)$. The posterior expected loss of decision φ equals $\varphi(y) p^0_{q,\Lambda} + \phi(1 - \varphi(y)) p^1_{q,\Lambda}(y)$, so that Bayes rules are of the form

$$\varphi^B(q, \Lambda) = \begin{cases} 1 \text{ if } g(y) > \frac{q}{\phi(1-q)} \int f_\theta(y) d\Lambda(\theta) \\ \kappa \text{ if } g(y) = \frac{q}{\phi(1-q)} \int f_\theta(y) d\Lambda(\theta),\ 0 \leq \kappa \leq 1 \\ 0 \text{ if } g(y) < \frac{q}{\phi(1-q)} \int f_\theta(y) d\Lambda(\theta) \end{cases}$$

mimicking the form of the Neyman-Pearson test in (9). Minimizing posterior expected loss for each draw y yields a decision that minimizes prior weighted risk, that is $\inf_\varphi r(\varphi, q, \Lambda) =$

$r(\varphi^B(q,\Lambda),q,\Lambda)$.

Now consider an ε-ALFD Λ^* in the sense of Definition 1, and set $\phi = \alpha/(1-\bar{\pi})$. Then $R_1(\varphi_{\Lambda^*}) = \alpha$, and since φ_{Λ^*} is of level α under H_{0,Λ^*}, $\int R_0(\varphi_{\Lambda^*},\theta)d\Lambda^*(\theta) = \alpha$. Furthermore, let $0 \leq q^* \leq 1$ solve $\frac{q^*}{1-q^*} = \phi\,\mathrm{cv}^*$, so that $\varphi_{\Lambda^*} = \varphi^B(q^*,\Lambda^*)$. Thus $\bar{V} \geq \inf_\varphi r(\varphi,q^*,\Lambda^*) = r(\varphi_{\Lambda^*},q^*,\Lambda^*) = q^*\alpha + (1-q^*)\alpha = \alpha$. By definition of an ε-ALFD, the adjusted test $\varphi^\varepsilon_{\Lambda^*}$ has size α and power within ε of $\bar{\pi}$. Thus $\sup_{\theta\in\Theta_0} R_0(\varphi^\varepsilon_{\Lambda^*},\theta) \leq \alpha$ and $R_1(\varphi^\varepsilon_{\Lambda^*}) \leq \alpha + \alpha\varepsilon/(1-\bar{\pi})$, so that the maximal risk of $\varphi^\varepsilon_{\Lambda^*}$ exceeds the lower bound of α by at most $\alpha\varepsilon/(1-\bar{\pi})$. In this sense, $\varphi^\varepsilon_{\Lambda^*}$ is an approximate minimax decision rule.

Minimax rules are inherently pessimistic, and they might be considered unattractive if they are rationalized by an unreasonable distribution for θ. This can be assessed for a given test $\varphi^\varepsilon_{\Lambda^*}$ derived with the algorithm developed in the next section by inspecting the ALFD Λ^*. From a Bayesian perspective, the ALFD might be used as a prior selection device, which guarantees attractive frequentist properties of Bayes rule $\varphi^\varepsilon_{\Lambda^*} = \varphi^B(q^{*\varepsilon},\Lambda^*)$, where $q^{*\varepsilon}$ solves $\frac{q^{*\varepsilon}}{1-q^{*\varepsilon}} = \phi\,\mathrm{cv}^{*\varepsilon}$.

The exact least favorable distribution (should it exist) arises naturally in this decision theoretic perspective by considering nature's best strategy: the largest risk that nature can induce is $\underline{V} = \sup_{q,\Lambda}\inf_\varphi r(\varphi,q,\Lambda)$. If the maximin theorem holds (a sufficient condition is finiteness of Θ_0; see, for instance, Theorem 2.9.1 in Ferguson (1967)), then the game has value $V = \bar{V} = \underline{V}$. Furthermore, there exists a least favorable prior $(q^\dagger,\Lambda^\dagger)$ such that $\varphi^\dagger = \varphi^B(q^\dagger,\Lambda^\dagger)$ achieves the maximin risk V. Its level $\alpha(\varphi^\dagger) = V = \sup_{\theta\in\Theta_0}\int\varphi^\dagger f_\theta d\nu$ is endogenously determined by the structure of the problem (f_θ and g) and the loss function parameter ϕ. Also note that $V = r(\varphi^\dagger,q^\dagger,\Lambda^\dagger)$ implies that the least favorable prior $(q^\dagger,\Lambda^\dagger)$ has positive mass only on points that lead to maximum risk.

3 Numerical Determination of Nearly Optimal Tests

We now discuss the suggested numerical algorithm to determine an ε-ALFD for small ε, and thus a nearly WAP maximizing test.

In a first step, it is useful to approximate H_0 by a set of finitely many distributions for Y. A straightforward way to achieve this is to simply discretize Θ_0 into a finite number of values. As long as Θ_0 is compact, some continuity implies that controlling size on a fine enough grid on Θ_0, $\{\theta_i\}_{i=1}^M \subset \Theta_0$ leads to at worst minimally oversized tests under the unrestricted H_0. Note that discretizing Θ_0 is only one way to discretize H_0; alternatively, one can also specify a finite set of "base" distributions Ψ_i, $i = 1,\ldots,M$, each with support in Θ_0, so that the M possible densities for Y under H_0 are of the form $f_i = \int f_\theta d\Psi_i(\theta)$. This reduces to a grid on

Θ_0 if the Ψ_i represent point masses. It is advantageous to use non-degenerate Ψ_i's, which lead to smoother rejection regions for the same number M.

With H_0 discretized, Λ is described by a point in the M dimensional simplex $\Lambda = (\lambda_1, \ldots, \lambda_M)$. The Neyman-Pearson test (9) is of the form[2] $\varphi_\Lambda(y) = \mathbf{1}[g(y) > \mathrm{cv}_\Lambda \sum_{i=1}^M \lambda_i f_i(y)]$, with critical value cv_Λ determined by $\int (\sum_{i=1}^M \lambda_i f_i)\varphi_\Lambda d\nu = \alpha$. It is convenient to optimize in $\mathbb{R}^M$ instead of the M dimensional simplex, and to subsume the critical value cv_Λ into the weights. Thus, let $\mu_i = \ln(\mathrm{cv}_\Lambda \lambda_i) \in \mathbb{R}$ and $\mu = (\mu_1, \ldots, \mu_M)$, so that $\varphi_\mu = \mathbf{1}[g > \sum_{i=1}^M \exp(\mu_i) f_i]$. Our suggested approach is to simply repeatedly adjust μ as a function of the rejection probabilities of φ_μ: start with some $\mu^{(0)} \in \mathbb{R}^M$, and for a suitable $\omega > 0$, set

$$\mu_j^{(i+1)} = \mu_j^{(i)} + \omega\Big(\int \varphi_{\mu^{(i)}} f_j d\nu - \alpha\Big) \tag{10}$$

for $i = 1, \ldots, O$. An iteration of (10) increases the weights $\exp(\mu_j)$ on those f_j that lead to overrejections relative to α, and decreases the weights $\exp(\mu_j)$ for those that lead to an underrejection. With sufficiently many iterations on (10) the implied $\hat{\Lambda} \propto (\exp \mu_1^{(O)}, \ldots, \exp \mu_M^{(O)})$ then serves as a candidate for an ε-ALFD Λ^* in the sense of Definition 1. So it remains to check that the adjusted test $\varphi_{\hat{\Lambda}}^\varepsilon$ controls size under the original null hypothesis. This may be determined by a fine grid search;[3] if that search reveals overrejections, then the algorithm is restarted with a finer discretization of H_0 (larger M). More comments on the efficient computation of $\int \varphi_{\mu^{(i)}} f_j d\nu$ and a step-by-step description of the algorithm are in Appendix A.2.1.

We found that this algorithm reliably generates an ε-ALFD in all problems we considered. In our experience, the weights $\exp(\mu_j^{(O)})$ generated by (10) are numerically fairly insensitive to the starting value $\mu^{(0)}$, the tuning parameter ω, and the number of iterations O (as long as O is larger than, say, 300). Also, in simple problems where this is easy to assess (such as in the running example), different choices for the discretization of H_0 end up yielding tests $\varphi_{\Lambda^*}^\varepsilon$ with very similar critical regions. The number of densities in H_0 can be chosen fairly large (say, larger than 100) without much difficulty; in Müller and Watson (2013a), this algorithm was employed to determine an ε-ALFD in a problem involving a 3 dimensional nuisance parameter, using a discretization with $M = 204$ (strategically chosen) points.

The literature contains a number of alternative approaches to approximating least favorable distributions. One simple but powerful approach is exploit the linearity of the rejec-

[2] This holds at least as long all convex combinations of $g(Y)$ and $f_i(Y)$, $i = 1, \cdots, M$ have an absolutely continuous distribution, which is the case in all applications we consider.

[3] Elliott and Müller (2012) develop a technique for numerically checking size control without discretization of H_0.

tion probability $\int \varphi f_\theta d\nu$ as a function of φ to obtain a linear programming problem. See Krafft and Witting (1967) for a general discussion of the relationship between least favorable distributions and linear programming, and Chiburis (2009) and Moreira (2003) for recent implementations. A disadvantage is, however, that unless the sample space $\mathcal{Y}$ can usefully be partitioned into a small number of regions, both the primal and dual of the linear program are of (potentially very) high dimension.

Kempthorne (1987) provides a numerical algorithm to determine maximin rules for general decision problems, which could be adopted to the testing problem described in Section 2.6. But the algorithm involves repeated maximizations of convex combinations of null rejection probabilities, both as function of the weights, and as a function of null parameter values. A more attractive variant of determining maximin rules for discrete H_0 is developed in Chamberlain (2000): If H_0 consists of M possible distributions for Y, then the prior (q, Λ) can be represented as a point in the $M+1$ dimensional simplex. Chamberlain notes that the Bayes risk $r(\varphi, q, \Lambda)$ of the Bayes rule $\varphi = \varphi^B(q, \Lambda)$ is a concave function on this simplex. The least favorable distribution $(q^\dagger, \Lambda^\dagger)$ maximizes Bayes risk, so that it can be determined using concave programming. The level of the resulting maximin test is an endogenous function of the loss function parameter ϕ, however, so that the least favorable distribution would have to be determined repeatedly to obtain a test for a given level α.

A third approach is based on the observation that the least favorable prior $\Lambda^\dagger$ puts all its mass on the values of $\theta \in \Theta_0$ where $\varphi_{\Lambda^\dagger}$ has rejection probability equal to α. Thus, with H_0 discretized, the problem of determining $\Lambda^\dagger = (\lambda_1^\dagger, \ldots, \lambda_M^\dagger)$ can be cast as the nonlinear complementarity problem (NCP) $\lambda_i^\dagger \geq 0$, $\alpha - \int \varphi_{\Lambda^\dagger} f_i d\nu \geq 0$ and $\lambda_i^\dagger(\alpha - \int \varphi_{\Lambda^\dagger} f_i d\nu) = 0$, $i = 1, \ldots, M$. See Facchinei and Pang (2003) for a comprehensive overview and references on the literature on algorithms for NCPs. One possibility is to cast the NCP into a constrained nonlinear optimization problem, such as $\min_\Lambda \sum_i \mathbf{1}[\lambda_i > 0]\lambda_i^2(\alpha - \int \varphi_\Lambda f_i d\nu)^2$ subject to $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. The algorithm of Srianantha kumar and King (2006) builds on this approach.

4 Switching Tests

In most problems of interest, the nuisance parameter space is unbounded. This raises issues for the approach discussed so far both under the null and alternative hypothesis. Under the null hypothesis, discretizations of the null parameter space for non-compact Θ_0 are inherently coarse, complicating the algorithm discussed in the last section. Under the alternative, attempting to use a WAP criterion for a non-compact Θ_1 faces the problem that any inte-

grable F puts almost all weight on a compact subset. Thus, any such F effectively puts very little weight on some large region of Θ_1, leading to potentially poor power properties in that region.

To address this issue, note that for many nonstandard testing problems involving a nuisance parameter $\delta \in \mathbb{R}$, one can choose a parameterization in which the testing problem for large values of δ essentially reduces to a standard problem. For example, in the weak instrument problem with concentration parameter δ, a large δ implies that the instruments are "almost" strong; inference problems involving a local-to-unity parameter $\delta \geq 0$, such as predictive regressions studied in Cavanagh, Elliott, and Stock (1995) and Jansson and Moreira (2006), essentially reduce to standard stationary time series problems as $\delta \to \infty$; and similarly, in our running example the problem becomes standard as $\delta \to \infty$.

It seems sensible, therefore, to consider tests that essentially reduce to the "standard best test" in the standard problem when δ is large, and to employ the weighted average power criterion only to ensure near optimality for small δ. This has the additional advantage that size control only needs to be carefully checked in the non-standard case of small δ.

We proceed in three steps: First, we formally discuss the convergence to a standard problem. Second, we determine the nearly WAP maximizing tests in a class of tests that, by a functional form restriction, have nearly the same rejection properties as the standard best test for δ large. Third, we examine whether the restriction to this class is costly in terms of weighted average power.

4.1 Limiting Problem

Consider the testing problem (4) under a reparameterization of $\theta = (\beta', \delta')' \in \mathbb{R}^{k_\beta+1}$ in terms of $\Delta, d \in \mathbb{R}$, and $b \in \mathbb{R}^{k_\beta}$ via $\delta = r_\delta(\Delta, d)$ and $\beta = r_\beta(\Delta, b)$, where $\Delta \to \infty$ implies $\delta \to \infty$ for fixed d, and $b = 0$ implies $\beta = \beta_0$. Think of Δ as some approximate baseline value of δ, and of d as the deviation of δ from this baseline. Now construct a sequence of testing problems, indexed by n, by setting $\Delta = \Delta_n$, but for fixed $h = (b, d) \in \mathbb{R}^{k_\beta+1}$. Denote by $f_{n,h}$ the density of Y in this parameterization (and define it arbitrarily if the implied $\theta \notin \Theta_0 \cup \Theta_1$). Further, let $X \in \mathcal{X}$ be a random element with density $f_{X,h}$ relative to some dominating measure, and $h \in \mathbb{R}^{k_\beta+1}$. The following condition provides sufficient assumptions to ensure convergence as $\Delta_n \to \infty$ to the experiment of observing the single random element X.[4]

Condition 1 *(i) Suppose that f_{X,h_1} is absolutely continuous relative to f_{X,h_2}, for all $h_1, h_2 \in \mathbb{R}^{k_\beta+1}$.*

[4]See the proof of Lemma 2 for details on how Condition 1 implies convergence of experiments.

(ii) For all sequences $\Delta_n \to \infty$ and fixed finite subsets $H \subset \mathbb{R}^{k_\beta+1}$

$$\left\{\frac{f_{n,h}(Y)}{f_{n,0}(Y)}\right\}_{h\in H} \Rightarrow \left\{\frac{f_{X,h}(X)}{f_{X,0}(X)}\right\}_{h\in H} \tag{11}$$

where Y and X are distributed according to $f_{n,0}$ and $f_{X,0}$, respectively.

The experiment involving X is typically much easier than that involving Y; in most of our applications, it is simply an unrestricted Gaussian shift experiment (2). Denote by $\varphi_S^{\lim} : \mathcal{X} \mapsto [0,1]$ the "standard best test" of level α in the limiting problem of

$$H_0 : b = 0, d \in \mathbb{R} \quad \text{against} \quad H_1 : b \neq 0, d \in \mathbb{R} \tag{12}$$

based on the single observation X. Suppose further that there exists a test $\varphi_S : \mathcal{Y} \mapsto [0,1]$ in the original problem that has the same asymptotic rejection properties as $\varphi_S^{\lim}$ as $\Delta_n \to \infty$ for all fixed values of h.

Running example: Set $r_\beta(\Delta_n, b) = b$ and $r_\delta(\Delta_n, d) = \Delta_n + d$. For any fixed $h = (b,d) \in \mathbb{R}^2$ and all $\Delta_n \geq -d$

$$\log \frac{f_{n,h}(Y)}{f_{n,0}(Y)} = \begin{pmatrix} Y_\beta \\ Y_\delta - \Delta_n \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} b \\ d \end{pmatrix} - \tfrac{1}{2} \begin{pmatrix} b \\ d \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} b \\ d \end{pmatrix}. \tag{13}$$

Under $h = 0$, for any $\Delta_n > 0$, $(Y_\beta, Y_\delta - \Delta_n)' \sim \mathcal{N}(0, \Sigma)$. One can therefore apply the reasoning in the proof of Theorem 9.4 in van der Vaart (1998) to show that (11) holds with

$$X = \begin{pmatrix} X_b \\ X_d \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} b \\ d \end{pmatrix}, \Sigma\right). \tag{14}$$

As discussed in the introduction, the usual (admissible) test of $H_0 : b = 0, d \in \mathbb{R}$ versus $H_1 : |b| > 0, d \in \mathbb{R}$ in this limiting problem is of the form $\varphi_S^{\lim}(X) = \mathbf{1}[|X_b| > \text{cv}]$. The power properties of this test are obtained in the original problem as $\Delta_n \to \infty$ by the test $\varphi_S(Y) = \mathbf{1}[|Y_\beta| > \text{cv}]$. ▲

Weak instruments example: Consider inference in a linear regression with a single endogeneous variable and a weak instrument. In Hillier's (1990) and Chamberlain's (2007) parameterization, the problem becomes inference about $\beta \in (-\pi, \pi] = B$ based on the bivariate observation

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \delta \sin\beta \\ \delta \cos\beta \end{pmatrix}, I_2\right) \tag{15}$$

with $\delta \geq 0$ measuring the strength of the instrument. Now consider the hypothesis test $H_0 : \beta = \beta_0 < \pi$ against $H_a : \beta \neq \beta_0$, and set $\delta = \Delta_n + d$ and $\beta = \beta_0 + b\Delta_n^{-1}$.[5] Then under $h = 0$, as $\Delta_n \to \infty$

$$\begin{aligned} \log \frac{f_{n,h}(Y)}{f_{n,0}(Y)} &= -\tfrac{1}{2} \left\| \begin{pmatrix} Y_1 - (\Delta_n + d)\sin(\beta_0 + b\Delta_n^{-1}) \\ Y_2 - (\Delta_n + d)\cos(\beta_0 + b\Delta_n^{-1}) \end{pmatrix} \right\|^2 + \tfrac{1}{2} \left\| \begin{pmatrix} Y_1 - \Delta_n \sin\beta_0 \\ Y_2 - \Delta_n \cos\beta_0 \end{pmatrix} \right\|^2 \\ &\Rightarrow \begin{pmatrix} Z_b \\ Z_d \end{pmatrix}' \begin{pmatrix} b \\ d \end{pmatrix} - \tfrac{1}{2} \begin{pmatrix} b \\ d \end{pmatrix}' \begin{pmatrix} b \\ d \end{pmatrix} \end{aligned}$$

with $(Z_b, Z_d)' \sim \mathcal{N}(0, I_2)$, since $\Delta_n(\sin(\beta_0 + b\Delta_n^{-1}) - \Delta_n \sin\beta_0) \to b\cos\beta_0$, $d\sin(\beta_0 + b\Delta_n^{-1}) \to d\sin\beta_0$, $\Delta_n(\cos(\beta_0 + b\Delta_n^{-1}) - \cos\beta_0) \to -b\sin\beta_0$, $d\cos(\beta_0 + b\Delta_n^{-1}) \to d\cos\beta_0$ and $(Y_1 - \Delta_n \sin\beta_0, Y_2 - \Delta_n \cos\beta_0) \sim \mathcal{N}(0, I_2)$, so (11) holds with X as defined in (14) and $\Sigma = I_2$ by Theorem 9.4 of van der Vaart (1998). The power properties of the standard test $\varphi_S^{\lim}(X) = \mathbf{1}[|X_b| > \text{cv}]$ are obtained in the original problem as $\Delta_n \to \infty$ by the test $\varphi_S(Y) = \mathbf{1}[\hat{\delta}|\hat{\beta} - \beta_0| > \text{cv}]$, where $(\hat{\beta}, \hat{\delta})$ is the MLE in (15). ▲

The convergence of experiments implies that one cannot systematically outperform φ_S for large δ, at least as long as $\varphi_S^{\lim}$ is an admissible test in the limiting problem, as the following Lemma makes precise. Write $E_{(b,\Delta)}[\cdot]$ for integration with respect to the density of Y in the parameterization $\delta = r_\delta(\Delta, 0)$ (that is, $d = 0$) and $\beta = r_\beta(\Delta, b)$ defined above, and $E_h[\cdot]$ for integration with respect to $f_{X,h}$.

Lemma 2 *Let $\varphi_S^{\lim}$ be a level α test of (12) with rejection probability $E_{(b,d)}[\varphi_S^{\lim}(X)]$ that does not depend on d, and assume that $\varphi_S^{\lim}$ is admissible in the sense that $E_{(b_1,d)}[\varphi^{\lim}(X)] > E_{(b_1,d)}[\varphi_S^{\lim}(X)]$ for some $b_1 \neq 0$ and level α test $\varphi^{\lim}$ of (12) implies existence of $b_2 \neq 0$ where $E_{(b_2,d)}[\varphi^{\lim}(X)] < E_{(b_2,d)}[\varphi_S^{\lim}(X)]$. Under Condition 1, for any level α test φ in the original problem (4) and any $b_1 \neq 0$,*

$$\limsup_{\Delta \to \infty} (E_{(b_1,\Delta)}[\varphi(Y)] - E_{(b_1,\Delta)}[\varphi_S(Y)]) > 0$$

implies the existence of $b_2 \neq 0$ such that

$$\liminf_{\Delta \to \infty} (E_{(b_2,\Delta)}[\varphi(Y)] - E_{(b_2,\Delta)}[\varphi_S(Y)]) < 0.$$

[5] The restriction to $\beta_0 < \pi$ is only for notational convenience; the result still goes through with $\beta_0 = \pi$ by setting $\beta = -\pi + b\Delta_n^{-1} \in B$ if $b > 0$.

4.2 Switching Tests

The convergence to a "standard" experiment as $\delta \to \infty$ via Condition 1 suggests that the part of the alternative parameter space Θ_1 with δ large essentially corresponds to a standard testing problem. Accordingly, partition Θ_1 into a subset $\Theta_{1,S}$ corresponding to $\delta > \kappa_\delta$ for some large κ_δ, and a subset $\Theta_{1,N}$ that is "non-standard" $\delta \leq \kappa_\delta$, so that

$$\Theta_1 = \Theta_{1,S} \cup \Theta_{1,N}.$$

The convergence to the experiment involving X discussed above implies that for large enough κ_δ, the test φ_S has a rejection profile arbitrarily close to $\varphi_S^{\lim}$. It thus makes sense to pick κ_δ just large enough for this to become a good enough approximation. Furthermore, in light of Lemma 2, it is then sensible to restrict attention to tests that have (nearly) the same power as φ_S on $\Theta_{1,S}$, so the WAP criterion is only relevant for $\Theta_{1,N}$.

A straightforward strategy to achieve power that nearly equals the power of φ_S on $\Theta_{1,S}$ is to consider tests of the following "switching" form

$$\varphi_{N,S,\chi}(y) = \chi(y)\varphi_S(y) + (1 - \chi(y))\varphi_N(y) \tag{16}$$

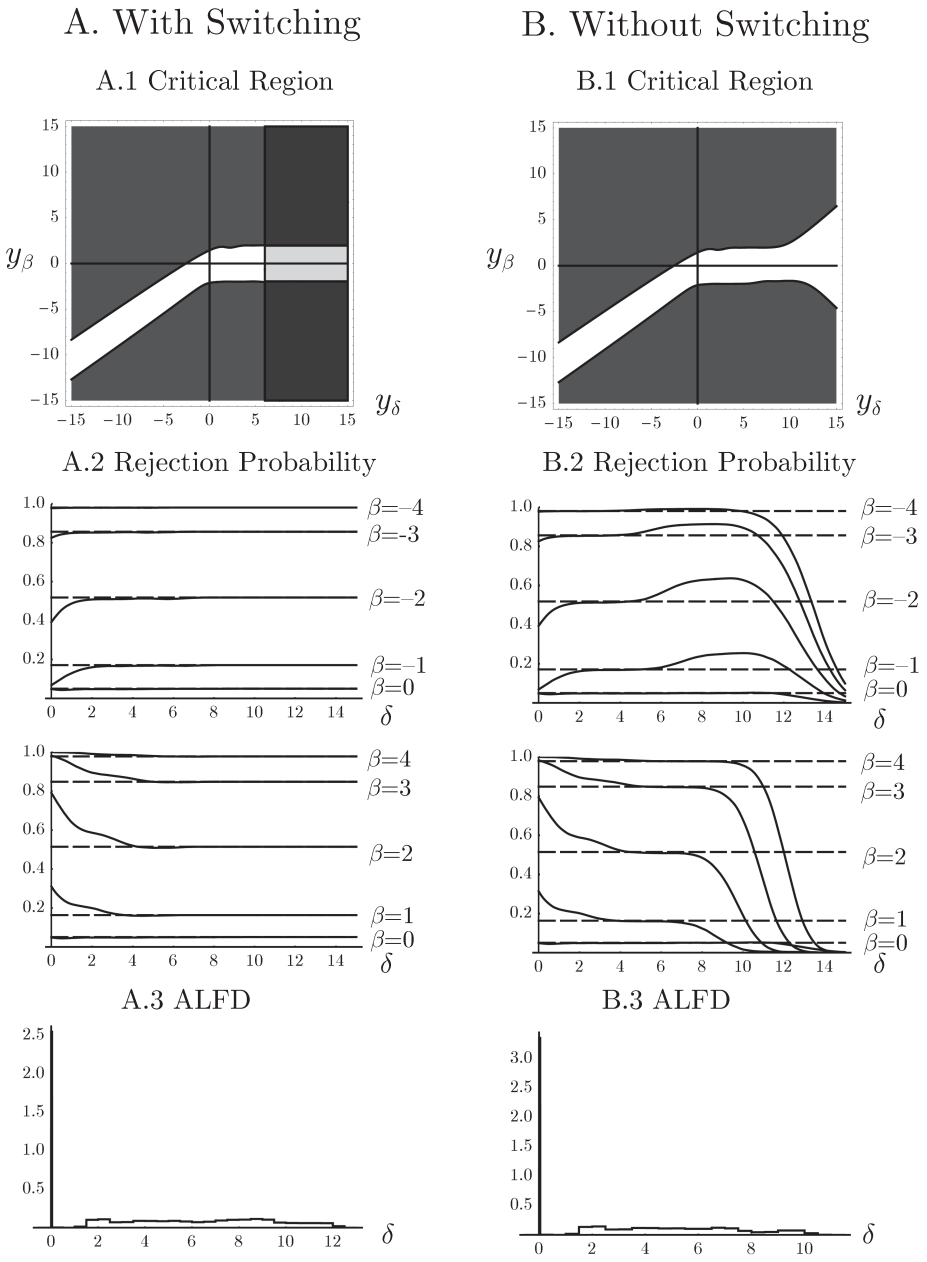
where φ_S is the standard test from the limiting problem and $\chi : \mathcal{Y} \mapsto [0,1]$ is a switching function chosen so that $\varphi_{N,S,\chi}(y) = \varphi_S(y)$ with probability close to one for all $\theta \in \Theta_{1,S}$. In our applications, we choose χ of the form $\chi(y) = \mathbf{1}[\hat{\delta} > \kappa_\chi]$, for some κ_χ. The value of κ_χ is chosen sufficiently smaller than κ_δ so that with probability very close to one, $\hat{\delta} > \kappa_\chi$ whenever $\delta > \kappa_\delta$.[6]

The function $\varphi_N : \mathcal{Y} \mapsto [0,1]$ is determined to maximize WAP relative to F. A straightforward generalization of last section's results for ε-ALFD tests yields the approximately best weighted average test, say $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}(y)$, for tests of the form (16). Details are provided in the appendix. In the next subsection we consider the efficiency of $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ within a wider class tests than (16), but before that discussion it is useful to return to the running example.

Running example, ctd: We choose $\kappa_\delta = 9$, so that $\Theta_{1,N}$ consists of parameter values $\theta = (\beta, \delta)$ with $\beta \neq 0$ and $0 \leq \delta \leq 9$. The weighting function F is such that δ is uniformly distributed on $[0,9]$, and β takes on the values -2 and 2 with equal probability. The two alternatives are thus treated symmetrically and the values ± 2 are chosen so that the test achieves approximately 50% power (following the rationale given in King (1987)). The switching function is specified as $\chi(y) = \mathbf{1}[\hat{\delta} > 6]$, where $\hat{\delta} = y_\delta$.

[6] A necessary condition for tests of the form (16) to control size is that the "pure switching test" $\tilde{\varphi} = \chi\varphi_S$ (ie, (16) with $\varphi_N = 0$) is of level α. Depending on the problem and choice of κ_χ, this might require a slight modification of the most natural choice for φ_S to ensure actual (and not just approximate) size control of φ_S for all $\delta > \kappa_\delta$.

Figure 1: Positive Nuisance Parameter



Notes: Darker shades for $y_\delta \geq 6$ in panel A.1 indicate the part of the acceptance and critical region imposed by the switching rule. In panels A.2 and B.2, solid lines are the rejection probability of the nearly optimal tests $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ (panel A) and $\varphi^{\varepsilon}_{\Lambda^*}$ (panel B), and dashed lines are for the usual test that ignores Y_δ, $\varphi_S(y) = \mathbf{1}[|y_\beta| > 1.96]$.

Figure 1 summarizes the results for the running example with $\rho = 0.7$ for tests of level $\alpha = 5\%$. The ALFD was computed using $\varepsilon = 0.005$, so that the power of the nearly optimal tests differs from the power bound by less than 0.5 percentage points. The number of Monte Carlo draws under the null and alternative hypotheses in the algorithm are chosen so that Monte Carlo standard errors are approximately 0.1%. Panel A shows results for $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$, and for comparison panel B shows results for the WAP maximizing test with the same weighting function F with support on $\Theta_{1,N}$, but without the constraint to the switching form (16).

The white and light gray band in the center of panel A.1 is the acceptance region of the nearly optimal test $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$, with the light gray indicating the acceptance region conditional on switching ($|y_\beta| \leq 1.96$ and $y_\delta \geq 6$). The dark shades show the critical region, with the darker shade indicating the critical region conditional on switching ($|y_\beta| > 1.96$ and $y_\delta \geq 6$). The critical region is seen to evolve smoothly as the test switches at $y_\delta = 6$, and essentially coincides with the standard test φ_S for values of y_δ as small as $y_\delta = 3$. This suggests that other choices of κ_δ and κ_χ (and corresponding F with δ uniform on $[0, \kappa_\delta]$) would yield similar WAP maximizing tests; unreported results show that this is indeed the case. As y_δ becomes negative the critical region is approximately $|y_\beta - \rho y_\delta| > 1.96\sqrt{1-\rho^2}$, which is recognized as the critical region of the uniformly best unbiased test for $\delta = 0$ known.

Panel A.2 shows power (plotted as a function of δ) for selected values of β. The solid curves show the power of the nearly optimal test and the dashed lines shows the power of the standard test φ_S. The figures show that power is asymmetric in β, with substantially lower power for negative values of β when δ is small; this is consistent with the critical region shown in panel A.1 where negative values of β and small values of δ make it more likely that y falls in the lower left quadrant of panel A.1. Because weighted average power is computed for uniformly distributed $\beta \in \{-2, 2\}$ and $\delta \in [0, 9]$, the optimal test maximizes the average of the power curves for $\beta = -2$ and $\beta = 2$ in A.2 over $\delta \in [0, 9]$. Weighted average power of $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ is higher than the power of φ_S for all pairs of values for β shown in the figure.

Panels B show corresponding results for the nearly optimal test $\varphi^{\varepsilon}_{\Lambda^*}$ that does not impose switching to a standard test, computed using the algorithm as described in Section 3.3. Because F only places weight on values of δ that are less than 9, this test sacrifices power for values of $\delta > 9$ to achieve more power for values of $\delta \leq 9$. The differences between the power function for $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ (shown in panel A) and $\varphi^{\varepsilon}_{\Lambda^*}$ (shown in panel B) highlights the attractiveness of switching to a standard test: it allows F to be chosen to yield high average power in the non-standard portion of the parameter space (small values of δ) while maintaining good power properties in other regions.

Panels A.3 and B.3 show the ALFDs underlying the two tests, which are mixtures of uniform baseline densities f_i used in the calculations. We emphasize that the ALFDs are

not direct approximations to the least favorable distributions, but rather are distributions that produce tests with nearly maximal weighted average power. ▲

4.3 Power Bounds under Additional Constraints on Power

The test $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ (nearly) coincides with φ_S when $\theta \in \Theta_{1,S}$, and thus is (nearly) as powerful as the standard best test $\varphi^{\lim}_S$ in the limiting problem in that part of the parameter space; moreover $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ comes close to maximizing WAP on $\Theta_{1,N}$ among all tests of the form (16). A natural question is whether this class is restrictive, in the sense that there exist tests outside this class that have better WAP on $\Theta_{1,N}$. We investigate this in this section by computing a WAP upper bound for any test that satisfies the level constraint and achieves prespecified power on $\Theta_{1,S}$.

To begin, decompose the alternative into two hypotheses corresponding to $\Theta_{1,N}$ and $\Theta_{1,S}$

$$H_{1,N} : \theta \in \Theta_{1,N} \text{ and } H_{1,S} : \theta \in \Theta_{1,S}.$$

Let $\pi_S(\theta) = \int \chi\varphi_S f_\theta dv$ denote the rejection frequency for the test $\tilde{\varphi} = \chi\varphi_S$, where as above, χ is the switching function and φ_S is the standard test. The test $\tilde{\varphi}$ is a useful benchmark for the performance of any test on $\Theta_{1,S}$, since it is a feasible level α test of H_0 against H_1 with power $\pi_S(\theta)$ under $H_{1,S}$ that is very close to the power of the standard test φ_S, which in turn has power close to the admissible test $\varphi^{\lim}_S$ in the limiting problem. Now, consider tests that (a) are of level α under H_0, (b) maximize WAP under $H_{1,N}$ relative to the weighting function F in (8), and (c) achieve power of at least $\pi_S(\theta)$ under $H_{1,S}$. Using an argument similar to that underlying Lemma 1, a bound on the power of level α tests of H_0 against $H_{1,F}$ (cf. (8)) with power of at least $\pi_S(\theta)$ for $\theta \in \Theta_{1,S}$ can be constructed by considering tests that replace the composite hypotheses H_0 and $H_{1,S}$ with simple hypotheses involving mixtures. Thus, let Λ_0 denote a distribution for θ with support in Θ_0 and let Λ_1 denote a distribution with support in $\Theta_{1,S}$, and consider the simple hypotheses

$$\begin{aligned} H_{0,\Lambda_0} &: Y \text{ has density } f_{0,\Lambda_0} = \int f_\theta d\Lambda_0(\theta) \\ H_{1,\Lambda_1} &: Y \text{ has density } f_{1,\Lambda_1} = \int f_\theta d\Lambda_1(\theta). \end{aligned}$$

Let $\bar{\pi}_S = \int \pi_S(\theta) d\Lambda_1(\theta)$ denote the weighted average of the power bound under $H_{1,S}$. The form of the best tests for these simple hypotheses is described by a generalized Neyman-Pearson Lemma.

Lemma 3 *Suppose there exist* $\text{cv}_0 \geq 0$, $\text{cv}_1 \geq 0$ *and* $0 \leq \kappa \leq 1$ *such that the test*

$$\varphi^{NP} = \begin{cases} 1 \text{ if } g + \text{cv}_1 f_{1,\Lambda_1} > \text{cv}_0 f_{0,\Lambda_0} \\ \kappa \text{ if } g + \text{cv}_1 f_{1,\Lambda_1} = \text{cv}_0 f_{0,\Lambda_0} \\ 0 \text{ if } g + \text{cv}_1 f_{1,\Lambda_1} < \text{cv}_0 f_{0,\Lambda_0} \end{cases}$$

satisfies $\int \varphi^{NP} f_{0,\Lambda_0} d\nu = \alpha$, $\int \varphi^{NP} f_{1,\Lambda_1} d\nu \geq \bar{\pi}_S$ *and* $\text{cv}_1(\int \varphi^{NP} f_{1,\Lambda_1} d\nu - \bar{\pi}_S) = 0$. *Then for any other test satisfying* $\int \varphi f_{0,\Lambda_0} d\nu \leq \alpha$ *and* $\int \varphi f_{1,\Lambda_1} d\nu \geq \bar{\pi}_S$, $\int \varphi^{NP} g d\nu \geq \int \varphi g d\nu$.

Proof. *If* $\text{cv}_1 = 0$, *the result follows from the Neyman-Pearson Lemma. For* $\text{cv}_1 > 0$, *by the definition of* φ^{NP}, $\int (\varphi^{NP} - \varphi)(g + \text{cv}_1 f_{1,\Lambda_1} - \text{cv}_0 f_{0,\Lambda_0}) d\nu \geq 0$. *By assumption* $\int (\varphi^{NP} - \varphi) f_{1,\Lambda_1} d\nu \leq 0$ *and* $\int (\varphi^{NP} - \varphi) f_{0,\Lambda_0} d\nu \geq 0$. *The result now follows from* $\text{cv}_0 \geq 0$ *and* $\text{cv}_1 \geq 0$. ■

As in Lemma 1, the power of the Neyman-Pearson test for the simple hypotheses provides an upper bound on power against $H_{1,F}$ when H_0 and $H_{1,S}$ are composite.

Lemma 4 *Let* φ^{NP} *denote the Neyman-Pearson test defined in Lemma 3, and let* φ *denote any test that satisfies (a)* $\sup_{\theta \in \Theta_0} \int \varphi f_\theta d\nu \leq \alpha$ *and (b)* $\inf_{\theta \in \Theta_{1,S}} \left[\int \varphi f_\theta d\nu - \pi_S(\theta) \right] \geq 0$. *Then* $\int \varphi^{NP} g d\nu \geq \int \varphi g d\nu$.

Proof. *Because* $\int \varphi f_{0,\Lambda_0} d\nu \leq \alpha$ *and* $\int \varphi f_{1,\Lambda_1} d\nu \geq \int \pi_S(\theta) d\Lambda_1(\theta) = \bar{\pi}_S$ *the result follows from Lemma 3.* ■

Of course, to be a useful guide for gauging the efficiency of any proposed test, the power bound should be as small as possible. For given α and power function $\pi_S(\theta)$ of $\tilde{\varphi} = \varphi_S \chi$, the numerical algorithm from the last section can be modified to compute distributions Λ_0^* and Λ_1^* that approximately minimize the power bound. Details are provided in the appendix. We use this algorithm to assess the efficiency of the level α switching test $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$, which satisfies the power constraint of Lemma 4 by construction, as $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}(y) \geq \tilde{\varphi}(y)$ for all y.

Running example, ctd: The power bound from Lemma 4 evaluated at Λ_0^* and Λ_1^* is 53.5%. Thus, there does not exist 5% level test with WAP larger than 53.5% and with at least as much power as the test $\tilde{\varphi}(y) = \chi(y)\varphi_S(y) = \mathbf{1}[\hat{\delta} > 6]\mathbf{1}[|y_\beta| > 1.96]$ for alternatives with $\delta \geq 9$. Since $\varphi_{\Lambda^*,S,\chi}$ of panel A of Figure 1 is strictly more powerful than $\tilde{\varphi}$, and it has WAP of 53.1%, it is thus also nearly optimal in the class of 5% level tests that satisfy this power constraint. ▲

5 Applications

In this section we apply the algorithm outlined above (with the same parameters as in the running example) to construct nearly weighted average power maximizing 5% level tests for

five non-standard problems. In all of these problems we set $\varepsilon = 0.005$, so that the ALFD test has power within 0.5% of the power bound for tests of the switching rule form (16) (if applicable). Unreported results show that the weighted average power of the resulting tests is also within 0.0065 of the upper bound on tests of arbitrary functional form under the power constraint described in Section 4.3 above.[7] The supplementary appendix contains further details on the computations in each of the problems, and the supplementary materials contain tables and Matlab programs to implement these nearly optimal tests for a wide range of parameter values.

5.1 The Behrens-Fisher Problem

Suppose we observe i.i.d. samples from two normal populations $x_{1,i} \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $i = 1, \ldots, n_1$ and $x_{2,i} \sim \mathcal{N}(\mu_2, \sigma_2^2)$, $i = 1, \ldots, n_2$, where $n_1, n_2 \geq 2$. We are interested in testing $H_0 : \mu_1 = \mu_2$ without knowledge of σ_1^2 and σ_2^2. This is the "Behrens-Fisher" problem, which has a long history in statistics. It also arises as an asymptotic problem when comparing parameters across two potentially heterogeneous populations, where the information about each population is in the form of n_1 and n_2 homogeneous clusters to which a central limit theorem can be applied (see Ibragimov and Müller (2010, 2013)).

Let $\bar{x}_j = n_j^{-1} \sum_{i=1}^{n_j} x_{j,i}$ and $s_j^2 = \frac{1}{n_j - 1} \sum_{i=1}^{n_j} (x_{j,i} - \bar{x}_j)^2$ be the sample means and variances for the two groups $j = 1, 2$, respectively. It is readily seen that the four dimensional statistic $(\bar{x}_1, \bar{x}_2, s_1, s_2)$ is sufficient for the four parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2)$. Imposing invariance to the transformations $(\bar{x}_1, \bar{x}_2, s_1, s_2) \to (c\bar{x}_1 + m, c\bar{x}_2 + m, cs_1, cs_2)$ for $m \in \mathbb{R}$ and $c > 0$ further reduces the problem to the two dimensional maximal invariant Y

$$Y = (Y_\beta, Y_\delta) = \left(\frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}, \log(\frac{s_1}{s_2}) \right)$$

whose density is derived in the supplementary appendix (cf. Linnik (1966) andTsui and Weerahandi (1989). Note that Y_β is the usual two-sample t-statistic which converges to $\mathcal{N}(0, 1)$ under the null hypothesis as $n_1, n_2 \to \infty$. The distribution of Y only depends on the two parameters $\beta = (\mu_1 - \mu_2)/\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}$ and $\delta = \log(\sigma_1/\sigma_2)$, and the hypothesis problem becomes

$$H_0 : \beta = 0, \, \delta \in \mathbb{R} \quad \text{against} \quad H_1 : \beta \neq 0, \, \delta \in \mathbb{R}. \tag{17}$$

[7] In some of these examples, we restrict attention to tests that satisfy a scale or location invariance property. Our ALFD test then comes close to maximizing weighted average power among all invariant tests.

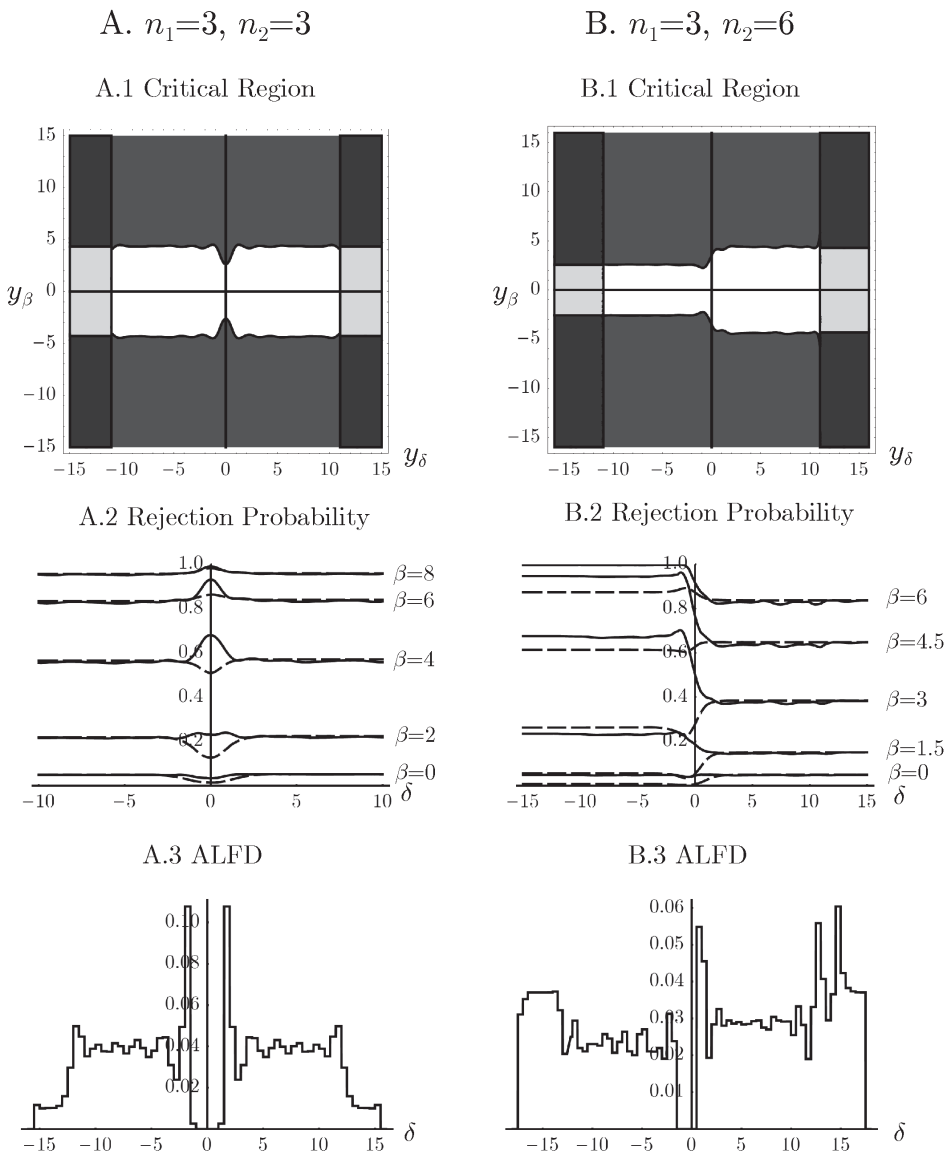
While the well-known two-sided test of Welch (1947) with "data dependent degrees of freedom" approximately controls size for sample sizes as small as $\min(n_1, n_2) = 5$, (Wang (1971) and Lee and Gurland (1975)), it is substantially over-sized when $\min(n_1, n_2) \leq 3$; moreover, its efficiency properties are unknown. Thus, we employ the algorithm described above to compute nearly optimal tests for $n_1 \in \{2, 3\}$ and $n_1 \leq n_2 \leq 12$; these are described in detail in the supplementary materials. In the following, we focus on the two cases $(n_1, n_2) \in \{(3, 3), (3, 6)\}$.

To implement the algorithm, we choose F as uniform on $\delta \in [-9, 9]$ and $\beta = \{-3, 3\}$. The appendix shows that as $|\delta| \to \infty$, the experiment converges to a one sample normal mean problem (since one of the samples has negligible variance).[8] In the limiting problem, the standard test is simply the one sample t-test with $n_1 - 1$ or $n_2 - 1$ degrees of freedom, depending on the sign of δ. Thus, $\varphi_S(y) = \mathbf{1}[y_\delta > 0]\mathbf{1}[|y_\beta| > T_{n_1-1}(0.975)] + \mathbf{1}[y_\delta < 0]\mathbf{1}[|y_\beta| > T_{n_2-1}(0.975)]$, where $T_n(x)$ is the x^{th} quantile of a Student-t distribution with n degrees of freedom. We use the switching function $\chi(y) = \mathbf{1}[|y_\delta| > 6]$. We compare the power of the resulting $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ test to the "conservative" test obtained by using the 0.975 quantile of a student-t distribution with degrees of freedom equal to $\min(n_1, n_2) - 1$, which is known be of level α (cf. Mickey and Brown (1966)).

Results are shown in Figure 2, where panel A shows results for $(n_1, n_2) = (3, 3)$ and panel B shows results for $(n_1, n_2) = (3, 6)$. Looking first at panel A, the critical region transitions smoothly across the switching boundary. In the non-standard part ($|y_\delta| < 6$) the critical region is much like the critical region of the standard test $\mathbf{1}[|y_\beta| > T_2(0.975)]$ for values of $|y_\delta| > 2$, but includes smaller values of $|y_\beta|$ when y_δ is close to zero. Evidently, small values of $|y_\delta|$ suggest that the values of σ_1 and σ_2 are close, essentially yielding more degrees of freedom for the null distribution of y_β. This feature of the critical region translates in the greater power for $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ than the conservative test when δ is close to zero. (See panel A.2). Panel B shows results when n_2 is increased to $n_2 = 6$. Now, the critical region becomes "pinched" around $y_\delta \approx -1$ apparently capturing a trade-off between a relatively small value of s_1 and n_1. Panel B.2 shows a power function that is asymmetric in δ, where the test has more power when the larger group has larger variance. Finally, the conservative test has a null rejection rejection frequency substantially less than 5% when $\delta < 0$ and weighted average power substantially below the nearly optimal test.

[8] Strictly speaking, there are two limit experiments, one as $\delta \to \infty$, and one as $\delta \to -\infty$.

Figure 2: Behrens-Fisher Problem



Notes: Darker shades for $|y_\delta| \geq 6$ in panels A.1 and B.1 indicate the part of the acceptance and critical region imposed by the switching rule. In panels A.2 and B.2, solid lines are the rejection probability of the nearly optimal test $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$, and dashed lines are for the usual t-test with critical value computed from the Student-t distribution with $n_1 - 1$ degrees of freedom.

5.2 Inference about the Break Date in a Time Series Model

In this section we consider tests for the break date τ in the parameter of a time series model with T observations. A leading example is a one-time shift by the amount η of the value of a regression coefficient, as studied in Bai (1994, 1997). Bai's asymptotic analysis focusses on breaks that are large relative to sampling uncertainty by imposing $T^{1/2}|\eta| \to \infty$. As discussed in Elliott and Müller (2007), this "large break" assumption may lead to unreliable inference in empirically relevant situations.

Under an alternative embedding for moderately sized breaks $T^{1/2}\eta \to \delta \in \mathbb{R}$, the parameter δ becomes a nuisance parameter that remains relevant even asymptotically. As a motivating example, suppose the mean of a Gaussian time series shifts at some date τ by the amount η,

$$y_t = \mu + \mathbf{1}[t \geq \tau]\eta + \varepsilon_t, \quad \varepsilon_t \sim iid\mathcal{N}(0,1)$$

and the aim is to conduct inference about the break date τ. As is standard in the structural break literature, assume that the break does not happen close to the beginning and end of the sample, that is with $\beta = \tau/T$, $\beta \in B = [0.15, 0.85]$. Restricting attention to translation invariant tests ($\{y_t\} \to \{y_t + m\}$ for all m) requires that tests are a function of the demeaned data $y_t - \bar{y}$. Partial summing the observations yields

$$T^{-1/2}\sum_{t=1}^{\lfloor sT \rfloor}(y_t - \bar{y}) \sim G(s) = W(s) - sW(1) - \delta(\min(\beta, s) - \beta s) \tag{18}$$

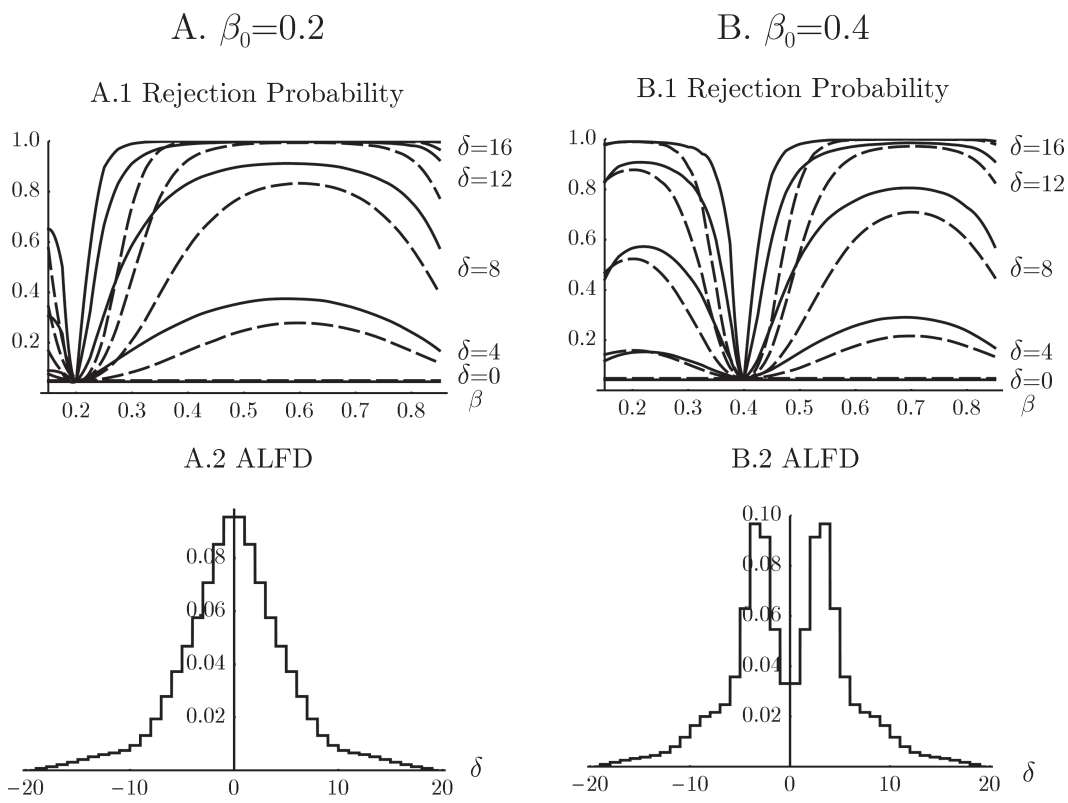
for $s = j/T$ and integer $1 \leq j \leq T$, where W is a standard Wiener process. This suggests that asymptotically, the testing problem concerns the observation of the Gaussian process G on the unit interval, and the hypothesis of interest concerns the location β of the kink in its mean. Elliott and Müller (2014) formally show that this is indeed the relevant asymptotic experiment for a moderate structural break in a well behaved parametric time series model. By Girsanov's Theorem, the Radon-Nikodym derivative of the measure of G in (18) relative to the measure ν of the standard Brownian Bridge, evaluated at G, is given by

$$f_\theta(G) = \exp[-\delta G(\beta) - \tfrac{1}{2}\delta^2\beta(1-\beta)]. \tag{19}$$

For $|\delta| > 20$, the discretization of the break date β becomes an important factor in this limiting problem, even with 1,000 step approximations to Wiener processes. Since these discretizations errors are likely to dominate the analysis with typical sample sizes for even larger δ, we restrict attention to $\delta \in \Delta = [-20, 20]$, so that the hypothesis are

$$H_0 : \beta = \beta_0,\ \delta \in \Delta \quad \text{against} \quad H_1 : \beta \neq \beta_0,\ \delta \in \Delta.$$

Figure 3: Break Date



Notes: In panels A.1 and B.1, solid lines are the rejection probability of the nearly optimal test $\varphi^{\varepsilon}_{\Lambda^*}$, and dashed lines are for Elliott and Müller's (2007) test that imposes an additional invariance.

To construct the ALFD test we choose F so that β is uniform on B and δ is $\mathcal{N}(0, 100)$, truncated to Δ. Results are shown in Figure 3. Panel A shows results for $\beta_0 = 0.2$, where panel A.1 plots power as a function of β for five values of δ; panel B shows analogous results for $\beta_0 = 0.4$. (Since G is a continuous time stochastic process, the sample space is of infinite dimension, so it is not possible to plot the critical region.) Rejection probabilities for a break at $\beta_0 > 0.5$ are identical to those at $1 - \beta_0$.

Also shown in the figures are the corresponding power functions from the test derived in Elliott and Müller (2007) that imposes the additional invariance

$$G(s) \to G(s) + c(\min(\beta_0, s) - \beta_0 s) \qquad \text{for all } c. \tag{20}$$

This invariance requirement eliminates the nuisance parameter δ under the null, and thus leads to a similar test. But the transformation (20) is not natural under the alternative,

leaving scope for reasonable and more powerful tests that are not invariant. Inspection of Figure 3 shows that the nearly optimal test $\varphi^{\varepsilon}_{\Lambda*}$ has indeed substantially larger power for most alternatives. Also, power is seen to be small when β is close to either β_0 or the endpoints, as this implies a mean function close to what is specified under the null hypothesis.

5.3 Predictive Regression with a Local-To-Unity Regressor

A number of macroeconomic and finance applications concern the coefficient γ on a highly persistent regressor x_t in the model

$$\begin{aligned} y_t &= \mu + \gamma x_{t-1} + \varepsilon_{y,t} \\ x_t &= r x_{t-1} + \varepsilon_{x,t}, \;\; x_0 = 0 \end{aligned} \tag{21}$$

where $E(\varepsilon_{y,t}|\{\varepsilon_{x,t-j}\}_{j=1}^{t-1}) = 0$, so that the first equation is a predictive regression. The persistence in x_t is often modelled as a local-to-unity process (in the sense of Bobkoski (1983), Cavanagh (1985), Chan and Wei (1987) and Phillips (1987)) with $r = r_T = 1 - \delta/T$. Interest focuses on a particular value of γ given by $H_0 : \gamma = \gamma_0$ (where typically $\gamma_0 = 0$). When the long-run covariance between ε_y and ε_x is non-zero, the usual t-test on γ is known to severely overreject unless δ is very large.

After imposing invariance to translations of y_t, $\{y_t\} \to \{y_t + m\}$, and an appropriate scaling by the (long-run) covariance matrix of $(\varepsilon_{y,t}, \varepsilon_{x,t})'$, the asymptotic inference problem concerns the likelihood ratio process f_θ of a bivariate Gaussian continuous time process G,

$$f_\theta(G) = K(G) \exp[\beta Y_1 - \delta Y_2 - \tfrac{1}{2}\left(\beta + \frac{\rho}{\sqrt{1-\rho^2}}\delta\right)^2 Y_3 - \tfrac{1}{2}\delta^2 Y_4] \tag{22}$$

where β is proportional to $T(\gamma - \gamma_0)$, $\rho \in (-1, 1)$ is the known (long-run) correlation between $\varepsilon_{x,t}$ and $\varepsilon_{y,t}$, $\theta = (\beta, \delta)' \in \mathbb{R}^2$ is unknown, and the four dimensional sufficient statistic $Y = (Y_1, Y_2, Y_3, Y_4)$ has distribution

$$\begin{aligned} Y_1 &= \int_0^1 W^{\mu}_{x,\delta}(s) dW_y(s) + \left(\beta + \frac{\rho}{\sqrt{1-\rho^2}}\delta\right) \int_0^1 W^{\mu}_{x,\delta}(s)^2 ds \\ Y_2 &= \int_0^1 W_{x,\delta}(s) dW_{x,\delta}(s) - \frac{\rho}{\sqrt{1-\rho^2}} Y_1 \\ Y_3 &= \int_0^1 W^{\mu}_{x,\delta}(s)^2 ds, \qquad Y_4 = \int_0^1 W_{x,\delta}(s)^2 ds \end{aligned}$$

with W_x and W_y are independent standard Wiener processes, and the Ornstein-Uhlenbeck process $W_{x,\delta}$ solves $dW_{x,\delta}(s) = -\delta W_{x,\delta}(s) ds + dW_x(s)$ with $W_{x,\delta}(0) = 0$, and $W^{\mu}_{x,\delta}(s) = W_{x,\delta}(s) - \int_0^1 W_{x,\delta}(t) dt$ (cf. Jansson and Moreira (2006)).

Ruling out explosive roots, $\delta \geq 0$, the one-sided asymptotic inference problem is

$$H_0 : \beta = 0, \delta \geq 0 \quad \text{against} \quad H_1 : \beta > 0,\ \delta \geq 0. \tag{23}$$

While several methods have been developed that control size in (23) (leading examples include Cavanagh, Elliott, and Stock (1995) and Campbell and Yogo (2006)), there are fewer methods with demonstrable optimality. Stock and Watson (1996) numerically determine a weighed average power maximizing test within a parametric class of functions $\mathbb{R}^4 \mapsto \{0, 1\}$, and Jansson and Moreira (2006) derive the best conditionally unbiased tests of (23), conditional on the specific ancillary (Y_3, Y_4). However, Jansson and Moreira (2006) report that Campbell and Yogo's (2006) test has higher power for most alternatives. We therefore compare the one-sided ALFD test to this more powerful benchmark.

In the appendix we show that in a parameterization with $\delta = \Delta_n - d\sqrt{2\Delta_n}$ and $\beta = b\sqrt{2\Delta_n/(1-\rho^2)}$, the experiment of observing G converges as $\Delta_n \to \infty$ to the unrestricted two dimensional Gaussian shift experiment. A natural way to obtain a test with the same asymptotic power function as $\delta \to \infty$ is to rely on the usual maximum likelihood t-test (with observed information). From (22), the MLE is given by

$$\hat{\beta} = \frac{Y_1}{Y_3} - \frac{\rho}{\sqrt{1-\rho^2}}\hat{\delta}, \qquad \hat{\delta} = -\frac{Y_2 + \frac{\rho}{\sqrt{1-\rho^2}}Y_1}{Y_4}$$

and the the standard test becomes $\varphi_S(Y) = \mathbf{1}[\hat{\beta}/\sqrt{Y_3^{-1} + \frac{\rho^2}{1-\rho^2}Y_4^{-1}} > \mathrm{cv}_S]$.

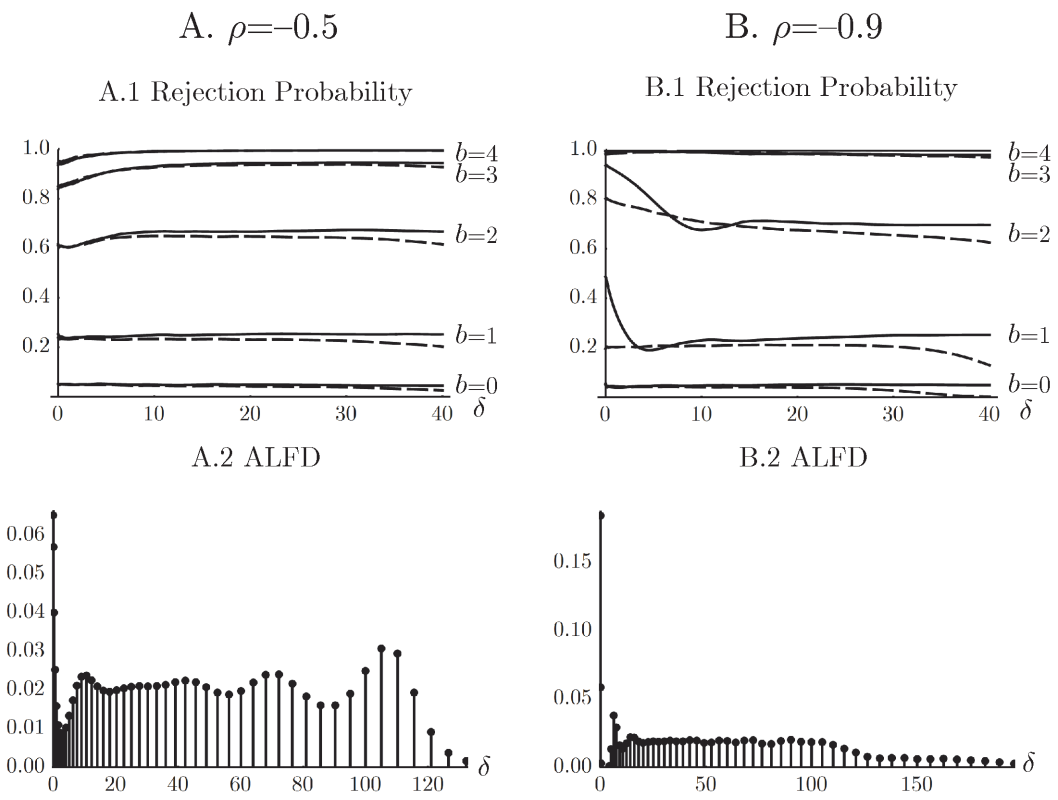
For numerical convenience, we use a discrete weighting function F with equal mass on 51 pairs of values (β, δ), where $\delta \in \{0, 0.25^2, 0.5^2, \ldots, 12.5^2\}$ and the corresponding values for β equal

$$\beta = b\sqrt{\frac{2\delta + 6}{1-\rho^2}} \text{ for } \delta > 0 \tag{24}$$

for $b = 1.645$. These alternatives are chosen so that power against each point in F is roughly 50%. (The spacing of the mass points for δ and the choice of β correspond to an approximately uniform weighting over d and an alternative of 1.645 standard deviations for b in the limiting experiment.) We use the switching function $\chi(Y) = \mathbf{1}[\hat{\delta} \geq 130]$. The critical value cv_S in φ_S equals the usual 5% level value of 1.645 when $\rho \geq 0$, but we choose $\mathrm{cv}_S = 1.70$ when $\rho < 0$. This slight adjustment compensates for the heavier tail of the t-test statistic for moderate values of δ and negative ρ.

Figure 4 compares the power of resulting nearly optimal test has close to the test developed by Campbell and Yogo (2006) under the practically relevant values of $\rho = -0.5$ and $\rho = -0.9$. Because the Campbell and Yogo (2006) test utilizes a confidence set for r with

Figure 4: Predictive Regression with a Local-To-Unity Regressor



Notes: In panels A.1 and B.1, solid lines are the rejection probability of the nearly optimal test $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$, and dashed lines are for Campbell and Yogo's (2006) test. Since the latter is constructed under the assumption that $\delta \leq 50$, we only report its rejection probability for $\delta \in [0, 50]$. Alternatives are parameterized as in (24), with $b \in \{0, 1, 2, 3, 4\}$.

correct coverage only when r is close to unity (see Mikusheva (2007) and Phillips (2014)) Figure 4 plots power for δ in the restricted range $0 \leq \delta \leq 40$, and over this range the optimal test nearly uniformly dominates the alternative test. For values of $\delta > 40$ (so that r is much less than unity), the power of the Campbell and Yogo (2006) falls dramatically, while the power of the optimal test increases slightly from its value with $\delta = 40$, and smoothly transitions to the power of φ_S. Unreported results show similar results for positive values of ρ.

5.4 Testing the Value of a Set-Identified Parameter

The asymptotic problem introduced by Imbens and Manski (2004) and further studied by Woutersen (2006), Stoye (2009) and Hahn and Ridder (2011) involves a bivariate observation

$$Y = \begin{pmatrix} Y_l \\ Y_u \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_l \\ \mu_u \end{pmatrix}, \begin{pmatrix} \sigma_l^2 & \rho\sigma_l\sigma_u \\ \rho\sigma_l\sigma_u & \sigma_u^2 \end{pmatrix} \right)$$

where $\mu_l \leq \mu_u$, and the elements $\sigma_l, \sigma_u > 0$ and $\rho \in (-1, 1)$ of the covariance matrix are known. The object of interest is μ, which is only known to satisfy

$$\mu_l \leq \mu \leq \mu_u. \tag{25}$$

Without loss of generality, suppose we are interested in testing $H_0 : \mu = 0$ (the test of the general hypothesis $\mu = \mu_0$ is reduced to this case by subtracting μ_0 from Y_l and Y_u). Whilst under the null hypothesis the inequality (25) holds if and only if $\mu_l/\sigma_l \leq 0 \leq \mu_u/\sigma_u$, under the alternative the normalized means μ_l/σ_l and μ_u/σ_u may no longer satisfy the ordering $\mu_l/\sigma_l \leq \mu_u/\sigma_u$. It is thus not possible to reduce this problem to a single known nuisance parameter ρ without loss of generality. In the sequel, we demonstrate our approach when $\sigma_l = \sigma_u = 1$ and various values of ρ.

It is useful to reparametrize (μ_l, μ_u) in terms of $(\beta, \delta_L, \delta_P) \in \mathbb{R}^3$ as follows: Let $\delta_L = \mu_u - \mu_l$ be the length of the identified set $[\mu_l, \mu_u]$, let $\beta = \mu_l$ if $\mu_l > 0$, $\beta = \mu_u$ if $\mu_u < 0$, and $\beta = 0$ otherwise, and let $\delta_P = -\mu_l$, so that under the null hypothesis δ_P describes the position of 0 in the identified set. In this parametrization, the hypothesis testing problem becomes

$$H_0 : \beta = 0,\ \delta_L \geq 0,\ \delta_P \in [0, \delta_L] \quad \text{against} \quad H_1 : \beta \neq 0,\ \delta_L \geq 0. \tag{26}$$

The appendix shows that the limiting problem as $\delta_L \to \infty$ becomes a simple one-sided testing problem in the Gaussian shift experiment (2) with unrestricted nuisance parameter space. Thus, in this limiting problem, the standard test can be written as $\varphi_S(y) = \mathbf{1}[y_l >$

1.645 or $y_u < -1.645$]. We switch to this standard test according to $\chi(y) = \mathbf{1}[\hat{\delta}_L > 6]$, where $\hat{\delta}_L = Y_u - Y_l \sim \mathcal{N}(\delta_L, 2(1-\rho))$. The weighting function F is chosen to be uniform on $\delta_L \in [0, 9]$, with equal mass on the two points $\beta \in \{-2, 2\}$.

Note that (26) has a two-dimensional nuisance parameter under the null hypothesis, as neither the length $\delta_L = \mu_u - \mu_l$ nor the distance δ_P of μ_l from zero is specified under H_0. It is reasonable to guess, though, that the least favorable distribution only has mass at $\delta_P \in \{0, \delta_L\}$, so that one of the endpoints of the interval coincides with the hypothesized value of μ. Further, the problem is symmetric in these these two values for δ_P. In the computation of the ALFD, we thus impose $\delta_P \in \{0, \delta_L\}$ with equal probability, and then check that the resulting test $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ does indeed control size also for $\delta_P \in (0, \delta_L)$.

Figure 5 shows results for two values of ρ. Looking first at the critical regions, when y_u is sufficiently large (say $y_u > 2$), the test rejects when $y_l > 1.645$, and similarly when y_l is sufficiently negative. The upper left-hand quadrant of the figures in panels A.1 and B.1 show the behavior of the test when the observations are inverted relative to their mean values, $y_l > y_u$. In that case, the test rejects unless $y_l + y_u$ is close to zero. Panels A.2 and B.2 compare the power of the ALFD test $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ to the test $\varphi_{\text{Stoye}}(y) = \mathbf{1}[y_l > 1.96$ or $y_u < -1.96]$, which is large sample equivalent to Stoye's (2009) suggestion under local asymptotics. Note that this test has null rejection probability equal to 5% when $\delta_L = 0$ and $\delta_P \in \{0, \delta_L\}$. Not surprisingly $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ dominates φ_{Stoye} when δ_L is large, but it also has higher power when δ_L is small and $\rho = 0.5$ (because when δ_L is small, the mean of Y_l and Y_u is more informative about μ than either Y_l or Y_u unless ρ is close to 1).

5.5 Regressor Selection

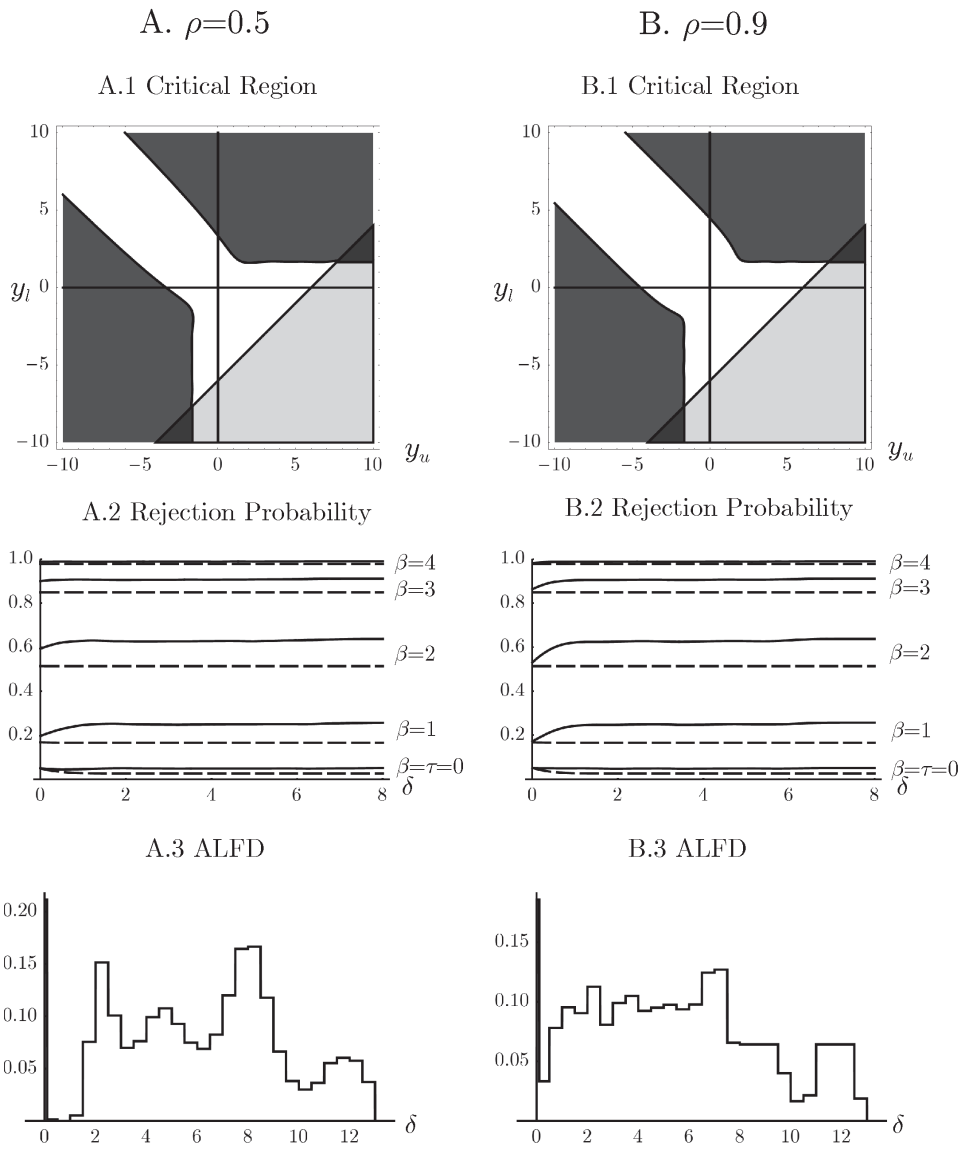
As in Leeb and Pötscher (2005), consider the bivariate linear regression

$$y_i = \gamma x_i + \eta z_i + \varepsilon_i, \;\; i = 1, \ldots, n, \;\; \varepsilon_i \sim \mathcal{N}(0, \sigma^2) \tag{27}$$

where σ^2 is known. We are interested in testing $H_0 : \gamma = \gamma_0$, and η is a nuisance parameter. Suppose there is substantial uncertainty whether the additional control z_i needs to be included in (27), that is $\eta = 0$ is deemed likely, but not certain. Denote by $(\hat{\gamma}, \hat{\eta})$ the OLS estimators from the "long" regression of y_i on (x_i, z_i). Let $\beta = n^{1/2}(\gamma - \gamma_0)$, $\delta = n^{1/2}\eta$, $(Y_\beta, Y_\delta) = n^{1/2}(\hat{\gamma} - \gamma_0, \hat{\eta})$, and for notational simplicity, assume that the regressors and σ^2 have been scale normalized so that

$$Y = \begin{pmatrix} Y_\beta \\ Y_\delta \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \beta \\ \delta \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \tag{28}$$

Figure 5: Set-Identified Parameter



Notes: Darker shades for $y_u + y_l \geq 6$ in panels A.1 and B1 indicate the part of the acceptance and critical region imposed by the switching rule. In panels A.2 and B.2, solid lines are the rejection probability of the nearly optimal tests $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$, and dashed lines are for Stoye's (2009) test $\varphi_{\text{Stoye}}(y) = \mathbf{1}[y_l > 1.96 \text{ or } y_u < -1.96]$.

where $-\rho$ is the known sample correlation between x_i and z_i. Note that with the Gaussian assumption about ε_i, Y is a sufficient statistic for the unknown parameters (β, δ).

For $\delta = 0$ known, the statistic $Y_\beta - \rho Y_\delta$ is more informative about β than Y_β. Intuitively, $Y_\beta - \rho Y_\delta$ is the (rescaled) regression coefficient in the "short" regression of y_i on x_i, omitting z_i, while Y_β corresponds to the coefficient in the "long" regression. Ideally, one would like to let the data decide whether indeed $\delta = 0$, so that one can appropriately base inference on $Y_\beta - \rho Y_\delta$, or on Y_β. As reviewed by Leeb and Pötscher (2005), however, data-dependent model selection procedures do not perform uniformly well for all values of η, even in large samples, so that optimal inference is not obtained in this manner.

As one possible notion of optimality, suppose that we seek a test of $H_0 : \beta = 0$ that is as powerful as possible when $\delta = 0$, but under the constraint that the test controls size for all values of $\delta \in \mathbb{R}$. The idea is that we want to maximize power in the a priori likely case of $\delta = 0$, while at the same time controlling the null rejection probability even if $\delta \neq 0$.

Consider first the one-sided problem. With F degenerate at $\beta_1 > 0$, we obtain the hypothesis test

$$H_0 : \beta = 0,\ \delta \in \mathbb{R} \quad \text{against} \quad H_{1,F} : \beta = \beta_1,\ \delta = 0. \tag{29}$$

Note that rejecting for large values of Y_β is the Neyman-Pearson test of $H_{1,F}$ against the single null hypothesis $H_0^s : (\beta, \delta) = (0, \delta_0)$, where $\delta_0 = -\rho\beta_1$. Since any level α test of (29) is also of level α under H_0^s, the uniformly most powerful one-sided test of (29) thus rejects for large values of Y_β (cf. Proposition 15.2 in van der Vaart (1998)). Thus, as long as one insists on uniform size control, the question of best one-sided inference about β has a straightforward answer: simply rely on the coefficient estimate of the long regression.

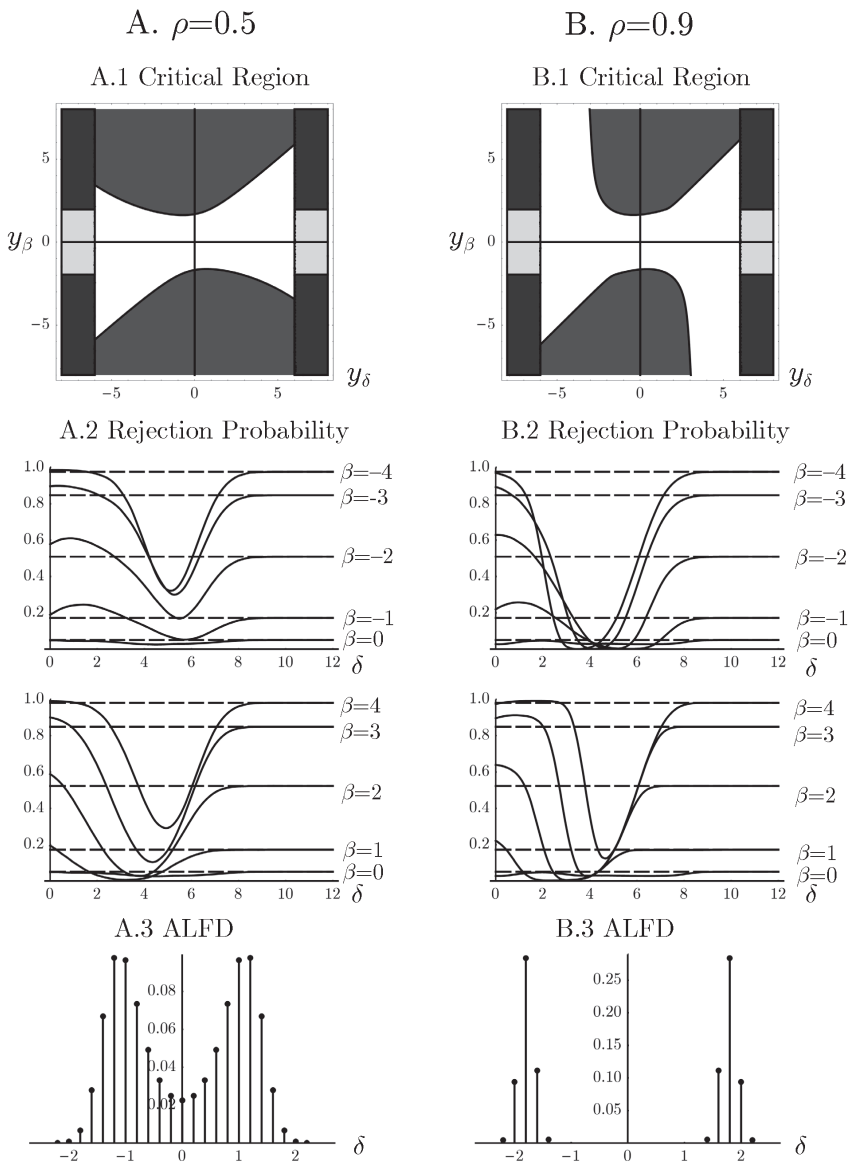
Now consider the two-sided problem. It is known that rejecting for large values of $|Y_\beta|$ yields the uniformly most powerful test among tests that are unbiased for all values of $\delta \in \mathbb{R}$ (cf. problem 1 on page 226 of van der Vaart (1998)). But with a focus on power at the point $\delta = 0$, this might be considered a too restrictive class of tests. Thus, we consider the unconstrained problem of maximizing weighted average power in the hypothesis testing problem

$$H_0 : \beta = 0,\ \delta \in \mathbb{R} \quad \text{against} \quad H_1 : \beta \neq 0,\ \delta = 0 \tag{30}$$

and choose a weighting function F that puts equal mass at the two points $\{-2, 2\}$. For large $|Y_\delta|$ we switch to the standard test $\varphi_S(y) = \mathbf{1}[|y_\beta| > 1.96]$ via $\chi(y) = \mathbf{1}[|y_\delta| > 6]$. Unreported results show that imposing this switching rule leads to no discernible loss in power when $\delta = 0$. At the same time, this switching rule leads to much higher power when $|\delta|$ is large.

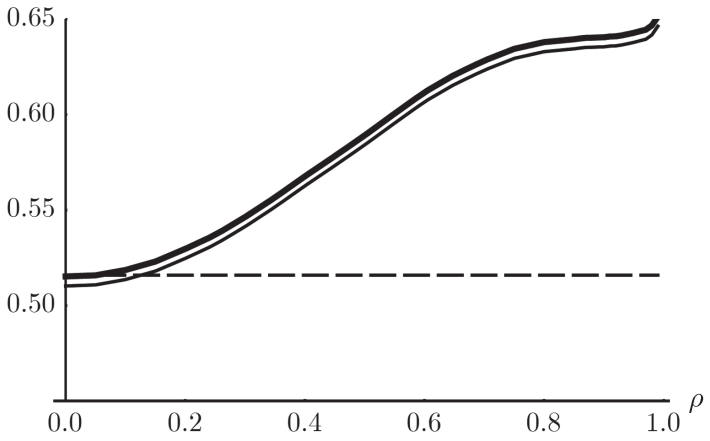
By construction, the weighted average power at $\delta = 0$ of $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ in Figure 6 is nearly the

Figure 6: Regressor Selection



Notes: Darker shade for $|y_\delta| \geq 6$ in panels A.1 and B.1 is the part of the critical region imposed by the switching rule. In panels A.2 and B.2, solid lines are the rejection probability of the nearly optimal tests $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$, and dashed lines are for the usual test that ignores Y_δ, $\varphi_S(y) = 1[|y_\beta| > 1.96]$.

Figure 7: Weighted Average Power in Regressor Selection Problem as Function of ρ



Notes: Thick solid line is power bound, thin solid line is power of 5% level test, and dashed line is power of usual test that ignores Y_δ, $\varphi_S(y) = \mathbf{1}[|y_\beta| > 1.96]$.

largest possible among all 5% valid tests. To get a more comprehensive view of the potential gains in power as a function of ρ, Figure 7 depicts the power bound, the power of $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ and the power of φ_S against $\delta = 0$ and $\beta \in \{-2, 2\}$. The experiment (28) becomes more informative about β as ρ increases, and correspondingly, the power bound is an increasing function of ρ.[9] It is striking, though, how flat the power bound becomes once $\rho \geq 0.75$. The gains in power at $\delta = 0$ over the standard test φ_S are never larger than 12 percentage points, and the test $\varphi^{\varepsilon}_{\Lambda^*,S,\chi}$ described in the supplementary materials comes very close to fully exploiting the available information.

6 Conclusion

Many statistics have been explored as the basis for constructing valid tests in non-standard econometric problems: maximum likelihood estimators, quasi likelihood ratio statistics, moment based criterion functions, statistics involving pretests, etc. Similarly, appropriate critical values may be obtained by various methods: by searching for the largest appropriate quantile of the test statistic under the null hypothesis, by arguing for the validity of bootstrap or subsampling methods, or by refining these approaches with Bonferroni inequalities

[9] Adding mean-zero Gaussian noise to Y_δ and an appropriate rescaling yields an equivalent experiment with smaller $|\rho|$.

or pretests.

The numerical determination of an approximately least favorable null distribution suggested here is, in general, not substantially harder, either conceptually or computationally. And, in contrast to these other approaches, it delivers a test that is nearly optimal in the sense that it comes close to maximizing a weighted average power criterion. What is more, once the approximate least favorable distribution is determined, the test is straightforward to implement in practice, compared to, say, methods that require resampling. Also, the low upper bound on weighted average power implied by the approximately least favorable distribution can be used to argue for the near optimality of a convenient test statistic, such as the quasi likelihood ratio statistic. For these reasons, the algorithm suggested here seems a practically useful approach to deal with non-standard testing problems.

The six applications considered in this paper have a one-dimensional nuisance parameter, and Müller (2012) and Müller and Watson (2013b) discuss additional applications with a one-dimensional nuisance parameter. While a high dimensional nuisance parameter generally poses numerical difficulties for the algorithm, Müller and Watson (2013a) successfully implement the algorithm in a problem with a three-dimensional nuisance parameter. The relatively hardest part in the computations is the numerical check of size control, but unless the problem has special features, the difficulty of this step is common to all approaches to constructing nontrivial tests.

A Appendix

A.1 Proof of Lemma 2

For any h_0, we have from (11)

$$\frac{f_{n,h_0}(Y)}{f_{n,0}(Y)} \Rightarrow \frac{f_{X,h_0}(X)}{f_{X,0}(X)}$$

so that f_{n,h_0} is contiguous to $f_{n,0}$ by LeCam's first lemma. A general version of LeCam's third lemma (see, for instance van der Vaart (2002)) thus implies that (11) also holds with Y and X distributed according to f_{n,h_0} and f_{X,h_0}, respectively. This establishes convergence of experiments via Definition 9.1 of van der Vaart (1998).

With $\epsilon = \limsup_{\Delta\to\infty}(E_{(b_1,\Delta)}[\varphi(Y)] - E_{(b_1,\Delta)}[\varphi_S(Y)]) > 0$, pick Δ_n with $\Delta_n \to \infty$ such that $\lim_{n\to\infty}(E_{(b_1,\Delta_n)}[\varphi(Y)] - E_{(b_1,\Delta_n)}[\varphi_S(Y)]) = \epsilon$. With this definition of Δ_n, $E_{(b,\Delta_n)}[\cdot]$ corresponds to integration w.r.t. $f_{n,h}$, where $h = (b, 0) \in \mathbb{R}^{k_\beta+1}$, denoted by $E_{n,h}[\cdot]$. By assumption about φ_S and $\varphi_S^{\lim}$, $E_{n,h_1}[\varphi_S(Y)] \to E_{h_1}[\varphi_S^{\lim}(X)]$, where $h_1 = (b_1, 0)$. Thus, $E_{n,h_1}[\varphi(Y)] \to E_{h_1}[\varphi_S^{\lim}(X)] + \epsilon$. Further, by contiguity, also $E_{n,h}[\varphi(Y)]$ converges for all fixed h. By Theorem 15.1 of van der Vaart (1998), there exists a test $\varphi^{\lim}$ with a power function $E_h[\varphi^{\lim}(X)]$ equal to $\lim_{n\to\infty} E_{n,h}[\varphi(Y)]$. The test $\varphi^{\lim}$ is of level α, as $E_{(0,\delta_n)}[\varphi(Y)] \leq \alpha$ for all n implies $\lim_{n\to\infty} E_{(0,\delta_n)}[\varphi(Y)] \leq \alpha$. Further, since $\varphi_S^{\lim}$ is admissible with a power function that does not depend on d, and $\varphi^{\lim}$ has higher power than $\varphi_S^{\lim}$ at $b = b_1$, there must exist b_2 such that with $h_2 = (b_2, 0)$, $E_{h_2}[\varphi^{\lim}(X)] < E_{h_2}[\varphi_S^{\lim}(X)]$. The conclusion follows from $E_{n,h_2}[\varphi(Y)] \to E_{h_2}[\varphi^{\lim}(X)]$ and $E_{n,h_2}[\varphi_S(Y)] \to E_{h_2}[\varphi_S^{\lim}(X)]$.

A.2 Algorithm Details

A.2.1 Details of the Algorithm Described in Section 3

Repeated application of (10) requires evaluation of the rejection probability $\int \varphi_\mu f_j d\nu$. This can be implemented straightforwardly by Monte Carlo integration, using N_0 independent draws $Y_{j,l}$ from f_j, $l = 1, \ldots, N_0$, for a total of $M \cdot N_0$ independent draws. Thus, one simple estimator of $\int \varphi_\mu f_j d\nu$ is $N_0^{-1} \sum_{l=1}^{N_0} \mathbf{1}[g(Y_{j,l}) > \sum_{i=1}^{M} \exp(\mu_i) f_i(Y_{j,l})]$. Note that the $M \cdot N_0$ scalar random variables $g(Y_{i,l})$ and $M^2 \cdot N_0$ variables $f_i(Y_{j,l})$ can be computed and stored once, prior to any fixed point iteration. But with f_i readily available, it makes sense to improve the estimator of $\int \varphi_\mu f_j d\nu$ via importance sampling, that is to use

$$\widehat{\mathrm{RP}}_j(\mu) = (MN_0)^{-1} \sum_{k=1}^{M} \sum_{l=1}^{N_0} \frac{f_j(Y_{k,l})}{\bar{f}(Y_{k,l})} \mathbf{1}[g(Y_{k,l}) > \sum_{i=1}^{M} \exp(\mu_i) f_i(Y_{k,l})] \tag{31}$$

where $\bar{f}(y) = M^{-1} \sum_{j=1}^{M} f_j(y)$. This has the advantage that with a finer discretization of H_0 (larger M), one may decrease N_0 for the same level of Monte Carlo accuracy, as the draws from neighboring densities become relatively more informative for the rejection probability under f_j.

The ultimate algorithm consists of the following 8 steps.

1. For each k, $k = 1, \ldots, M$, generate N_0 draws $Y_{k,l}$, $l = 1, \ldots, N_0$, with density f_k. The draws $Y_{k,l}$ are independent across k and l.

2. Compute and store $g(Y_{k,l})$, $f_j(Y_{k,l})$ and $\bar{f}(Y_{k,l})$, $j,k = 1,\ldots,M$, $l = 1,\ldots,N_0$.

3. Set $\mu^{(0)} = (-2,\ldots,-2) \in \mathbb{R}^M$.

4. Compute $\mu^{(i+1)}$ from $\mu^{(i)}$ via $\mu_j^{(i+1)} = \mu_j^{(i)} + \omega(\widehat{\mathrm{RP}}_j(\mu^{(i)}) - \alpha)$ and $\omega = 2$, and repeat this step $O = 600$ times. Denote the resulting element in the simplex by $\hat{\Lambda}^* = (\hat{\lambda}_1^*,\ldots,\hat{\lambda}_M^*)$, where $\hat{\lambda}_j^* = \exp(\mu_j^{(O)}) / \sum_{k=1}^M \exp(\mu_k^{(O)})$.

5. Compute the number cv such that the test $\varphi_{\hat{\Lambda}^*}$ is exactly of (Monte Carlo) level α when Y is drawn from the mixture $\sum_{i=1}^M \hat{\lambda}_i^* f_i$, that is solve $\sum_{j=1}^M \hat{\lambda}_j^* (\widehat{\mathrm{RP}}_j^*(\mathrm{cv}) - \alpha) = 0$, where $\widehat{\mathrm{RP}}_j^*(\mathrm{cv})$ is the Monte Carlo estimate (31) of the rejection probability of the test $\mathbf{1}[g > \mathrm{cv} \sum_{i=1}^M \hat{\lambda}_i^* f_i]$ under f_j.

6. Compute the (estimate of) the power bound $\bar{\pi}$ of $\varphi_{\hat{\Lambda}^*} = \mathbf{1}[g > \mathrm{cv} \sum_{i=1}^M \hat{\lambda}_i^* f_i]$ via $N_1^{-1} \sum_{l=1}^{N_1} \varphi_{\hat{\Lambda}^*}(Y_l)$, where Y_l are N_1 i.i.d. draws of Y with density g.[10]

7. Compute the number $\mathrm{cv}^\varepsilon > \mathrm{cv}$ such that the test $\varphi_{\hat{\Lambda}^*}^\varepsilon = \mathbf{1}[g > \mathrm{cv}^\varepsilon \sum_{i=1}^M \hat{\lambda}_i^* f_i]$ has (estimated) power $\bar{\pi} - \varepsilon$, using the same estimator as in Step 6.

8. Check size control of $\varphi_{\hat{\Lambda}^*}^\varepsilon$ by evaluating its null rejection probabilities on a fine grid of H_0, using estimates of null rejection probabilities of the form (31). If $\varphi_{\hat{\Lambda}^*}^\varepsilon$ is found to overreject, restart the algorithm at Step 1 with a finer discretization of H_0 (larger M). Otherwise, $\hat{\Lambda}^*$ satisfies the definition of an ε-ALFD, $\Lambda^* = \hat{\Lambda}^*$.

With $N_0 = 20,000$ and $N_1 = 100,000$ (so that Monte Carlo standard deviations of $\widehat{\mathrm{RP}}_j(\mu)$ are about 0.1%) and $M = 50$ the algorithm takes about one minute on a modern PC, and it spends most of the time on Steps 2 and 8.

A.2.2 Details of the Algorithm Used to Compute ε-ALFD Switching Tests

First note the following result that extends Lemma 1 to tests of the switching form (16).

Lemma 5 *For given χ and φ_S, let SW be the set of tests of the form (16). Let $\varphi_{\Lambda,S,\chi} \in SW$ be of size α under $H_{0,\Lambda}$ with φ_Λ of the Neyman-Pearson form*

$$\varphi_\Lambda = \begin{cases} 1 \text{ if } g(y) > \mathrm{cv} \int f_\theta(y) d\Lambda(\theta) \\ \kappa \text{ if } g(y) = \mathrm{cv} \int f_\theta(y) d\Lambda(\theta) \\ 0 \text{ if } g(y) < \mathrm{cv} \int f_\theta(y) d\Lambda(\theta) \end{cases}$$

for some $\mathrm{cv} \geq 0$ *and* $0 \leq \kappa \leq 1$. *Then for any test $\varphi \in SW$ that is of level α under H_0, $\int \varphi_{\Lambda,S,\chi} g d\nu \geq \int \varphi g d\nu$.*

[10] It makes sense to choose $N_1 > N_0$, as the importance sampling in (31) leads to a relatively smaller Monte Carlo standard error.

Proof. Note that by construction, $\int(\varphi_{\Lambda,S,\chi}-\varphi)(g-\mathrm{cv}\int f_\theta d\Lambda(\theta))d\nu \geq 0$. Since φ is of level α under H_0, $\int \varphi f_\theta d\nu \leq \alpha$ for all $\theta \in \Theta_0$. Therefore, $\int\int \varphi f_\theta d\nu d\Lambda(\theta) = \int \varphi(\int f_\theta d\Lambda(\theta))d\nu \leq \alpha$, where the equality follows from Fubini's Theorem. Thus $\int(\varphi_{\Lambda,S,\chi}-\varphi)(\int f_\theta d\Lambda(\theta))d\nu \geq 0$, and the result follows. ■

Given this result, the only change to the algorithm of Section **??** is the replacement of φ_Λ and $\varphi^\varepsilon_\Lambda$ by $\varphi_{\Lambda,S,\chi}$ and $\varphi^\varepsilon_{\Lambda,S,\chi}$, respectively. Specifically, the (estimated) rejection probability (31) now reads

$$\widehat{\mathrm{RP}_j}(\mu) = (MN_0)^{-1}\sum_{k=1}^{M}\sum_{l=1}^{N_0}\frac{f_j(Y_{k,l})}{\bar{f}(Y_{k,l})}(\chi(Y_{k,l})\varphi_S(Y_{k,l}) + (1-\chi(Y_{k,l}))\mathbf{1}[g(Y_{k,l}) > \sum_{i=1}^{M}\exp(\mu_i)f_i(Y_{k,l})])$$

and in Steps 6 and 7, $\bar{\pi}$ and $\bar{\pi}-\varepsilon$ are the (estimated) powers of the tests $\varphi_{\hat{\Lambda},S,\chi} = \chi\varphi_S+(1-\chi)\mathbf{1}[g > \mathrm{cv}\sum_{i=1}^{M}\hat{\lambda}_i f_i]$ and $\varphi^\varepsilon_{\hat{\Lambda},S,\chi} = \chi\varphi_S + (1-\chi)\mathbf{1}[g > \mathrm{cv}^\varepsilon\sum_{i=1}^{M}\hat{\lambda}_i f_i]$, respectively. Correspondingly, the size check in Step 8 also concerns $\varphi^\varepsilon_{\hat{\Lambda},S,\chi}$.

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Supplementary Appendix to

Nearly Optimal Tests when a Nuisance Parameter is Present Under the Null Hypothesis

by Graham Elliott, Ulrich K. Müller and Mark W. Watson

B Details of the Algorithm Used to Compute the Power Bounds in Section 4.3

Similar to the discussion in Section 3, discretize $\Theta_{1,S}$ by defining M_1 base distributions $\Psi_{1,i}$ with support in $\Theta_{1,S}$, and denote $f_{1,i} = \int f_\theta d\Psi_{i,1}$. The constraint $\inf_{\theta\in\Theta_{1,S}}\left[\int \varphi f_\theta d\nu - \pi_S(\theta)\right] \geq 0$ on φ thus implies $\int \varphi f_{1,i} d\nu \geq \int \tilde{\varphi} f_{1,i} d\nu$, $i = 1, \ldots, M_1$. For notational consistency, denote the discretization of Θ_0 by $f_{0,i}$, $i = 1, \ldots, M_0$. Let $\mu = (\mu_0', \mu_1')' \in \mathbb{R}^{M_0} \times \mathbb{R}^{M_1}$, and consider tests of the form

$$\varphi_\mu = \mathbf{1}[g + \sum_{i=1}^{M_1} \exp(\mu_{1,i}) f_{1,i} > \sum_{i=1}^{M_0} \exp(\mu_{0,i}) f_{0,i}].$$

The algorithm is similar to the one described in Section **??**, but based on the iterations

$$\begin{aligned}
\mu_{0,j}^{(i+1)} &= \mu_{0,j}^{(i)} + \omega(\int \varphi_{\mu^{(i)}} f_{0,j} d\nu - \alpha),\ j = 1, \ldots, M_0 \\
\mu_{1,j}^{(i+1)} &= \mu_{1,j}^{(i)} - \omega(\int \varphi_{\mu^{(i)}} f_{1,j} d\nu - \int \tilde{\varphi} f_{1,i} d\nu),\ \ j = 1, \ldots, M_1.
\end{aligned}$$

More explicitly, the importance sampling estimators for $\int \varphi_{\mu^{(i)}} f_{0,j} d\nu$ and $\int \varphi_{\mu^{(i)}} f_{1,j} d\nu$ are given by

$$\begin{aligned}
\widehat{\mathrm{RP}}_{0,j}(\mu) &= (M_0 N_0)^{-1} \sum_{k=1}^{M_0} \sum_{l=1}^{N_0} \frac{f_{0,j}(Y_{k,l}^0)}{\bar{f}_0(Y_{k,l}^0)} \mathbf{1}[g(Y_{k,l}^0) + \sum_{i=1}^{M_1} \exp(\mu_{1,i}) f_{1,i}(Y_{k,l}^0) > \sum_{i=1}^{M_0} \exp(\mu_{0,i}) f_{0,i}(Y_{k,l}^0)] \\
\widehat{\mathrm{RP}}_{1,j}(\mu) &= (M_1 N_0)^{-1} \sum_{k=1}^{M_1} \sum_{l=1}^{N_0} \frac{f_{1,j}(Y_{k,l}^1)}{\bar{f}_1(Y_{k,l}^1)} \mathbf{1}[g(Y_{k,l}^1) + \sum_{i=1}^{M_1} \exp(\mu_{1,i}) f_{1,i}(Y_{k,l}^1) > \sum_{i=1}^{M_0} \exp(\mu_{0,i}) f_{0,i}(Y_{k,l}^1)]
\end{aligned}$$

where $\bar{f}_0(y) = M_0^{-1} \sum_{j=1}^{M_0} f_{0,j}(y)$ and $\bar{f}_1(y) = M_1^{-1} \sum_{j=1}^{M_1} f_{1,j}(y)$, and $Y_{k,l}^0$ and $Y_{k,l}$ are N_0 i.i.d. draws from density $f_{0,k}$ and $f_{1,k}$, respectively. For future reference, for two given points $\hat{\Lambda}_0^* = (\hat{\lambda}_{0,1}^*, \ldots, \hat{\lambda}_{0,M_0}^*)$ and $\hat{\Lambda}_1^* = (\hat{\lambda}_{1,1}^*, \ldots, \hat{\lambda}_{1,M_1}^*)$ in the M_0 and M_1 dimensional simplex, respectively, define

$$\widehat{\mathrm{RP}}_{0,j}(\mathrm{cv}_0, \mathrm{cv}_1) = (M_0 N_0)^{-1} \sum_{k=1}^{M_0} \sum_{l=1}^{N_0} \frac{f_{0,j}(Y_{k,l}^0)}{\bar{f}_0(Y_{k,l}^0)} \mathbf{1}[g(Y_{k,l}^0) + \mathrm{cv}_0 \sum_{i=1}^{M_1} \hat{\lambda}_{1,i}^* f_{1,i}(Y_{k,l}^0) > \mathrm{cv}_1 \sum_{i=1}^{M_0} \hat{\lambda}_{1,i}^* f_{0,i}(Y_{k,l}^0)]$$

$$\widehat{\mathrm{RP}}_{1,j}(\mathrm{cv}_0,\mathrm{cv}_1) = (M_1N_0)^{-1}\sum_{k=1}^{M_1}\sum_{l=1}^{N_0}\frac{f_{1,j}(Y_{k,l}^1)}{\bar{f}_1(Y_{k,l}^1)}\mathbf{1}[g(Y_{k,l}^1)+\mathrm{cv}_0\sum_{i=1}^{M_1}\hat{\lambda}_{1,i}^* f_{1,i}(Y_{k,l}^0) > \mathrm{cv}_1\sum_{i=1}^{M_0}\hat{\lambda}_{1,i}^* f_{0,i}(Y_{k,l}^1)]$$

$$\widehat{\mathrm{RP}}_{g}(\mathrm{cv}_0,\mathrm{cv}_1) = N_1^{-1}\sum_{l=1}^{N_1}\mathbf{1}[g(Y_l)+\mathrm{cv}_0\sum_{i=1}^{M_1}\hat{\lambda}_{1,i}^* f_{1,i}(Y_l) > \mathrm{cv}_1\sum_{i=1}^{M_0}\hat{\lambda}_{1,i}^* f_{0,i}(Y_l)]$$

where Y_l are N_1 i.i.d. draws from density g. The algorithm now proceeds in the following steps:

1. For each k, $k=1,\ldots,M_0$, generate N_0 i.i.d. draws $Y_{k,l}^0$, $l=1,\ldots,N_0$, with density $f_{0,k}$, and for each $k=1,\ldots,M_1$, generate N_0 i.i.d. draws $Y_{k,l}^1$, $l=1,\ldots,N_0$ with density $f_{1,k}$. The draws $Y_{k,l}^0$ and $Y_{k,l}^1$ are independent across k and l.

2. Compute and store $g(Y_{k,l})$, $f_{0,j}(Y_{k,l}^0)$, $\bar{f}_0(Y_{k,l}^0)$, $j,k=1,\ldots,M_0$, $l=1,\ldots,N_0$, as well as $f_{1j}(Y_{k,l}^1)$ and $\bar{f}_1(Y_{k,l}^1)$, $j,k=1,\ldots,M_1$, $l=1,\ldots,N_0$.

3. Compute the (estimated) power $\pi_j \approx \int\tilde{\varphi}f_{1,j}d\nu$ of $\tilde{\varphi}=\chi\varphi_S$ under $f_{1,j}$ via $\pi_j = (M_1N_0)^{-1}\sum_{k=1}^{M_1}\sum_{l=1}^{N_0}\frac{f_{1,j}(Y_{k,l}^1)}{\bar{f}_1(Y_{k,l}^1)}\chi(Y_{k,l}^1)\varphi_S(Y_{k,l}^1)$, $j=1,\ldots,M_1$.

4. Set $\mu^{(0)}=(-2,\ldots,-2)\in\mathbb{R}^{M_0+M_1}$.

5. Compute $\mu^{(i+1)}$ from $\mu^{(i)}$ via $\mu_{0,j}^{(i+1)} = \mu_{0,j}^{(i)} + \omega(\widehat{\mathrm{RP}}_{0,}(\mu^{(i)})-\alpha)$, $j=1,\ldots,M_0$ and $\mu_{1,j}^{(i+1)} = \mu_{1,j}^{(i)} - \omega(\widehat{\mathrm{RP}}_{1,j}(\mu^{(i)})-\pi_j)$, $j=1,\ldots,M_1$ with $\omega=2$, and repeat this step $O=600$ times. Denote the resulting elements in the M_0 and M_1 dimensional simplex by $\hat{\Lambda}_0^*=(\hat{\lambda}_{0,1}^*,\ldots,\hat{\lambda}_{0,M_0}^*)$ and $\hat{\Lambda}_1^*=(\hat{\lambda}_{1,1}^*,\ldots,\hat{\lambda}_{1,M_1}^*)$, where $\hat{\lambda}_{0,j}^* = \exp(\mu_{0,j}^{(O)})/\sum_{k=1}^{M_0}\exp(\mu_{0,k}^{(O)})$ and $\hat{\lambda}_{1,j}^* = \exp(\mu_{1,j}^{(O)})/\sum_{k=1}^{M_1}\exp(\mu_{1,k}^{(O)})$.

6. Compute the number $\mathrm{cv}_{0,0}^*$ such that the test $\mathbf{1}[g > \mathrm{cv}_{0,0}^*\sum_{i=1}^{M_0}\hat{\lambda}_{0,i}^* f_{0,i}]$ is exactly of (Monte Carlo) level α under the mixture $\sum_{j=1}^{M_0}\hat{\lambda}_{0,j}^* f_{0,j}$, that is solve $\sum_{j=1}^{M_0}\hat{\lambda}_{0,j}^*\widehat{\mathrm{RP}}_{0,j}(\mathrm{cv}_{0,0},0)=\alpha$ for $\mathrm{cv}_{0,0}$. If the resulting test has power under the mixture $\sum_{j=1}^{M_1}\hat{\lambda}_{1,j}^* f_{1,j}$ larger than $\sum_{j=1}^{M_1}\hat{\lambda}_{1,j}^*\pi_j$, that is if $\sum_{j=1}^{M_1}\hat{\lambda}_{1,j}^*(\widehat{\mathrm{RP}}_{1,j}(\mathrm{cv}_{0,0}^*,0)-\pi_j)\geq 0$, then the power constraint doesn't bind, and the power bound is given by $\widehat{\mathrm{RP}}_g(\mathrm{cv}_{0,0}^*,0)$.

7. Otherwise, compute the two numbers cv_0^* and cv_1^* such that the test $\mathbf{1}[g+\mathrm{cv}_1^*\sum_{i=1}^{M_1}\hat{\lambda}_{1,i}^* f_{1,i} > \mathrm{cv}_0^*\sum_{i=1}^{M_0}\hat{\lambda}_{0,i}^* f_{0,i}]$ is of (Monte Carlo) level α under the mixture $\sum_{j=1}^{M_0}\hat{\lambda}_{0,j}^* f_{0,j}$, and of power equal to $\sum_{j=1}^{M_1}\hat{\lambda}_{1,j}^*\pi_j$ under the mixture $\sum_{j=1}^{M_1}\hat{\lambda}_{1,j}^* f_{1,j}$, that is solve the two equations $\sum_{j=1}^{M_0}\hat{\lambda}_{0,j}^*\widehat{\mathrm{RP}}_{0,j}(\mathrm{cv}_0,\mathrm{cv}_1)=\alpha$ and $\sum_{j=1}^{M_1}\hat{\lambda}_{1,j}^*\widehat{\mathrm{RP}}_{1,j}(\mathrm{cv}_0,\mathrm{cv}_1)=\sum_{j=1}^{M_1}\hat{\lambda}_{1,j}^*\pi_j$ for $(\mathrm{cv}_0,\mathrm{cv}_1)\in\mathbb{R}^2$. The power bound is then given by $\widehat{\mathrm{RP}}_g(\mathrm{cv}_0^*,\mathrm{cv}_1^*)$.

C Additional Details for the Applications

The following Lemma is useful for obtaining closed form expressions in many of the applications.

Lemma 6 *For $c > 0$, $\int_{-\infty}^{a} \exp[sd - \frac{1}{2}s^2c^2]ds = \sqrt{2\pi}c^{-1}\exp[\frac{1}{2}d^2/c^2]\Phi(ac - d/c)$, where Φ is the c.d.f. of a standard normal.*

Proof. Follows from "completing the square". ■

In all applications, the M base distributions on Θ_0 are either uniform distributions, or point masses. Size control is always checked by computing the Monte Carlo rejection probability at all δ that are end or mid-points of these intervals, or that are simple averages of the adjacent locations of point masses, respectively (this check is successful in all applications). The power bound calculations under the power constraint of Section 4.3 use the same $M_0 = M$ base distributions under the null, and the M_1 base distributions with support on $\Theta_{1,S}$ all set β to the same value as employed in F, and use the same type of base distribution on δ as employed in the discretization of Θ_0.

C.1 Running Example

The base distributions on Θ_0 are the uniform distributions on the intervals $\{[0, 0.04], [0, .5], [.5, 1], [1, 1.5], \ldots, [12, 12.5]\}$. The base distributions on $\Theta_{1,S}$ have $\beta \in \{-2, 2\}$ and δ uniform on the intervals $\{[9, 9.5], [9.5, 10], \ldots, [13, 13.5]\}$.

C.2 Behrens-Fisher

Limit Experiment and Standard Best Test:

We analyze convergence as $\delta \to \infty$, that is as $\sigma_2/\sigma_1 \to 0$. The convergence as $\delta \to -\infty$ follows by the same argument.

Consider the 4 dimensional observation $\tilde{Y} = (\bar{x}_1, \bar{x}_2, s_1, s_2)$, with density

$$\frac{\sqrt{n_1 n_2}}{\sigma_1^2\sigma_2^2}\phi(\frac{\bar{x}_1 - \mu_1}{\sigma_1/\sqrt{n_1}})\phi(\frac{\bar{x}_2 - \mu_2}{\sigma_2/\sqrt{n_2}})f_{n_1}(\frac{s_1}{\sigma_1})f_{n_2}(\frac{s_2}{\sigma_2})$$

where ϕ is the density of a standard normal, and f_n is the density of a chi-distributed random variable with $n-1$ degrees of freedom, divided by $\sqrt{n-1}$. Now set $\mu_2 = 0$, so that $b = \beta = (\mu_1 - \mu_2)/\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}$ implies $\mu_1 = b\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}$. Also, set $\sigma_1 = \exp(d)$ and $\sigma_2 = \exp(-\Delta_n)$, so that $\delta = \log(\sigma_1/\sigma_2) = \Delta_n + d$. This is without loss of generality as long as one restricts attention to tests that are invariant to the transformations described in the main text.

Let $f_{n,h}$ be the density of $\tilde{Y}$ in this parameterization, where $h = (b, d)$. Further, let $f_{X,h}$ be the density of the bivariate vector $X = (X_b, X_d)$ where $X_b \sim \mathcal{N}(b\exp(d), \exp(2d))$ and X_d is an independently chi-distributed random variable with $n-1$ degrees of freedom, divided by $\sqrt{n-1}$. With $\tilde{Y}$ distributed according to $f_{n,0}$, and X distributed according to $f_{X,0}$, we find for any finite set $H \subset \mathbb{R}^2$

$$\left\{\frac{f_{n,h}(\tilde{Y})}{f_{n,0}(\tilde{Y})}\right\}_{h\in H} = \left\{\frac{\exp(2d)\phi(\frac{\bar{x}_1 - b\sqrt{\exp(2d)/n_1 + \exp(-2\Delta_n)/n_2}}{\exp(d)/\sqrt{n_1}})f_{n_1}(\frac{s_1}{\exp(d)})}{\phi(\frac{\bar{x}_1}{1/\sqrt{n_1}})f_{n_1}(s_1)}\right\}_{h\in H}$$

$$\Rightarrow \quad \left\{ \frac{\exp(2d)\phi(\frac{X_b - b\exp(d)/\sqrt{n_1}}{\exp(d)/\sqrt{n_1}}) f_{n_1}(\frac{X_d}{\exp(d)})}{\phi(\frac{X_b}{1/\sqrt{n_1}}) f_{n_1}(X_d)} \right\}_{h\in H} = \left\{ \frac{f_{X,h}(X)}{f_{X,0}(X)} \right\}_{h\in H}$$

so that Condition 1 is satisfied. Thus, tests of $H_0 : b = 0$ against $H_1 : b \neq 0$ based on X form an upper bound on the asymptotic power as $\Delta_n \to \infty$ of invariant tests based on $\tilde{Y}$. The standard (and admissible) test $\varphi_S^{\lim}$ based on X is the usual t-test $\mathbf{1}[|X_b|/X_d > \text{cv}]$. Further, a straightforward calculation shows that the invariant test $\varphi_S = \mathbf{1}[|\frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}| > \text{cv}] = \mathbf{1}[|Y_\beta| > \text{cv}]$ has the same asymptotic rejection probability as $\varphi_S^{\lim}$ for all fixed values of h.

Computational details:

It is computationally convenient to consider the one-to-one transformation $(t, r) = ((\bar{x}_1 - \bar{x}_2)/s_2, s_1/s_2) = (\sqrt{\frac{e^{2Y_\delta}}{n_1} + \frac{1}{n_2}} Y_\beta, e^{Y_\delta})$ with parameters $\eta = \mu_1 - \mu_2 = \beta\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}$ and $\omega = \sigma_1/\sigma_2 = \exp(\delta)$. A transformation of variable calculation shows that the density of (t, r) is given by

$$f(t,r) = \frac{(n_1-1)^{n_1/2}(n_2-1)^{n_2/2}\omega}{r^2\Gamma(\frac{1+n_1}{2})\Gamma(\frac{1+n_2}{2})} \sqrt{\frac{n_1 n_2}{\pi(n_1+\omega^2 n_2)}} \left(\frac{r}{\omega}\right)^{n_1} 2^{(1-n_1-n_2)/2} \exp[-\tfrac{1}{2}\frac{\eta^2 n_1 n_2}{n_1+n_2\omega^2}]$$
$$\times \int_0^\infty s^{n_1+n_2-2} \exp[\frac{2\eta n_1 n_2 st - s^2((n_2-1)n_2\omega^4 + n_1^2 r^2 - n_2\omega^2 + n_1(n_2\omega^2(1+r^2+t^2) - \omega^2 - r^2))/\omega^2}{2(n_1+n_2\omega^2)}] ds$$

where Γ denotes the Gamma function. The integral is recognized as being proportional to the $(n_1 + n_2 - 2)$th absolute moment of a half normal. In particular, for $c > 0$, $\int_0^\infty \exp[-\frac{1}{2}s^2c^2]s^n ds = 2^{\frac{n-1}{2}}\Gamma(\frac{1+n}{2})c^{-(n+1)}$, and following Dhrymes (2005),

$$\int_0^\infty \exp[sd - \tfrac{1}{2}s^2c^2]s^n ds = \exp[\tfrac{1}{2}\frac{d^2}{c^2}]\frac{d^n}{c^{2n+1}} \sum_{l=0}^{n} \binom{n}{l} (-\frac{c}{d})^l I_l(d/c^2)$$

where

$$\begin{aligned} I_l(h) &= \sqrt{2\pi}\int_{-\infty}^{h} \phi(z) z^l dz \\ &= 2^{(l-1)/2}((-1)^l\Gamma(\frac{1+l}{2}) + (h/|h|)^{l+1}(\Gamma(\frac{1+l}{2}) - \tilde{\Gamma}(\frac{1+l}{2}, \frac{h^2}{2}))) \end{aligned}$$

with ϕ the pdf of a standard normal, and $\tilde{\Gamma}$ the incomplete Gamma function.

The base distributions on Θ_0 are uniform distributions for δ on the intervals $\{[-12.5, -12], [-12, -11.5], \ldots, [12, 12.5]\}$, and the base distributions on $\Theta_{1,S} = \{(\beta, \delta\} : |\delta| > 9\}$ have δ uniform on $\{[-14, -13.5], [-13.5, -13], \ldots, [-9.5, -9]\} \cup \{[9, 9.5], [9.5, 10], \ldots, [14.5, 15]\}$. The corresponding integrals are computed via Gaussian quadrature using 10 nodes (for this purpose the integral under the alternative is split up in intervals of length 2). For $n_1 = n_2$, symmetry around zero is imposed in the calculation of the ALFD.

C.3 Break Date

Wiener processes are approximated with 1,000 steps. Symmetry around zero is imposed in the calculation of the ALFD, and the set of base distribution for $|\delta|$ contains uniform distributions on $\{[0,1],[1,2],\ldots,[19,20]\}$.

C.4 Predictive Regression

Limit Experiment:

As in the main text, let $\delta = r_\delta(\Delta_n, d) = \Delta_n - \sqrt{2\Delta_n}d$ and $\beta = r_\beta(\Delta_n, \beta) = \sqrt{2\Delta_n/(1-\rho^2)}b$. After some algebra, under $h = 0$

$$\begin{aligned}\ln \frac{f_{n,h}(G)}{f_{n,0}(G)} &= \sqrt{2\Delta_n}\left(\frac{(b-d\rho)\int_0^1 W^\mu_{x,\Delta_n}(s)dW_y(s)}{\sqrt{1-\rho^2}} + d\int_0^1 W_{x,\Delta_n}(s)dW_x(s)\right)\\ &\quad -\Delta_n\left(d^2\int_0^1 W_{x,\Delta_n}(s)^2ds + \frac{(b-d\rho)^2}{1-\rho^2}\int_0^1 W^\mu_{x,\Delta_n}(s)^2ds\right).\end{aligned}$$

Now suppose the following convergence holds as $\Delta_n \to \infty$

$$\begin{pmatrix} \sqrt{2\Delta_n}\int_0^1 W_{x,\Delta_n}(s)dW_x(s) \\ \sqrt{2\Delta_n}\int_0^1 W^\mu_{x,\Delta_n}(s)dW_y(s) \\ 2\Delta_n\int_0^1 W_{x,\Delta_n}(s)^2ds \\ 2\Delta_n\int_0^1 W^\mu_{x,\Delta_n}(s)^2ds \end{pmatrix} \Rightarrow \begin{pmatrix} Z_x \\ Z_y \\ 1 \\ 1 \end{pmatrix} \tag{32}$$

where Z_x and Z_y are independent $\mathcal{N}(0,1)$. Then, as $\Delta_n \to \infty$

$$\begin{aligned}\ln \frac{f_{n,h}(G)}{f_{n,0}(G)} &\Rightarrow -\tfrac{1}{2}\frac{b^2 - 2bd\rho + d^2 - 2(b-\rho d)\sqrt{1-\rho^2}Z_y - 2d(1-\rho^2)Z_x}{1-\rho^2}\\ &= \begin{pmatrix} X_b \\ X_d \end{pmatrix}'\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}^{-1}\begin{pmatrix} b \\ d \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} b \\ d \end{pmatrix}'\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}^{-1}\begin{pmatrix} b \\ d \end{pmatrix}\end{aligned}$$

where $X_b = \rho Z_x + \sqrt{1-\rho^2}Z_y$ and $X_d = Z_x$, and Condition 1 follows from the continuous mapping theorem.

To establish (32) note that

$$\begin{aligned}\int_0^1 W^\mu_{x,\Delta_n}(s)^2ds &= \int_0^1 W_{x,\Delta_n}(s)^2ds - \left(\int_0^1 W_{x,\Delta_n}(s)ds\right)^2\\ \int_0^1 W_{x,\Delta_n}(s)dW_x(s) &= \tfrac{1}{2}(W_{x,\Delta_n}(1)^2 - 1) + \Delta_n\int W_{x,\delta}(s)^2ds\\ \int_0^1 W^\mu_{x,\Delta_n}(s)dW_y(s) &= \int_0^1 W_{x,\Delta_n}(s)dW_y(s) - W_y(1)\int_0^1 W_{x,\Delta_n}(s)ds.\end{aligned}$$

Thus, with $t = (t_1,\ldots,t_4)$ and $\mathrm{i} = \sqrt{-1}$

$$\phi_n(t) = E[\exp[\mathrm{i}t'\begin{pmatrix} \sqrt{2\Delta_n}\int_0^1 W_{x,\Delta_n}(s)dW_x(s) \\ \sqrt{2\Delta_n}\int_0^1 W^\mu_{x,\Delta_n}(s)dW_y(s) \\ 2\Delta_n\int_0^1 W_{x,\Delta_n}(s)^2ds \\ 2\Delta_n\int_0^1 W^\mu_{x,\Delta_n}(s)^2ds \end{pmatrix}]]$$

$$= E[\exp[\mathfrak{i}\begin{pmatrix} -\sqrt{2\Delta_n}t_2 \\ \sqrt{2\Delta_n}t_2 \\ 2\Delta_n t_3 + 2\Delta_n t_4 + \sqrt{2}\Delta_n^{3/2}t_1 \\ \sqrt{\Delta_n/2}t_1 \\ -2\Delta_n t_4 \end{pmatrix}' \begin{pmatrix} W_y(1)\int_0^1 W_{x,\Delta_n}(s)ds \\ \int_0^1 W_{x,\Delta_n}(s)dW_y(s) \\ \int_0^1 W_{x,\Delta_n}(s)^2 ds \\ W_{x,\Delta_n}(1)^2 \\ \left(\int_0^1 W_{x,\Delta_n}(s)ds\right)^2 \end{pmatrix} - \mathfrak{i}t_1\sqrt{\Delta_n/2}]]$$

Note that

$$E[E[\exp[\mathfrak{i}\begin{pmatrix} -\sqrt{2\Delta_n}t_2 \\ \sqrt{2\Delta_n}t_2 \end{pmatrix}' \begin{pmatrix} W_y(1)\int_0^1 W_{x,\Delta_n}(s)ds \\ \int_0^1 W_{x,\Delta_n}(s)dW_y(s) \end{pmatrix}]|W_x]]$$

$$= E[\exp[-\tfrac{1}{2}\begin{pmatrix} -\sqrt{2\Delta_n}t_2 \\ \sqrt{2\Delta_n}t_2 \end{pmatrix}' \begin{pmatrix} \left(\int_0^1 W_{x,\Delta_n}(s)ds\right)^2 & \left(\int_0^1 W_{x,\Delta_n}(s)ds\right)^2 \\ \left(\int_0^1 W_{x,\Delta_n}(s)ds\right)^2 & \int_0^1 W_{x,\Delta_n}(s)^2 ds \end{pmatrix} \begin{pmatrix} -\sqrt{2\Delta_n}t_2 \\ \sqrt{2\Delta_n}t_2 \end{pmatrix}]]$$

Thus

$$\phi_n(t) = E[\exp[\begin{pmatrix} 2\Delta_n t_3\mathfrak{i} + 2\Delta_n t_4\mathfrak{i} + \sqrt{2}\Delta_n^{3/2}t_1\mathfrak{i} - \Delta_n t_2^2 \\ \sqrt{\Delta_n/2}t_1\mathfrak{i} \\ -2\Delta_n t_4\mathfrak{i} + \Delta_n t_2^2 \end{pmatrix}' \begin{pmatrix} \int_0^1 W_{x,\Delta_n}(s)^2 ds \\ W_{x,\Delta_n}(1)^2 \\ \left(\int_0^1 W_{x,\Delta_n}(s)ds\right)^2 \end{pmatrix} - \mathfrak{i}t_1\sqrt{\Delta_n/2}]]$$

$$= E[\exp[\begin{pmatrix} l_{n,1} \\ l_{n,2} \\ l_{n,3} \end{pmatrix}' \begin{pmatrix} \int_0^1 W_{x,\Delta_n}(s)^2 ds \\ W_{x,\Delta_n}(1)^2 \\ \left(\int_0^1 W_{x,\Delta_n}(s)ds\right)^2 \end{pmatrix} - \mathfrak{i}t_1\sqrt{\Delta_n/2}]]$$

$$= \det(I_2 - 2V(\gamma_n)\Omega_n)^{-1/2}\exp[-\mathfrak{i}t_1\sqrt{\Delta_n/2} - \tfrac{1}{2}(\gamma - \Delta_n)]$$

where $\gamma_n = \sqrt{\Delta_n^2 - 2l_{n,1}}$, $\Omega_n = \mathrm{diag}(l_{n,2} + \frac{1}{2}(\gamma_n - \Delta_n), l_{n,3})$ and

$$V(\gamma) = \int \begin{pmatrix} e^{-\gamma(1-s)} \\ \frac{1-e^{-\gamma(1-s)}}{\gamma} \end{pmatrix} \begin{pmatrix} e^{-\gamma(1-s)} \\ \frac{1-e^{-\gamma(1-s)}}{\gamma} \end{pmatrix}' ds$$

and the third equality applies Lemma 1 of Elliott and Müller (2006). Let $\Upsilon_n = \mathrm{diag}(1, \sqrt{\Delta_n})$. A calculation now shows that, as $\Delta_n \to \infty$

$$\begin{aligned} \Upsilon_n V(\gamma_n)\Upsilon_n &\to 0 \\ \Upsilon_n^{-1}\Omega_n\Upsilon_n^{-1} &= O(1) \\ -\mathfrak{i}t_1\sqrt{\Delta_n/2} - \tfrac{1}{2}(\gamma_n - \Delta_n) &\to -\tfrac{1}{2}t_1^2 - \tfrac{1}{2}t_2^2 + t_3\mathfrak{i} + t_4\mathfrak{i} \end{aligned}$$

so that $\phi_n(t)$ converges pointwise to the characteristic function of the right hand side of (32), which proves (32).

Computational details:

Ornstein-Uhlenbeck and stochastic integrals are approximated with 1,000 steps. The base distributions on Θ_0 are point masses at the points $\delta \in \{0^2, .5^2, ..., 14.25^2\}$, and the base distributions on $\Theta_{1,S}$ are point masses on $\delta \in \{160, 165, \ldots, 190\}$, with the corresponding value of β as in (24) with $b = 1.645$.

C.5 Set Identified Parameter

Limit Experiment:

We consider convergence for $\beta \geq 0$ as $\Delta_L \to \infty$, the convergence for $\beta \leq 0$ follows analogously.

Set $\beta = b$, $\delta_P = d_P$ and $\delta_L = \Delta_n + d_L$, so that in this parameterization, $\mu_l = \tau(b, d_P) = \mathbf{1}[b > 0]b - \mathbf{1}[b = 0]d_P$ and $\mu_u = \Delta_n + d_L + \tau(b, d_P)$. For any fixed $h = (b, d_L, d_P) \in \mathbb{R}^2 \times [0, \infty)$, as $\Delta_n \to \infty$

$$\log \frac{f_{n,h}(Y)}{f_{n,0}(Y)} = \begin{pmatrix} Y_l \\ Y_u - \Delta_n \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} \tau(b, d_P) \\ \tau(b, d_P) + d_L \end{pmatrix} - \tfrac{1}{2} \begin{pmatrix} \tau(b, d_P) \\ \tau(b, d_P) + d_L \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} \tau(b, d_P) \\ \tau(b, d_P) + d_L \end{pmatrix}.$$

Because $(Y_l, Y_u - \Delta_n)' \sim \mathcal{N}(0, \Sigma)$ for $h = 0$ as $\Delta_n \to \infty$, Theorem 9.4 in van der Vaart (1998) implies that Condition 1 holds with

$$X = \begin{pmatrix} X_b \\ X_d \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \tau(b, d_P) \\ \tau(b, d_P) + d_L \end{pmatrix}, \Sigma \right).$$

The test of $H_0 : b = 0$, $(d_L, d_P) \in \mathbb{R} \times [0, \infty)$ against $H_1 : b > 0$, $d_L \in \mathbb{R}$ in this limiting experiment thus corresponds to $H_0 : E[X_b] \leq 0$ against $H_1 : E[X_b] > 0$, with $E[X_d]$ unrestricted under both hypotheses. The uniformly best test is thus given by $\varphi_S^{\lim}(x) = \mathbf{1}[x_b > \text{cv}]$: This follows by the analytical least favorable distribution result employed below (29) assuming $d_P = 0$ known, and since $\varphi_S^{\lim}$ is of level α also for $d_P > 0$, putting all mass at $d_P = 0$ is also least favorable in this more general testing problem.

A test with the same asymptotic rejection probability for any fixed h is given by $\varphi_S(y) = \mathbf{1}[y_l > \text{cv}]$.

Computational details:

The base distributions on Θ_0 have δ_L uniform on the intervals $\{[0, 0.1], [0, .5], [.5, 1], [1, 1.5], \ldots, [12.5, 13]\}$, with δ_P an equal probability mixture on the two points $\{0, \delta_L\}$. The base distributions on $\Theta_{1,S}$ have δ_L uniform on the intervals $\{[9, 9.25], [9.25, 9.5], \ldots, [11.75, 12]\}$.

C.6 Regressor Selection

Symmetry around zero is imposed in the computation of the ALFD. The base distributions on Θ_0 are point masses at $|\delta| \in \{0, .2, .4, \ldots, 9\}$.