Spatial Unit Roots*

Ulrich K. Müller and Mark W. Watson
Department of Economics
Princeton University

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Abstract

This paper proposes a model for, and investigates the consequences of, strong spatial dependence in economic variables. Our approach and findings echo those of the corresponding "unit root" time series literature: We suggest a model for spatial $I(1)$ processes, and establish a functional central limit theorem that justifies a large sample Gaussian process approximation for such processes. We further generalize the $I(1)$ model to a spatial "local-to-unity" model that exhibits weak mean reversion. We characterize the large sample behavior of regression inference with spatial $I(1)$ variables, and establish that spurious regression is as much a problem with spatial $I(1)$ data as it is with time series $I(1)$ data. We develop asymptotically valid spatial unit root tests, stationarity tests, and inference methods for the local-to-unity parameter. Finally, we use simulations to study strategies for valid inference in regressions with persistent ($I(1)$ or local-to-unity) spatial data, such as spatial analogues of first-differencing transformations.

Keywords: spatial correlation, spurious regression, Lévy-Brownian motion, functional central limit theorem
JEL: C12, C20

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1 Introduction

Serial correlation complicates inference in time series regressions. When the serial correlation in the regressors and regression errors is weak, that is $I(0)$, inference can proceed as with i.i.d. sampling after using HAC/HAR standard errors that incorporate adjustments for serial correlation. However, when the serial correlation is strong, that is $I(1)$, HAC/HAR inference fails and OLS produces "spurious regressions" (Granger and Newbold (1974)) with estimators and test statistics behaving in non-standard ways (Phillips (1986)). Panel (a) of Figure 1 illustrates this well-known phenomenon: the realization of two independent random walks of length $n = 250$ are strongly correlated in sample, with a corresponding Newey and West (1987) t-statistic that is highly significant.

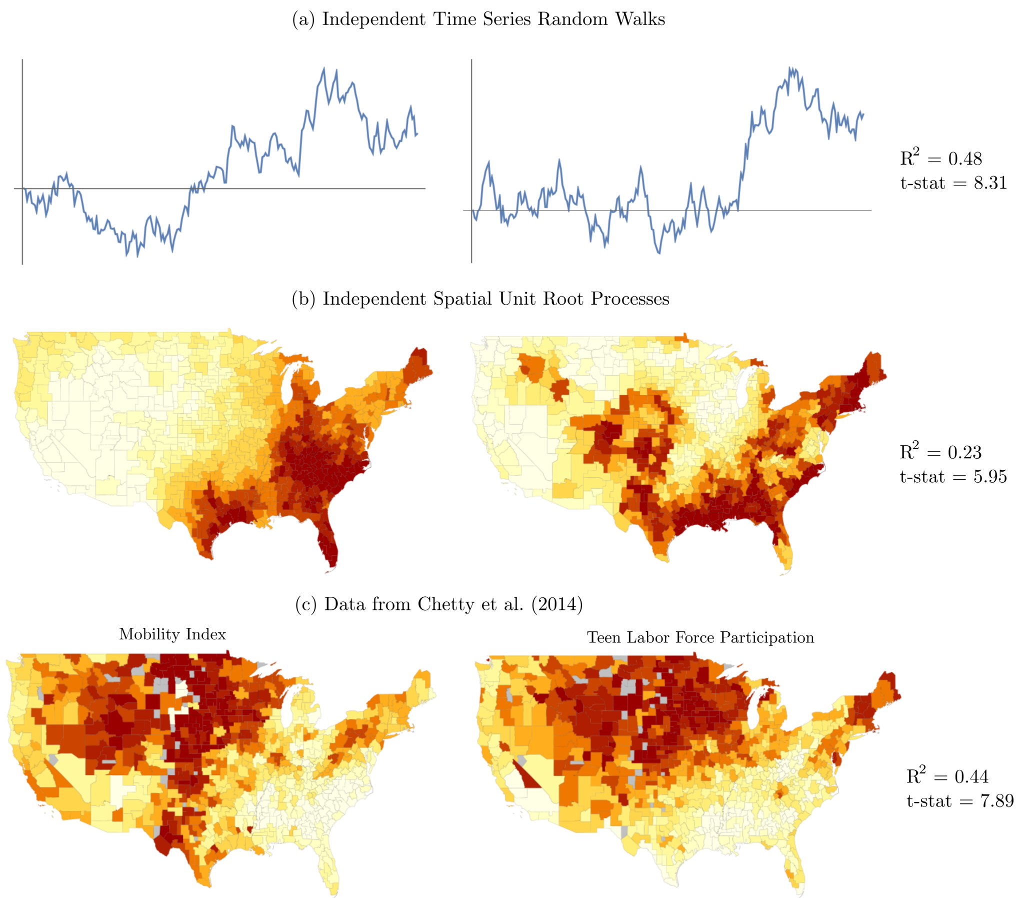
Variables measured over points in space exhibit correlation patterns that in many ways are analogous to serial correlation in time series, and this correlation also complicates inference in spatial regressions. There is a reasonably well-developed literature on HAC/HAR corrections that are required in spatial regressions with weakly dependent regressors and errors.[1] However, much less is known about the implications of strong spatial correlation despite evidence suggesting its presence in many empirical applications in economics (Kelly (2019, 2020)). Panel (b) of Figure 1 illustrates the issue: the realization of two independent spatial "unit root" processes with values for each of the $n = 722$ commuter zones in the 48 contiguous U.S. states are strongly correlated in sample, and a t-statistic that is clustered by U.S. states is highly significant.

This raises several questions. What is a natural spatial analogue of an $I(1)$ time series process, such as the process in Figure 1 (b)? Do such processes systematically induce spuriously significant regression coefficients? How can one test for $I(1)$ spatial persistence? Is there a spatial analogue to the "first-differencing" transformation in time series that eliminates $I(1)$ persistence?

To address these issues, we introduce some basic concepts for the analysis of highly persistent spatial data. In particular, we define a class of spatial $I(1)$ processes, and derive a corresponding functional central limit theorem (FCLT). With those in hand, the analysis of spurious regressions with spatial $I(1)$ processes becomes straightforward. In time series applications, researchers routinely investigate the persistence of their data using unit-root tests,

[1] Conley (1999) is a leading example of spatial HAC inference. See Müller and Watson (2022a, 2022b) for a discussion of the post-Conley literature and new suggestions for inference in regression models with weak spatial dependence.

Figure 1: Strongly Dependent Data in Time and Space



stationarity tests, confidence intervals for large autoregressive roots, half-lives of shocks and so forth. Corresponding tests and methods are not available for spatial data, and they are also developed here. Spurious regression in time series can be avoided by using first-differences of $I(1)$ processes. Empirical researchers using spatial data often allow for regional fixed effects and use clustered standard errors. Are these effective in avoiding spurious regression effects, and are there better methods? This paper also takes up that question.

Throughout the paper we use spatial data and regressions from Chetty, Hendren, Kline, and Saez (2014) to illustrate the issues and methods. These authors construct an index of intergenerational mobility for commuting zones in the United States, and study its relationship to other socioeconomic factors using bivariate regressions with standard errors clustered by U.S. states. As an example, Figure 1 (c) plots their mobility index along with the teenage labor force participation rate. The apparent similarity of these data with the simulated data of panel (b) highlights the empirical relevance of the issues and methods presented here.

Much of our analysis parallels the analysis of persistent time series, but there is a notable difference worth highlighting at the outset. Time series analysis typically studies observations, say y_t, observed at equidistant points in time, $t = 1, 2, 3, ...$ where t indexes months, quarters, years, etc. Economic variables observed in space are typically not so neatly arranged. For example, geographical data may be collected at potentially arbitrary locations s_l within a given region such as a U.S. state, and each state has its own unique shape. For the analysis to be useful in a wide range of spatial applications, we posit a model that assigns values to all locations that may potentially be observed. Thus, for the general problem with d spatial dimensions, we begin with a stochastic process $Y(s)$ over $s \in \mathbb{R}^d$, where $d = 2$ in the geography example. When $d = 1$, s could index time, so this is a time series model where $Y(s)$ is a continuous time process and where the sample data correspond to realizations of $y_l = Y(s_l)$ observed at potentially irregularly spaced points $s_l \in \mathbb{R}$. More abstractly, as discussed in Conley (1999), "locations" might index an economic characteristic and the "economic distance" between locations measures the dissimilarity of the characteristic. We thus follow the geostatistical tradition of positing a continuous parameter model of spatial variation, rather than, say, model dependence by spatial autoregressive (SAR) models of Cliff and Ord (1974) and Anselin (1988).[2]

The roadmap of the paper is a follows. Section 2 provides our definition of a spatial $I(1)$ process. In time series models ($d = 1$ in our notation), the canonical $I(1)$ process is a Wiener process. Lévy-Brownian motion is a useful generalization of the Wiener process for $d > 1$, and Section 2 begins by reviewing its properties. In standard time series models, more general $I(1)$ processes can be constructed by replacing the white noise increments of a random walk with a weakly correlated stationary series. For example, stationary ARMA(p, q) noise yields an ARIMA$(p, 1, q)$ process. Section 2 similarly defines the spatial $I(1)$ process by replacing the white noise innovations in the moving average representation of Lévy-Brownian motion with a weakly dependent stationary spatial process.

An important insight from time series analysis is that the large sample distributions of functions of $I(1)$ processes can be approximated by the distributions of corresponding functions of Wiener processes. The functional central limit theorem (FCLT) is the key driver of such approximations, and it provides the basis for large-sample inference using statistics

[2]Gelfand, Diggle, Guttorp, and Fuentes (2010) provide a useful overview. There is a small literature on unit roots and spurious regression in SAR models that we discuss in Section 5.

constructed from realizations of $I(1)$ processes. Section 2 provides a FCLT that is applicable to spatial $I(1)$ processes. We also show how to appropriately generalize the $I(1)$ model to a spatial "local-to-unity" process and provide a corresponding FCLT result about its large sample behavior.

Armed with the tools from Section 2, Section 3 studies regressions involving spatial $I(1)$ variables, specifically models where the regressors and dependent variable are independent $I(1)$ processes. The section shows that many of the key results from the spurious time series regression (cf., Phillips (1986)) carry over to the spatial case. For example, OLS regression coefficients and the regression R^2 are not consistent, but have limiting distributions that can be represented by functions of Lévy-Brownian motion. Regression F-statistics (including HAC and clustered versions) diverge to infinity. The bottom line is that researchers should be wary of spurious regressions using spatial data, just as they are using time series data.

Section 4 takes up the problem of conducting inference about the degree of spatial persistence in a scalar variable. In particular, we construct spatial analogues of the time series "low-frequency" unit root and stationary tests of Müller and Watson (2008). In addition, we derive a confidence interval for the mean reversion parameter in the spatial local-to-unity model, analogous to the time series work by Stock (1991). We also consider versions of these tests that can be applied to residuals of a regression, yielding spatial analogues of the residual-based cointegration tests of Engle and Granger (1987).

First-differencing an $I(1)$ time series yields an $I(0)$ process, so spurious time series regressions can be avoided by taking first differences of $I(1)$ variables. Section 5 uses simulations to study several spatial differencing methods including nearest-neighbor differences, local demeaning, local fixed effects and a GLS transformation. When combined with spatial HAR standard errors, several of these differencing methods mitigate or eliminate spurious regression problems. We find the GLS transformation to be particularly effective. In contrast, inference methods that rely on clustered standard errors perform poorly.

Section 6 offers some concluding remarks. The appendix contains all proofs.

2 Spatial $I(1)$ Processes and Their Limits

This section is divided into five subsections. The first defines some notation for the spatial environment. The second reviews Lévy-Brownian motion, a spatial generalization of the Wiener process. The third provides the definition of a spatial $I(1)$ process, and the fourth provides a corresponding functional central limit theorem. The final subsection presents a spatial generalization of the time-series local-to-unity model which serves as a benchmark mean-reverting, but highly persistent spatial process.

2.1 Set-up and Notation

Our analysis requires three ingredients: (1) the spatial sampling region under consideration, denoted by $\mathcal{S}$; (2) the observed locations, $s_l \in \mathcal{S}$; and (3) the stochastic process Y, that is defined on $\mathcal{S}$. Taken together, these ingredients describe the observations

$$y_l = Y(s_l) \text{ for } l = 1, ..., n. \tag{1}$$

We discuss the sampling region and observed locations in this subsection. The stochastic process Y is discussed in the following two subsections.

We utilize a large-sample framework and assume that the locations s_l, $l = 1, \dots, n$ are non-stochastic (or, equivalently, are independent of all other random elements). The locations are allowed to depend on n in a double-array fashion, but we abstract from this dependence in the notation. We assume the following regularity condition:[3]

Condition 1. *(a) The locations s_l are elements of $\mathcal{S}_n = \lambda_n \mathcal{S}^0 = \{s : \lambda_n^{-1} s \in \mathcal{S}^0\}$ for some fixed and compact set $\mathcal{S}^0 \subset \mathbb{R}^d$ and deterministic non-decreasing positive real sequence λ_n.*

(b) The empirical cumulative distribution function G_n of $\{\lambda_n^{-1} s_l\}_{l=1}^n \subset \mathcal{S}^0$ converges to G, $G_n(s) \to G(s)$ for all $s \in \mathcal{S}^0$, with G an absolutely continuous distribution with support $\mathcal{S}^0$.

A familiar example helps clarify the sampling framework: consider a regularly spaced time series process observed at time periods $l = 1, ..., n$, so that $s_l = l$. In this example, the sampling region can be represented as $\mathcal{S}_n = [0, n]$, with a domain increasing at the rate $\lambda_n = n$. Thus, $\lambda_n^{-1} s_l = l/n$ and $\mathcal{S}^0 = [0, 1]$. The empirical distribution of the locations is $G_n(s) = n^{-1} \lfloor sn \rfloor \to s$ for $s \in [0, 1]$, so that G is the uniform distribution. Condition 1 extends this example to a general spatial setting with a general prototypical sampling region $\mathcal{S}^0 \subset \mathbb{R}^d$ that grows at an arbitrary rate λ_n.

[3] This coincides with Lahiri's (2003) large-sample framework, except that he replaces Condition 1 with an assumption that the locations are i.i.d. draws from the distribution G.

2.2 Lévy-Brownian Motion

Consider the usual time series $I(1)$ process $y_t = \sum_{s=1}^{t} u_s$, $t = 1, \ldots, n$, where u_t is mean zero, covariance stationary and weakly dependent (that is, u_t is $I(0)$). A standard time series FCLT implies that $n^{-1/2} y_{\lfloor \cdot n \rfloor} \Rightarrow \omega W(\cdot)$, where W is a standard Wiener process on the unit interval $[0, 1]$. For this reason, Wiener processes play a key role in the asymptotic analysis of inference involving $I(1)$ time series. Moreover, if $n^{-1/2} y_t = \omega W(t/n)$ holds exactly, then y_t is a Gaussian random walk. Thus, Wiener processes represent the canonical $I(1)$ time series model, and the FCLT shows that other $I(1)$ processes behave similarly to this canonical model in a well-defined sense.

With this in mind, we begin by defining the generalization of the Wiener process to the spatial case. In the next subsection we discuss more general spatial $I(1)$ processes.

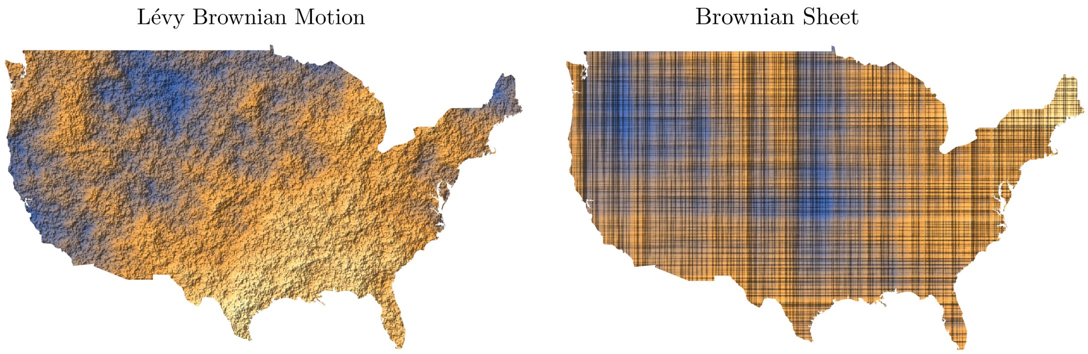
An attractive generalization of the Wiener process to the spatial case is *Lévy-Brownian motion* $L(s)$, $s \in \mathbb{R}^d$ (Lévy (1948)), which plays a corresponding important role in our analysis of $I(1)$ spatial variables. Lévy-Brownian motion is a zero-mean Gaussian process with domain $\mathbb{R}^d$ and covariance function

$$\mathbb{E}[L(s)L(r)] = \tfrac{1}{2}(|s| + |r| - |s - r|) \tag{2}$$

with $|x| = \sqrt{x'x}$ for $x \in \mathbb{R}^d$, so in particular, $\mathrm{Var}(L(s)) = |s|$ and $\mathrm{Var}(L(s) - L(r)) = |s - r|$. When $d = 1$ and $s, r \geq 0$, the covariance function (2) simplifies to $\mathbb{E}[L(s)L(r)] = \min(s, r)$, the covariance function of a Wiener process. More generally, for any d, the process obtained along a line in $\mathbb{R}^d$, $W_{a,b}(s) = L(a + bs) - L(a)$, $a, b \in \mathbb{R}^d$, $|b| = 1$, $s \in \mathbb{R}$ is a Wiener process. Thus, L is a natural embedding of the canonical time series model of strong persistence to the spatial case. Notice that Lévy-Brownian motion is *isotropic*, that is, $\mathrm{Var}(L(s) - L(r))$ depends on s, r only through $|s - r|$, so Lévy-Brownian motion is invariant to rotations of the spatial axes, that is $L(Os) \sim L(s)$ for any $d \times d$ rotation matrix O.

The left panel of Figure 2 plots a realization of L on the sampling region $\mathcal{S}_n$ representing the 48 contiguous U.S. States. The right panel shows a realization of another generalization of the Wiener process to $d > 1$, the *Brownian sheet* $\int_{\mathbb{R}^d} \mathbf{1}[0 \leq r \leq s] dW(r)$, $s \geq 0$, where the inequality $0 \leq r \leq s$ is to be understood element by element. The Brownian sheet is not isotropic, as is apparent from the sample realization. We therefore find Lévy-Brownian motion a more appealing generalization of the Wiener process for most applications, and thus define $y_l = L(s_l)$ as the *canonical unit root process* for $d > 1$.

Figure 2: Sample Realizations of Stochastic Processes for $d = 2$



2.2.1 Two Representations of Lévy-Brownian Motion

We take advantage of two representations for Lévy-Brownian motion, the Karhunen–Loève expansion and a spatial "moving average" representation. We discuss these in turn.

By Mercer's Theorem, the covariance kernel (2) evaluated at $s, r \in \mathcal{S}^0$ can be represented as

$$\mathbb{E}[L(s)L(r)] = \sum_{j=1}^{\infty} \nu_j \varphi_j(s)\varphi_j(r) \tag{3}$$

where (ν_j, φ_j) are eigenvalue/eigenfunction pairs with $\nu_j \geq \nu_{j+1} \geq 0$ and $\varphi_j : \mathcal{S}^0 \mapsto \mathbb{R}$ satisfying $\int \varphi_i(s)\varphi_j(s)dG(s) = \mathbf{1}[i = j]$. This spectral decomposition of the covariance kernel leads to a corresponding Karhunen–Loève expansion of L as the infinite sum

$$L(s) = \sum_{j=1}^{\infty} \nu_j^{1/2} \varphi_j(s)\xi_j,\ \xi_j \sim iid\mathcal{N}(0, 1) \tag{4}$$

where the right-hand side converges uniformly on $\mathcal{S}^0$ with probability one (cf. Theorem 3.1.2 of Adler and Taylor (2007)). This result generalizes the corresponding observation in Phillips (1998) about representations of the Wiener process in terms of stochastically weighted averages of deterministic series.

The spatial moving average representation represents Lévy-Brownian motion as a weighted average of spatial white noise. Recall that a Wiener process can trivially be written as an integral over white noise, $W(s) = \int_0^s dW(r)$. This can be generalized for Lévy-Brownian motion for all $d \geq 1$: from Lindstrøm (1993)

$$L(s) = \int h(r,s)dW(r) = \begin{cases} \int_0^s dW(r) \text{ for } d = 1 \\ \kappa_d \int_{\mathbb{R}^d} (|s-r|^{(1-d)/2} - |r|^{(1-d)/2}) dW(r) \text{ for } d > 1 \end{cases} \tag{5}$$

where $\kappa_d > 0$ is a scalar chosen so that $\mathrm{Var}(L(s)) = 1$ when $|s| = 1$.

2.3 Spatial $I(1)$ Processes

The commuter-zone data plotted in panel (b) of Figure 1 are realizations of Lévy-Brownian motion evaluated at the zone centers, while the data plotted in panel (c) are variables from Chetty, Hendren, Kline, and Saez (2014). To the naked eye, the long-range spatial correlation patterns in these figures are similar, suggesting that Lévy-Brownian motion may be a reasonable model for low-frequency correlation in socioecconomic spatial data. That said, the higher-frequency/short-range correlation patterns look different. In this section, we propose a generalization of Lévy-Brownian motion that inherits its long-range properties but allows for more flexible short-range correlation patterns. Following the notation used in time series, we call these (spatial) $I(1)$ processes.

In the standard time series case, $I(1)$ processes are defined as partial sums of a weakly dependent $I(0)$ process, say u_t, so that $y_t = \sum_{s=1}^t u_s$. Because spatial locations typically do not fall on a regular lattice, this definition does not naturally generalize. Instead, we utilize the moving average representation (5), replace the white noise innovations $dW(r)$ by a weakly dependent random field B, and define a *spatial* $I(1)$ *process* on $\mathcal{S}_n$ via

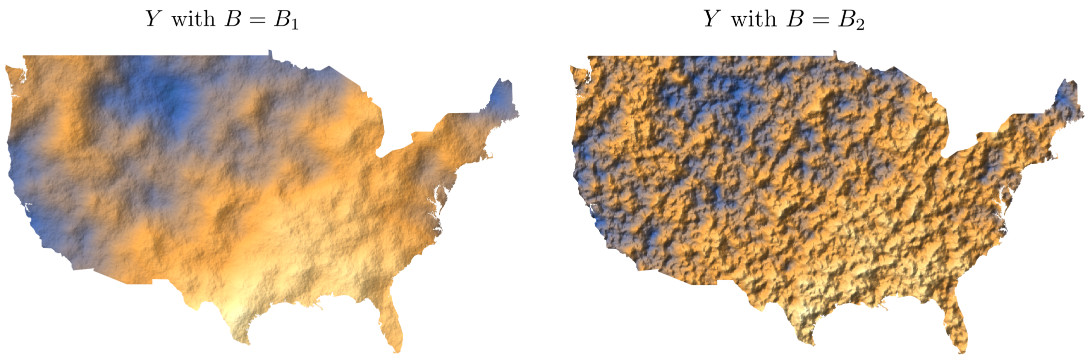
$$Y(s) = \int h(r,s)B(r)dr. \tag{6}$$

Note that if B is isotropic, then so is Y.

The integral $\int_{\mathbb{R}^d} |h(r,s)|dr$ does not exist for $d > 1$, so Y in (6) is not defined pathwise for every realization of B. However, $\int_{\mathbb{R}^d} h(r,s)^2 dr < \infty$, so under appropriate weak dependence conditions on B, the integral that defines Y can be shown to converge in a mean square sense. We make the following assumption.

Condition 2. *The mean-zero random field B with domain $\mathbb{R}^d$ is covariance stationary with $\mathbb{E}[B(s)B(r)] = \sigma_B(s-r)$ and $\int_{\mathbb{R}^d} \sigma_B(s)ds < \infty$, and B is such that for some $m > 2d$, $C_m > 0$*

Figure 3: Sample Realizations of Y with Different Underlying B



Notes: B_1 and B_2 are zero mean Gaussian processes with spectral densities $f_1(\omega) \propto 1/(|\omega|^2 + 100^2)^{3/2}$ and $f_2(\omega) \propto (|\omega|^2 + 50^2)^{3/2}/(|\omega|^2 + 200^2)^3$ for $\omega \in \mathbb{R}^2$, respectively, where the width of the contiguous U.S. is normalized to unity.

and any square integrable function $f : \mathbb{R}^d \mapsto \mathbb{R}$*,*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(r)B(r)dr\right)^{2m}\right] \leq C_m \left(\int_{\mathbb{R}^d} f(r)^2 dr\right)^m .$$

Lemma 1.8.4 of Ivanov and Leonenko (1989) implies that Condition 2 holds for a wide range of covariance stationary mixing random fields B.

Lemma 1. *Under Condition 2, for all* $d \geq 1$*,* $Y(\cdot)$ *exists on* $\mathcal{S}_n \subset \mathbb{R}^d$ *for all* n *and has continuous sample paths with probability one.*

Figure 3 plots realizations from two spatial $I(1)$ processes with B equal to two different isotropic Gaussian processes. These realizations were generated using the same underlying normal variables as the Lévy-Brownian motion plotted in Figure 2. Evidently, different B processes can induce quite different local behavior of Y, but with the same long-range behavior as Lévy-Brownian motion, a result formalized in the next subsection.

2.4 A Functional Central Limit Theorem

In the standard time series case, a functional central limit theorem (FCLT) yields $n^{-1/2}y_{\lfloor \cdot n \rfloor} = n^{-1/2}\sum_{t=1}^{\lfloor \cdot n \rfloor} u_t \Rightarrow \omega W(\cdot)$ for a covariance stationary and weakly dependent time series u_t, where $\omega^2 = \sum_{k=-\infty}^{\infty} \mathbb{E}[u_t u_{t-k}]$ is the so-called long-run variance of u_t. We now develop a similar result for the spatial $I(1)$ process $Y(\cdot)$ in (6).

The classic time series FCLT involves two rescalings: one that maps time into the unit interval, and one that shrinks the scale of y_t to compensate for its increasing variance. For the spatial $I(1)$ process in (6) we similarly define the process $Y_n^0(\cdot)$ on $\mathcal{S}^0$ via

$$Y_n^0(r) = \lambda_n^{-1/2} Y(\lambda_n r),\, r \in \mathcal{S}^0. \tag{7}$$

We make the following assumption about the process B.

Condition 3. *For some positive sequence $\zeta_n \to \infty$, let $\mathcal{R}_n = [-\zeta_n, \zeta_n]^d \subset \mathbb{R}^d$, and let $f_n : \mathbb{R}^d \mapsto \mathbb{R}$ be any sequence of functions such that $\limsup_{n\to\infty} \sup_{r\in\mathcal{R}_n} \lambda_n^{d/2} |f_n(r)| < \infty$ and $\mathrm{Var}[\int_{\mathcal{R}_n} f_n(r)B(r)dr] \to \sigma_0^2$. Then $\int_{\mathcal{R}_n} f_n(r)B(r)dr \Rightarrow \mathcal{N}(0, \sigma_0^2)$.*

The central limit theorems in Section 1.7 of Ivanov and Leonenko (1989) provide primitive mixing and moment conditions on B that imply Condition 3.

With this background, we can state a FCLT for spatial data.

Theorem 2. *Suppose Conditions 2 and 3 hold. If $\lambda_n \to \infty$, then $Y_n^0(\cdot) \Rightarrow \omega L(\cdot)$ on $\mathcal{S}^0$, where $\omega^2 = \int_{\mathbb{R}^d} \sigma_B(r)dr$.*

Remark 2.1. Under Condition 1, the rate λ_n governs the degree of "infill" versus "outfill" sampling. To see this, note that $\mathcal{S}_n = \lambda_n \mathcal{S}^0$ implies that the volume of $\mathcal{S}_n$ is proportional to λ_n^d , so the average number of observations per unit of volume is proportional to n/λ_n^d, and $\lambda_n \asymp n^{1/d}$ corresponds to pure outfill. The theorem holds for any mixture of infill and outfill sampling as long as the overall sampling region diverges, $\lambda_n \to \infty$.

Remark 2.2. It is well known that suitably scaled partial sums over rectangles of random variables defined on a lattice converge to a Brownian Sheet under suitable mixing and moment conditions; see, for instance, Deo (1975). In contrast, we are not aware of previous results that imply convergence to Lévy-Brownian motion.

2.5 Spatial Local-to-Unity Processes

A large time series literature, initiated by Cavanagh (1985), Chan and Wei (1987) and Phillips (1987), concerns a generalization of the $I(1)$ model to the weakly mean reverting local-to-unity (LTU) model. LTU models exhibit strong persistence, but are stationary, and serve as a convenient bridge between $I(0)$ and $I(1)$ processes. In time series, the LTU model is employed to derive confidence intervals for autoregressive roots near unity (Stock (1991)) and the associated half-life of shocks (Rossi (2005)), for local-power analysis of unit root tests (Elliott, Rothenberg, and Stock (1996)), to understand the implications of local-deviations from exact unit root specifications (Elliott (1998)), and for long-horizon economic forecasting (Müller and Watson (2016)), to mention just a handful of applications. Related spatial applications are easy to imagine and we consider some of these below. Here we provide the requisite spatial generalization of the LTU model.

In the time series LTU model y_t satisfies $n^{-1/2}(y_{\lfloor \cdot n \rfloor} - y_1) \Rightarrow \omega(J_c(\cdot) - J_c(0))$, with J_c a stationary Ornstein-Uhlenbeck (OU) process with covariance kernel $\mathbb{E}[J_c(s)J_c(r)] = \exp[-c|s-r|]/(2c)$, $c > 0$. Taking the limit of this covariance kernel shows that $J_c(\cdot) - J_c(0)$ converges to a Wiener process as $c \to 0$ (see Elliott (1999)).

To generalize the LTU model for $d > 1$, define J_c on $\mathbb{R}^d$ as the stationary and isotropic Gaussian process with covariance function $\mathbb{E}[J_c(s)J_c(r)] = \exp[-c|s-r|]/(2c)$, $c > 0$. This is recognized as a member of the Matérn class of covariance functions, with a spectral density proportional to $(|\omega|^2 + c^2)^{-(d+1)/2}$, $\omega \in \mathbb{R}^d$. As in the $d = 1$ model, $J_c(\cdot) - J_c(0)$ converges to $L(\cdot)$ as $c \to 0$ for any integer d. Also, along any line $J_c(a + bs)$, $a, b \in \mathbb{R}^d$, $|b| = 1$, $s \in \mathbb{R}$ is a standard OU process.

From equation of (3.2.8) of Matérn (1986), J_c has the moving average representation

$$J_c(s) = \int_{\mathbb{R}^d} h_c(r,s)dW(r) \tag{8}$$

with $h_c(r,s) = \kappa_{c,d}|s-r|^{(1-d)/4}K_{(1-d)/4}(c|s-r|)$ for a suitable choice of constant $\kappa_{c,d}$, where K_ν is the modified Bessel function of the second kind, $d \geq 1$.[4] We proceed as in the spatial $I(1)$ model (6) and replace the white noise term by the weakly dependent random field B,

$$Y_c(s) = \int_{\mathbb{R}^d} h_c(r,s)B(r)dr,$$

and define the *spatial local-to-unity process* on $\mathcal{S}_n$ as the sequence of processes Y_{c/λ_n}. In this definition, the parameter c/λ_n is a drifting sequence, generalizing the corresponding local-to-unity time series device in which the largest autoregressive root is parameterized as

[4]For $d = 1$, the usual one-sided (causal) representation for a stationary OU process is $J_c(s) = \int_{-\infty}^{s} e^{-c(s-r)}dW(r)$. Equation (8) is an alternative two-sided (non-causal) representation when $d = 1$.

$\rho_n = 1 - c/n$. The rate of this drift is such that the overall degree of mean reversion of

$$Y_{n,c}^0(r) = \lambda_n^{-1/2} Y_{c/\lambda_n}(\lambda_n r),\ r \in \mathcal{S}^0 \tag{9}$$

over the fixed set $\mathcal{S}^0$ converges as $n \to \infty$.

The appendix shows that under Condition 2, Y_{c/λ_n} exists on $\mathcal{S}_n$ for all n. Furthermore, under the conditions of Theorem 2, $Y_{n,c}^0$ in (9) satisfies $Y_{n,c}^0(\cdot) \Rightarrow \omega J_c(\cdot)$.

3 Spurious Regressions with Spatial $I(1)$ Variables

As a first application of the results in Section 2, consider the regression model

$$y_l = \alpha + x_l'\beta + u_l \tag{10}$$

for $l = 1, ..., n$, where $(y_l, x_l) = (Y(s_l), X(s_l)) \in \mathbb{R}^{p+1}$ follow $p+1$ independent spatial $I(1)$ processes. The FCLT in Theorem 2 allows for a straightforward spatial extension of the classic spurious time-series regression results in Phillips (1986).

Let $\tilde{y}_l = y_l - n^{-1}\sum_{\ell=1}^n y_\ell$ denote the demeaned value of y_l and similarly for x_l. Let $s_{\tilde{y}\tilde{y}} = n^{-1}\sum_{l=1}^n \tilde{y}_l^2$, $S_{\tilde{x}\tilde{x}} = n^{-1}\sum_{l=1}^n \tilde{x}_l\tilde{x}_l'$ and $S_{\tilde{x}\tilde{y}} = n^{-1}\sum_{l=1}^n \tilde{x}_l\tilde{y}_l$. The OLS estimator is $\hat{\beta} = S_{\tilde{x}\tilde{x}}^{-1}S_{\tilde{x}\tilde{y}}$, the regression $R^2 = S_{\tilde{x}\tilde{y}}'S_{\tilde{x}\tilde{x}}^{-1}S_{\tilde{x}\tilde{y}}/s_{\tilde{y}\tilde{y}}$, and the classical (non-spatial-correlation robust, homoskedastic) F-statistic for testing $H_0 : H\beta = 0$, where H is a non-stochastic matrix with $\operatorname{rank}(H) = m \le p$, is $F^{\text{Hom}} = \frac{n}{m}\hat{\beta}'H'(H'S_{\tilde{x}\tilde{x}}^{-1}H)^{-1}H\hat{\beta}/s_u^2$ with $s_u^2 = \frac{n}{n-p-1}(s_{\tilde{y}\tilde{y}} - S_{\tilde{x}\tilde{y}}'S_{\tilde{x}\tilde{x}}^{-1}S_{\tilde{x}\tilde{y}})$.

Suppose $(y_l, x_l) = (Y(s_l), X(s_l))$ follow spatial $I(1)$ processes with

$$\begin{bmatrix} Y_n^0(\cdot) \\ X_n^0(\cdot) \end{bmatrix} = \begin{bmatrix} \lambda_n^{-1/2}Y(\lambda_n\cdot) \\ \lambda_n^{-1/2}X(\lambda_n\cdot) \end{bmatrix} \Rightarrow \begin{bmatrix} Y^0(\cdot) \\ X^0(\cdot) \end{bmatrix} \tag{11}$$

on $\mathcal{S}^0$, where $[Y^0(\cdot), X^0(\cdot)']$ are $p+1$ independent and arbitrarily scaled Lévy-Brownian motions. Let $\tilde{Y}(\cdot) = Y^0(\cdot) - \int Y^0(r)dG(r)$ denote the demeaned version of Y^0 using spatial-weighted average demeaning, and define $\tilde{X}$ analogously.

Theorem 3. *Under Condition 1 and (11)*

(i) $\lambda_n^{-1}s_{\tilde{y}\tilde{y}} \Rightarrow \Xi_{\tilde{y}\tilde{y}} = \int \tilde{Y}^2(r)dG(r)$, $\lambda_n^{-1}S_{\tilde{x}\tilde{x}} \Rightarrow \Xi_{\tilde{x}\tilde{x}} = \int \tilde{X}(r)\tilde{X}(r)'dG(r)$ *and* $\lambda_n^{-1}S_{\tilde{x}\tilde{y}} \Rightarrow \Xi_{\tilde{x}\tilde{y}} = \int \tilde{X}(r)\tilde{Y}(r)dG(r)$,

(ii) $\hat{\beta} \Rightarrow \Xi_{\tilde{x}\tilde{x}}^{-1}\Xi_{\tilde{x}\tilde{y}}$,

(iii) $R^2 \Rightarrow \Xi'_{\tilde{x}\tilde{y}}\Xi^{-1}_{\tilde{x}\tilde{x}}\Xi_{\tilde{x}\tilde{y}}/\Xi_{\tilde{y}\tilde{y}}$,

(iv) $n^{-1}F^{\text{Hom}} \Rightarrow m^{-1}\Xi'_{\tilde{x}\tilde{y}}\Xi^{-1}_{\tilde{x}\tilde{x}}H'(H\Xi^{-1}_{\tilde{x}\tilde{x}}H')^{-1}H\Xi^{-1}_{\tilde{x}\tilde{x}}\Xi_{\tilde{x}\tilde{y}}/(\Xi_{\tilde{y}\tilde{y}} - \Xi'_{\tilde{x}\tilde{y}}\Xi^{-1}_{\tilde{x}\tilde{x}}\Xi_{\tilde{x}\tilde{y}})$.

An implication of part (iv) of Theorem 3 is that the classical F-test statistic diverges to infinity so that $\mathbb{P}(F^{\text{Hom}} > \text{cv}) \to 1$ for any $\text{cv} \geq 0$. We now show that this extends to statistics computed with heteroskedasticity and HAC-corrected standard errors (cf. Phillips (1998) for a corresponding time series result).

Consider the class of correlation-robust-HAC F-statistics

$$F^{\text{HAC}} = \frac{n}{m}\hat{\beta}'H'(HS^{-1}_{\tilde{x}\tilde{x}}\hat{\Omega}_n S^{-1}_{\tilde{x}\tilde{x}}H')^{-1}H\hat{\beta} \tag{12}$$

where $\hat{\Omega}_n$ is a kernel-based estimator of $\text{Var}\left(n^{-1/2}\sum_{l=1}^{n}\tilde{x}_l u_l\right)$ of the form

$$\hat{\Omega}_n = n^{-1}\sum_{l,\ell=1}^{n}\kappa\left(b_n(s_l - s_\ell)\right)e_l e'_\ell \tag{13}$$

with $e_l = \tilde{x}_l(\tilde{y}_l - \tilde{x}'_l\hat{\beta})$, b_n a bandwidth (that may depend both on $\{s_l\}$ and the data $\{(y_l, x_l)\}$) with $\lambda_n^{-1}b_n^{-1} = o_p(1)$ and $\kappa : \mathbb{R}^d \mapsto \mathbb{R}$ a kernel weighting function satisfying

$$\sup_r |\kappa(r)| = \bar{\kappa} < \infty, \qquad \lim_{\lambda\to\infty}\sup_{|a|=1}|\kappa(\lambda a)| = 0. \tag{14}$$

The assumption of $\lambda_n^{-1}b_n^{-1} = o_p(1)$ ensures that in large samples, $\hat{\Omega}_n$ in (13) puts negligible weight on pairs of locations with $\lambda_n^{-1}|s_l - s_\ell| > \varepsilon$, for all positive ε. Since $\lambda_n^{-1}s_l \in \mathcal{S}^0$ with $\mathcal{S}^0$ compact, this is necessary for a kernel estimator to be consistent under weak spatial dependence. These conditions are satisfied, for instance, for the spatial correlation robust estimator suggested in Conley (1999). As long as all locations are distinct, heteroskedasticity robust standard errors correspond to $\kappa(r) = \mathbf{1}[r = 0]$, which also satisfies (14).

Alternatively, researchers sometimes employ clustered standard errors over larger regions to account for spatial dependence. The corresponding F^{clust} statistic has the same form as F^{HAC} in (12) with $\hat{\Omega}_n$ replaced by $\hat{\Omega}_n^{\text{clust}} = n^{-1}\sum_{j=1}^{n_C}\left(\sum_{l\in C_j}e_l\right)\left(\sum_{l\in C_j}e_l\right)'$ where the partitions C_j of $\{1, 2, \ldots, n\}$ indicate membership in cluster $j = 1, \ldots, n_C$. With $|C_j|$ the number of observations in cluster j, we assume $\max_{1\leq j\leq n_C}|C_j|/n \to 0$ as $n \to \infty$. As discussed in Hansen and Lee (2019), page 270, this is necessary for the consistency of cluster robust inference under weak dependence.

Theorem 4. *Under Condition 1, (11) and (14),* $\mathbb{P}(F^{\text{HAC}} > \text{cv}) \to 1$ *and* $\mathbb{P}(F^{\text{clust}} > \text{cv}) \to 1$ *for any* $\text{cv} \geq 0$.

Remark 3.1. In contrast to spatial HAC inference, fixed-b type spatial HAR inference (Sun and Kim (2012), Bester, Conley, Hansen, and Vogelsang (2016)) does not lead to diverging F-statistics, and the spatial correlation robust inference derived in Müller and Watson (2022) explicitly accommodates some degree of "strong" persistence of the type exhibited by the spatial local-to-unity model for large enough c.

Remark 3.2. Theorems 3 and 4 also hold for local-to-unity processes, that is, if $[Y^0(\cdot), X^0(\cdot)]$ in (11) are $p+1$ independent processes of the type (8), with arbitrary and potentially different mean-reversion parameters c.

Remark 3.3. It follows from the Karhunen–Loève representation of L in (4) and the FCLT result in Theorem 2 that the coefficients of a regressions of $\lambda_n^{-1/2} y_l$ on the eigenfunctions $[\varphi_1(\lambda_n^{-1} s_l), \ldots, \varphi_p(\lambda_n^{-1} s_l)]$ converge to independent $\mathcal{N}(0, \omega^2 \nu_j)$ random variables. This generalizes the "understanding spurious regressions" result in Theorem 3.1 (a) of Phillips (1998) to the spatial case. More generally, the coefficients of a regression of $\lambda_n^{-1/2} y_l$ on smooth deterministic functions of $\lambda_n^{-1} s_l$, say $\psi(\lambda_n^{-1} s_l) \in \mathbb{R}^p$, converge to $\left(\int \psi(r)\psi(r)' dG(r)\right)^{-1} \omega \int \psi(r) L(r) dG(r)$ and are asymptotically significant as measured by a corresponding F^{Hom}, F^{HAC} or F^{clust} statistic. Kelly (2019) observes such a phenomenon empirically in a number of applications with spatial data.

4 Inference for Spatial Persistence

The autoregressive representation for a discrete-time $I(1)$ time series process has a unit root in its autoregressive polynomial, making it straightforward to test for $I(1)$ persistence using Dickey-Fuller or related unit root tests. Spatial $I(1)$ processes do not have an analogous autoregressive representation, so these tests do not directly generalize to spatial processes. Similarly, popular "stationarity" tests (e.g., Nyblom (1989), Kwiatkowski, Phillips, Schmidt, and Shin (1992), and Elliott and Müller (2006)) do not directly generalize. An alternative, non-regression based approach to learn about time series persistence is developed in Müller and Watson (2008). That approach is based on the properties of q suitably chosen weighted averages, and generalizes fairly directly to the spatial setting studied here.[5]

[5]Other approaches to testing for the presence of spatial correlation, such as Moran's (1950) I statistic or Geary's (1954) c, require the specification of a spatial weight matrix and test the null hypothesis of zero

The intuition underlying this approach is straightforward: The Karhunen–Loève expansion (4) implies that eigenfunction weighted averages of a Lévy-Brownian motion recover independent normal variates with a variance that is proportional to the eigenvalues. Focussing on the q eigenfunctions corresponding to the largest eigenvalues yields a set of independent normal random variables with sharply decaying variance. In contrast, when the data are i.i.d. Gaussian random variables, these weighted averages are i.i.d. normal random variables because of the orthogonality of the eigenfunctions. This difference in behavior may be used to empirically distinguish between these two canonical cases.

The FCLT result in Theorem 2 suggests that little is lost by developing the various tests for the canonical models from Section 2 with $y_l = L(s_l)$ or $y_l = J_c(s_l)$, respectively, and for clarity we take this approach in the Sections 4.1-4.5. Section 4.6 discusses the asymptotic validity of the tests under the more general assumptions for $I(1)$ and LTU processes given in Section 2, as well as for more general $I(0)$ processes by invoking the CLT of Lahiri (2003).

4.1 Dimension Reduction by Weighted Averages

The suggested tests depend on the data only through q weighted averages. This section establishes corresponding notation and briefly reviews standard results for hypothesis tests under Gaussianity.

Let $\mathbf{Y} = (y_1, \ldots, y_n)'$ and let $\mathbf{\Sigma}_L$ be the $n \times n$ covariance matrix of $\mathbf{Y}$ induced by Lévy-Brownian motion $y_l = L(s_l)$. We are interested in tests that are invariant to translation shifts $\mathbf{Y} \to \mathbf{Y}+a\mathbf{1}$, where $\mathbf{1}$ is a vector of ones. Such tests can be constructed from weighted averages of $\mathbf{Y}$ that sum to zero. Let $\mathbf{M} = \mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'$, and $\mathbf{MY}$ denote the demeaned values of y. With $\mathrm{Var}(\mathbf{Y}) = \mathbf{\Sigma}_L$ we have $\mathrm{Var}(\mathbf{MY}) = \mathbf{M\Sigma}_L\mathbf{M}$. Let $\mathbf{R}$ be the $n \times q$ matrix of eigenvectors of $\mathbf{M\Sigma}_L\mathbf{M}$ corresponding to the q largest eigenvalues, where $\mathbf{R}$ satisfies $n^{-1}\mathbf{R}'\mathbf{R}= \mathbf{I}_q$. The columns of $\mathbf{R}$ extract the q linear combinations of $\mathbf{MY}$ with the largest variance, that is $\mathbf{Z} = \mathbf{R}'\mathbf{MY} = \mathbf{R}'\mathbf{Y}$ are the q largest principal components of $\mathbf{MY}$ under $\mathbf{MY} \sim (\mathbf{0}, \mathbf{M\Sigma}_L\mathbf{M})$. As in Müller and Watson (2008), we treat $\mathbf{Z}$ as the effective observation, that is, we seek to conduct inference about the persistence properties of $\mathbf{Y}$ with a test that is a function of $\mathbf{Z}$ only.

Different models for persistence in $\mathbf{Y}$ imply different values for $\mathrm{Var}(\mathbf{Z}) = \mathbf{\Omega}$, and this means that we can discriminate between the models by testing hypotheses concerning the value of

spatial correlation.

$\mathbf{\Omega}$. Thus, consider the generic problem of testing $H_0 : \mathbf{\Omega} = \mathbf{\Omega}_0$ versus $H_a : \mathbf{\Omega} = \mathbf{\Omega}_a$ when $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Omega})$. A standard calculation shows that the most powerful level α scale invariant test rejects for large values of

$$\frac{\mathbf{Z}'\mathbf{\Omega}_0^{-1}\mathbf{Z}}{\mathbf{Z}'\mathbf{\Omega}_a^{-1}\mathbf{Z}} \tag{15}$$

with a critical value that equals the $1-\alpha$ quantile of (15) under the null distribution $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Omega}_0)$.

Inference of this type depends on q, the number of weighted averages included in $\mathbf{Z}$. The choice of q faces a classic efficiency vs. robustness trade-off: large q increases power, but at the expense of exploiting implications of the specific models of persistence over many weighted averages. In practice, a moderate value of q, say a number around 10-20, as in Müller and Watson (2008), yields a reasonable compromise: it is large enough to yield informative inference and yet does not overly stretch the asymptotic approximations of the FCLT in Theorem 2. We set $q = 15$ in our numerical analysis.

4.2 Tests of the $I(1)$ Null Hypothesis

With this background in place, consider the problem of testing the $I(1)$ null hypothesis against the local-to-unity alternative, where the canonical models are $y_l = L(s_l)$ and $y_l = J_c(s_l)$. This yields $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_L)$ and $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}(c))$, respectively, with the l, ℓ element of $\mathbf{\Sigma}(c)$ equal to $\exp[-c|s_l - s_\ell|]/(2c)$. Optimal tests in this problem are of the form (15) with $\mathbf{\Omega}_0 = \mathbf{\Omega}_L = \mathbf{R}'\mathbf{\Sigma}_L\mathbf{R}$ and $\mathbf{\Omega}_a = \mathbf{\Omega}(c_a) = \mathbf{R}'\mathbf{\Sigma}(c_a)\mathbf{R}$ for some $c_a > 0$. This yields the test statistic

$$\text{LFUR} = \frac{\mathbf{Z}'\mathbf{\Omega}_L^{-1}\mathbf{Z}}{\mathbf{Z}'\mathbf{\Omega}^{-1}(c_a)\mathbf{Z}}. \tag{16}$$

To determine a value of c_a that ensures good power for a wide range of values of c, we follow King (1987) and choose c_a such that a 5% level test has 50% power.[6] (We label the statistic "LFUR" because it is the spatial generalization of the low-frequency unit root test proposed in Müller and Watson (2008).)

[6] The well-known DF-GLS test of Elliott, Rothenberg, and Stock (1996) also requires a choice of c_a, which is chosen in the same fashion.

4.3 Tests of the $I(0)$ Null Hypothesis

Now consider a corresponding spatial stationarity test based on $\mathbf{Z}$. Here we seek a test of the null hypothesis that y_l exhibits weak spatial correlation, and this requires a definition of "weak" correlation. One useful gauge for the strength of correlation is whether HAR inference about the mean remains valid. The inference derived in Müller and Watson (2022) remains valid by construction in the $\mathbf{\Sigma}(c)$ model for (all large enough) values of c that induce an average pairwise correlation

$$\bar{\rho}(c) = \frac{1}{n(n-1)} \sum_{l \neq \ell} \exp[-c|s_l - s_\ell|], \tag{17}$$

of no more than $\overline{\rho} = 0.03$. Denote the corresponding cut-off value of c by $c_{0.03}$, that is, $\bar{\rho}(c_{0.03}) = 0.03$. The canonical version of the testing problem then becomes $H_0 : \mathbf{\Omega} = \mathbf{\Omega}(c)$, $c \geq c_{0.03}$ against $H_a : \mathbf{\Omega} = \mathbf{\Omega}(c) + g_a^2 \mathbf{\Omega}_L$, $g_a > 0$, where the form of the alternative, a sum of a stationary and $I(1)$ process, is standard in time series stationarity tests (for example, see Nyblom (1989) and Kwiatkowski, Phillips, Schmidt, and Shin (1992)). The larger the scale g_a of the Lévy-Brownian motion under the alternative, the easier it is to discriminate the two hypotheses, so g_a can again be chosen using the 50% power rule. The stationarity testing problem is complicated by the presence of the additional nuisance parameter c that indexes the covariance matrix $\mathbf{\Omega}(c)$ in both the null and alternative. Here numerical experimentation revealed that in many configurations of locations, picking $c = c_{0.001}$ under both H_0 and H_a works well in the sense of generating a test statistic (15) that has a 95% quantile that is fairly constant as a function of $c \geq c_{0.03}$. Thus, the stationary test rejects if

$$\text{LFST} = \frac{\mathbf{Z}'\mathbf{\Omega}(c_{0.001})^{-1}\mathbf{Z}}{\mathbf{Z}'[\mathbf{\Omega}(c_{0.001}) + g_a^2 \mathbf{\Omega}_L]^{-1}\mathbf{Z}} \tag{18}$$

exceeds the critical value cv^{LFST}, where the critical value is chosen to insure the correct size of the test for all values of $c \geq c_{0.03}$. More precisely, cv^{LFST} solves $\sup_{c \geq c_{0.03}} \mathbb{P}(\text{LFST} \geq \text{cv}^{\text{LFST}}) = \alpha$, where α is the level of the test and the probability is computed under $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Omega}(c))$. (We label the statistic "LFST" because it is the spatial generalization of the low-frequency stationarity test proposed in Müller and Watson (2008).)

Remark 4.1. Suppose the $p \times 1$ vector x_l is spatially cointegrated of order one with cointegrating vector β_0, that is, $\beta_0' x_l \sim I(0)$, but $\beta' x_l \sim I(1)$ for all β that are not proportional to

β_0. An asymptotic level $1-\alpha$ confidence set for β_0 can then be formed by collecting those values of b for which the level-α LFST test does not reject the $I(0)$ null hypothesis when applied to the series $b'x_l$. This is the spatial analogue of Wright's (2000) idea for inference about the cointegrating vector in time series; also see Müller and Watson (2013).

4.4 Confidence Sets for the Local-To-Unity Parameter and Half-Life of Persistence

A closely related problem is the construction of a confidence set for c, the parameter in the spatial local-to-unity model. As usual, a $100(1-\alpha)\%$ confidence set is given by the values of c_0 for which a family of α-level tests of $H_0 : c = c_0$ does not reject. What is more, if this family of tests is optimal against the alternative that c is drawn from some probability distribution Π, the classic result in Pratt (1961) implies that the resulting confidence interval has the smallest Π-weighted expected length.

An easily interpretable transformation of the parameter c is given by the half-life, that is the distance Δ at which the correlation $\exp[-c\Delta]$ is equal to $1/2$, which is $h(c) = \ln 2/c$. With Π such that the implied weighting of h is uniform on $[0, \Delta_{\max}]$ with $\Delta_{\max} = \max_{l,\ell} |s_l - s_\ell^0|$, the average length minimizing scale-invariant confidence interval collects the values of h_0 for which the test based on

$$\frac{\int_0^{\Delta_{\max}} \det(\mathbf{\Omega}(\ln 2/h))^{-1/2}(\mathbf{Z}'\mathbf{\Omega}(\ln 2/h)^{-1}\mathbf{Z})^{-q/2}dh}{(\mathbf{Z}'\mathbf{\Omega}(\ln 2/h_0)^{-1}\mathbf{Z})^{-q/2}} \tag{19}$$

does not exceed the $1-\alpha$ quantile of (19) under $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Omega}(\ln 2/h_0))$.

4.5 Residual Based Tests

Consider inference about the persistence properties of the disturbance u_l in a linear regression $y_l = x_l'\beta + u_l$ (where x_l may include a constant). The above results are then not directly applicable, since with β unknown, u_l is unobserved.

There is an easy solution to this problem if $\mathbf{u} = (u_1, \ldots, u_n)'$ is independent of $\mathbf{X} = (x_1, \ldots, x_n)'$. Namely, one can simply base inference on weighted averages of $\mathbf{Y}$ with weights that they are orthogonal to $\mathbf{X}$. Let $\mathbf{R}_X$ be the $n \times q$ matrix of the eigenvectors of $\mathbf{M}_X\mathbf{\Sigma}_L\mathbf{M}_X$ corresponding to the largest q eigenvalues, where $\mathbf{M}_X = \mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ and $n^{-1}\mathbf{R}_X'\mathbf{R}_X = \mathbf{I}_q$. Then by construction, $\mathbf{R}_X'\mathbf{X} = \mathbf{0}$, so that $\mathbf{Z}_X = \mathbf{R}_X'\mathbf{Y} = \mathbf{R}_X'\mathbf{u}$. With $\mathbf{u}$ independent of

$\mathbf{X}$, one can simply condition on the realization of $\mathbf{X}$, and apply the above tests with $\mathbf{Z}_X$ in place of $\mathbf{Z}$.

Remark 4.2. A noteworthy application is the test of the null hypothesis of no cointegration among the $p+1$ variables (x_l, y_l), that is, for the spatial analogue of Engle and Granger's (1987) residual based test of cointegration (see Phillips and Ouliaris (1990) for its asymptotic distribution).[7] To implement such a level α test of the null hypothesis of no spatial cointegration in practice, one computes the LFUR statistic (16) using $\mathbf{Z}_X$ in place of $\mathbf{Z}$, $\mathbf{\Omega}_0 = \mathbf{R}_X' \mathbf{\Sigma}_L \mathbf{R}_X$ and $\mathbf{\Omega}_1 = \mathbf{R}_X' \mathbf{\Sigma}(c_a) \mathbf{R}_X$, and compares it to $1-\alpha$ quantile of the statistic under $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_L)$.

4.6 Large-Sample Analysis

In Sections 4.1-4.5, the data were assumed to follow the canonical $I(1)$ or LTU processes with $y_l = L(s_l)$ or $y_l = J_c(s_l)$. The large-sample validity of these procedures under the general $I(1)$ and LTU processes defined in Section 2 follow from the FCLT result in Theorem 2. Furthermore, the validity of the LFST test under a more general $I(0)$ null hypothesis can be established by invoking the spatial CLT in Lahiri (2003). Details are provided in the Appendix. Here we highlight three features of that analysis.

Remark 4.3. The asymptotic analysis in Section 2 included the long-run standard deviation ω and the spatial scale factor λ_n, but they have been ignored in this section. This is due to two features of the proposed tests. First, the tests are scale invariant, so they are unaffected by any scaling of the data. Second, our choice for c_a, $c_{0.001}$ and g_a in the tests LFUR and LFST induce invariance to the scale of the locations $\{s_l\}$, and thus λ_n.

Remark 4.4. A complication in directly applying the FCLT, CLT and continuous mapping theorem to show the large-sample normality of $\mathbf{Z} = \mathbf{R}'\mathbf{Y}$ is that the eigenvector weights $\mathbf{R}$ depend on the n sample locations s_l. Lemma S.1 in the supplementary appendix studies the large-sample behavior of these eigenvectors and shows that they converge to the eigenfunctions of the covariance kernel of demeaned Lévy-Brownian motion in a well defined sense, building on closely related results of Rosasco, Belkin, and Vito (2010) and Müller and Watson (2022).

[7] In the canonical model the assumption is that under the null hypothesis, $(x_l', y_l) = (X(s_l)', Y(s_l)) = \Phi L_{X,Y}(s_l)$ where $L_{X,Y}$ is a vector of $p+1$ independent Lévy-Brownian Motions, and Φ is an arbitrary full rank $(p+1) \times (p+1)$ matrix. In this model, the assumption that u_l follows a Lévy-Brownian Motion and is independent of x_l follows directly by defining β as $\Sigma_{XX}^{-1}\Sigma_{XY}$ where Σ_{XX} and Σ_{XY} are the appropriate blocks of $\Sigma = \Phi\Phi'$.

Remark 4.5. The asymptotic validity of the residual based tests only requires a form of asymptotic independence of $\mathbf{u}$ and $\mathbf{X}$, and this independence always holds under the null hypothesis of no cointegration for the Engle and Granger (1987)-type test.

4.7 Spatial Correlation in the Chetty et al. (2014) Data

Chetty, Hendren, Kline, and Saez (2014) use administrative records on the incomes of more than 40 million children and parents to study intergenerational income mobility in the United States. They construct an index of mobility for each of the commuter zones in the United States and investigate the relationship between mobility and other factors by regressing their mobility index on variables such as racial segregation, school quality and so forth. They find large and statistically significant correlations between their absolute mobility index and many socioeconomic indicators. One might suspect that the variables used in their regressions are strongly spatially correlated, and in light of the spurious regression results of Section 3, this questions the validity of their inference results. This issue is taken up in Table 1.[8]

The first three columns in the table apply the tests outlined in this section to gauge the spatial correlation in the socioeconomic variables for the contiguous 48 states, which contain 722 of the 741 commuting zones used by Chetty et al. The results indicate that these variables exhibit substantial spatial correlation: the $I(1)$ null is rejected for only a handful of the variables, the $I(0)$ null is rejected for most, and the confidence intervals for the implied value of the half-life, h, while wide, suggest a high degree of spatial persistence. The remaining columns of the table investigate the robustness of the regression results reported in Chetty et al. to this spatial correlation. We discuss these columns after introducing additional analysis in the next section.

5 Regressions with Transformed Spatial Variables

To avoid spurious regression effects using $I(1)$ time series data, researchers routinely estimate regressions using first differences of the original variables and rely on HAC/HAR inference methods to account for any remaining $I(0)$ autocorrelation. The best approach for regressions involving spatial $I(1)$ variables is not so obvious. There are a several ways to spatially difference the data and confidence intervals can be constructed using various spatial HAC/HAR

[8]The variables are chosen from Figure VIII in Chetty, Hendren, Kline, and Saez (2014). The data are taken from their comprehensive replication materials.

Table 1: Variables and Regressions from Chetty et al. (2014)

Variable	Spatial Persistence Statistics			Regression of AMI onto Variable				
	p-value for Test of		Half-life	Levels			LBM-GLS	
	$I(1)$ Null	$I(0)$ Null	95% CI	R^2	$\hat{\beta}$ [95% CI] Cluster	p-value Resid. $I(1)$ Test	R^2	$\hat{\beta}$ [95% CI] C-SCPC
Absolute Mobility Index (AMI)	0.39	< 0.01	$[0.10,\infty]$	NA	NA	NA	NA	NA
Frac. Black Residents	0.11	< 0.01	$[0.03,\infty]$	0.36	-0.60 [-0.73,-0.47]	0.21	0.10	-0.43 [-0.71,-0.14]
Racial Segregation	0.01	0.12	[0.00,0.29]	0.14	-0.38 [-0.46,-0.29]	0.29	0.18	-0.24 [-0.35,-0.12]
Segregation of Poverty	0.29	0.03	$[0.06,\infty]$	0.18	-0.43 [-0.55,-0.30]	0.28	0.16	-0.21 [-0.38,-0.04]
Frac. < 15 Mins to Work	0.58	< 0.01	$[0.14,\infty]$	0.48	0.69 [0.55, 0.84]	0.14	0.16	0.37 [0.09, 0.66]
Mean Household Income	0.13	0.14	$[0.02,\infty]$	0.00	0.05 [-0.10, 0.19]	0.39	0.00	-0.02 [-0.25, 0.22]
Gini Coefficient	0.78	< 0.01	$[0.25,\infty]$	0.37	-0.60 [-0.78,-0.43]	0.24	0.10	-0.22 [-0.39,-0.04]
Top 1 Perc. Inc. Share	0.31	0.02	$[0.07,\infty]$	0.04	-0.21 [-0.36,-0.06]	0.37	0.02	-0.06 [-0.13,-0.00]
Student-Teacher Ratio	0.22	0.13	$[0.05,\infty]$	0.12	-0.35 [-0.55,-0.15]	0.45	0.03	-0.19 [-0.48, 0.11]
Test Scores (Inc. adjusted)	0.29	0.06	$[0.07,\infty]$	0.34	0.58 [0.40, 0.76]	0.42	0.28	0.41 [0.16, 0.66]
High School Dropout	0.09	0.02	$[0.03,\infty]$	0.34	-0.58 [-0.74,-0.41]	0.51	0.22	-0.29 [-0.55,-0.03]
Social Capital Index	0.72	< 0.01	$[0.22,\infty]$	0.41	0.64 [0.46, 0.82]	0.30	0.08	0.28 [-0.01, 0.58]
Frac. Religious	0.27	0.04	$[0.07,\infty]$	0.28	0.53 [0.36, 0.70]	0.26	0.14	0.32 [0.13, 0.51]
Violent Crime Rate	0.54	0.02	$[0.14,\infty]$	0.21	-0.45 [-0.67,-0.24]	0.35	0.04	-0.15 [-0.27,-0.02]
Frac. Single Mothers	0.18	< 0.01	$[0.05,\infty]$	0.59	-0.77 [-0.92,-0.63]	0.11	0.51	-0.61 [-0.94,-0.28]
Divorce Rate	0.05	0.17	$[0.02,\infty]$	0.27	-0.52 [-0.70,-0.33]	0.50	0.27	-0.39 [-0.64,-0.14]
Frac. Married	0.05	0.08	$[0.01,\infty]$	0.31	0.56 [0.43, 0.68]	0.22	0.31	0.35 [0.11, 0.59]
Local Tax Rate	0.02	0.23	[0.01,0.51]	0.12	0.35 [0.21, 0.48]	0.40	0.01	0.08 [-0.07, 0.23]
Colleges per Capita	0.24	0.07	$[0.06,\infty]$	0.06	0.24 [-0.01, 0.48]	0.28	0.00	0.02 [-0.22, 0.25]
College Tuition	0.38	< 0.01	$[0.09,\infty]$	0.00	-0.02 [-0.15, 0.11]	0.29	0.00	0.01 [-0.05, 0.08]
Coll. Grad. Rate (Inc. Adjusted)	0.04	0.03	[0.00,3.00]	0.02	0.15 [0.03, 0.28]	0.36	0.03	0.08 [0.01, 0.15]
Manufacturing Share	0.21	< 0.01	$[0.06,\infty]$	0.09	-0.30 [-0.47,-0.13]	0.37	0.01	0.06 [-0.11, 0.23]
Chinese Import Growth	0.02	0.07	[0.02,0.43]	0.03	-0.17 [-0.33,-0.02]	0.39	0.00	0.03 [-0.01, 0.06]
Teenage LFP Rate	0.51	< 0.01	$[0.12,\infty]$	0.44	0.66 [0.50, 0.82]	0.29	0.04	0.25 [-0.06, 0.57]
Migration Inflow	0.30	0.08	$[0.00,\infty]$	0.07	-0.27 [-0.42,-0.13]	0.32	0.04	-0.14 [-0.31, 0.04]
Migration Outlflow	0.35	0.01	$[0.08,\infty]$	0.03	-0.16 [-0.30,-0.02]	0.37	0.02	-0.09 [-0.19, 0.01]
Frac. Foreign Born	0.55	0.04	$[0.16,\infty]$	0.00	-0.03 [-0.16, 0.09]	0.40	0.02	-0.12 [-0.30, 0.06]

Notes: The first two columns show p-values for tests of the $I(1)$ and $I(0)$ null hypotheses using the statistics (16) and (18). The third column shows the 95% confidence for the half-life, $h_{1/2}$, constructed by inverting the tests in (19). The half-life values shown are relative to the largest distance between commuter zones in the dataset, which is approximately 2800 miles. The results for the levels regressions show the R^2 and estimated regression coefficients ($\hat{\beta}$) from regression of the Absolute Mobility Index (AMI) onto each of the variables in the table, with nominal 95% confidence intervals constructed using standard errors clustered by state; also shown are the p-values for the $I(1)$ and $I(0)$ tests applied to the residuals. The final two columns show the corresponding regression results using variables transformed using the LBM-GLS transformation, with 95% nominal confidence intervals constructed using C-SCPC. Results are based on commuting zones in the contiguous 48 U.S. states. The variables are standardized (in levels) to have mean zero and unit standard deviation.

or clustered standard errors. In this section we use simulations to study the properties of regressions estimated using a variety of spatial differencing methods and confidence intervals constructed using spatial HAR and cluster standard errors. We stress that the analysis in this sections relies on simulations rather than the kind of formal analysis that was the basis of earlier sections. In that sense, this section's conclusions are tentative; that said, we think they provide both direction and motivation for future analytic work.

The simulations used here focus on the Chetty et al. regressions shown in Table 1, and therefore involve univariate regressions $y_l = x_l\beta + u_l$, with locations fixed at the (centers of the) of the $n = 720$ commuting zones plotted in Figure 1.[9] In all experiments, the sample data (in levels) are standardized to have a zero sample mean and unit standard deviation so that intercepts are excluded from the regression and β is measured in standard deviation units. The experiments differ in the spatial persistence of the data, the method for spatial differencing the data and the standard errors used to construct test statistics and confidence intervals.

5.1 Data Transformations and Spatial Standard Errors

The experiments involve regressions estimated by OLS after transforming $\{y_l, x_l\}$ using the same spatial difference transformation. Denote the transformed data by $\{y_l^*, x_l^*\}$. We consider five methods, described here for y_l^*.

Levels: $y_l^* = y_l$. The resulting estimator is the OLS levels regression studied in Section 3.

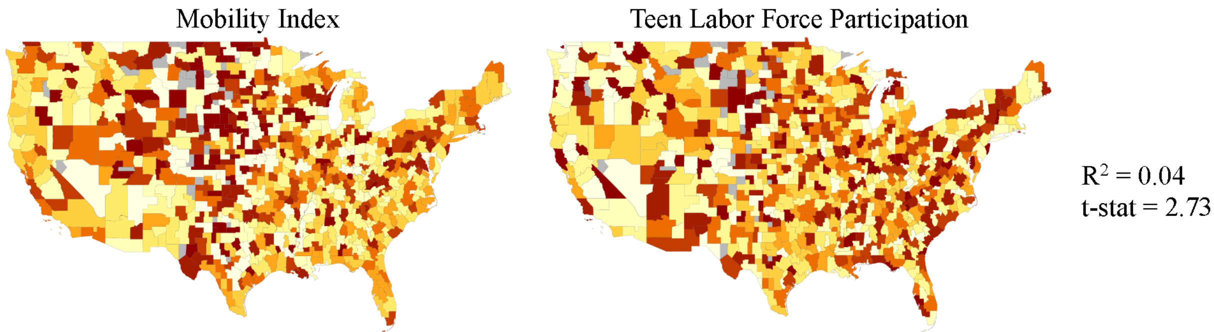
Nearest Neighbor (NN) Differences: $y_l^* = y_l - y_{\ell(l)}$, where $s_{\ell(l)}$ is the location closest to s_l.

Isotropic Differences: y_l^* is the deviation of y_l from the average value of y in a circular (isotropic) neighborhood of s_l with radius equal to b. Specifically, let $m_l = \sum_{j \neq l} \mathbf{1}[(|s_l - s_j| < b]$ denote the number of locations within a distance b of s_l and $\overline{y}_l(b) = m_l^{-1} \sum_{j \neq l} \mathbf{1}[|s_l - s_j| < b] y_j$ denote the average value of y at these locations. If $m_l > 0$, then $y_l^* = y_l - \overline{y}_l(b)$ and $y_l^* = 0$ otherwise (equivalently, the observation is dropped from the regression). In the experiments, we scale the locations so that $\max_{l,\ell} |s_l - s_\ell| = 1$ and use two values, $b = 0.04$ and $b = 0.08$. When $b = 0.04$, $m_l = 0$ for 1% of the locations, and no locations are dropped when $b = 0.08$.

Clustered Fixed Effects: Here the data are partitioned into m clusters and cluster-fixed effects are included in the regression. Equivalently, the regressions are estimated by OLS using

[9] To avoid arguably artificial effects in our spatial AR data generating process, the experiments exclude 2 of the 722 commuting zones because they correspond to islands (Nantucket, MA and Friday Harbor, WA) that are non-adjacent to other commuting zones.

Figure 4: LBM-GLS Transformed Data



data that is demeaned within each cluster. Our experiments use three choices of clusters. First, because the locations are commuting zones, we follow a common practice in applied work and cluster by U.S. state. Alternatively, we use a more agnostic approach that partitions the sampling region into clusters by applying the k-means algorithm to the locations $\{s_l\}$, and this is implemented for $m = 60$ and $m = 240$ clusters.

LBM-GLS: In the time series regression, first differences correspond to a GLS transformation under the canonical random walk model for $I(1)$ persistence. This motivates a GLS transformation based on Lévy-Brownian motion, the canonical spatial $I(1)$ model. Recall that the data used in the regression are demeaned, so that under the canonical model $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \mathbf{M}\mathbf{\Sigma}_L\mathbf{M})$, where $\mathbf{M} = \mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'$ and $\mathbf{\Sigma}_L$ is the covariance matrix induced by Lévy-Brownian motion. The corresponding GLS transform sets

$$\mathbf{Y}^* = (\mathbf{M}\mathbf{\Sigma}_L\mathbf{M})^{-1/2}\mathbf{Y} \tag{20}$$

where $(\mathbf{M}\mathbf{\Sigma}_L\mathbf{M})^{-1/2}$ is the Moore-Penrose generalized inverse of $(\mathbf{M}\mathbf{\Sigma}_L\mathbf{M})^{1/2}$.[10] In the canonical model, this GLS transformation converts $\mathbf{Y}$ into a set of demeaned i.i.d. random variables $\mathbf{Y}^*$. This is no longer true in the more general $I(1)$ model, but given the FCLT in Theorem 2, it is plausible that the LBM-GLS transformation induces enough stationarity for spatial HAR inference to be reliable. As an example, Figure 4 illustrates the GLS transformations for the levels data plotted in panel (c) of Figure 1 and shows that the LBM-GLS transformed data exhibit far less spatial persistence.

These estimators of β are used in conjunction with two types of standard errors. The first

[10]We choose the symmetric square root $(\mathbf{M}\mathbf{\Sigma}_L\mathbf{M})^{1/2}$ to ensure that $\mathbf{Y}^*$ shares the same spatial coordinate basis as $\mathbf{Y}$. This is useful when constructing spatial correlation standard errors for the transformed data.

are spatial HAR standard errors and critical values suggested by Müller and Watson (2023). Their so-called C-SCPC method is calibrated to control size under spatial dependence with an average pairwise correlation of no more than 0.03 (and, by "conditioning" on the regressor, it is by construction more conservative than the method developed in Müller and Watson (2022)). For the fixed effects estimator, we employ the cluster version of the C-SCPC method, and also report results using traditional clustered standard errors.

5.2 Data Generating Processes

The experiments differ in their distribution of $(\mathbf{Y}, \mathbf{X})$, where $\mathbf{Y}$ and $\mathbf{X}$ are independent and identically distributed and are generated by one of nine models.

The first four models (DGP1-DGP4) use $y_l = Y(s_l)$, where Y are $I(1)$ processes: DGP1 is Lévy-Brownian motion and DGP2-DGP4 use $Y \sim I(1)$ as in (6) with different models for B. DGP2 uses $B = J_c$ and $c = c_{0.01}$, so the average pairwise correlation (17) of $\{B(s_l)\}_{l=1}^n$ is $\bar{\rho} = 0.01$, and DGP3 is the same with $c = c_{0.03}$. For DGP4, B is a Gaussian process with Matérn covariance function equal to $\mathbb{E}[B(s)B(r)] = (1 + c\Delta + (c\Delta)^2/3)\exp(-c\Delta)$ for $\Delta = |s - r|$ and c such that the average pairwise correlation of $\{B(s_l)\}_{l=1}^n$ is $\bar{\rho} = 0.03$.

The next two models (DGP5 and DGP6) use $y_l = J_c(s_l)$ with $c = c_{0.03}$ in DGP5 and $c = c_{0.50}$ in DGP6. These models exhibit less than $I(1)$ persistence, much less so in DGP5, and are included to examine the potential effects of "over-differencing".

The final three models (DGP7-DGP9) generate highly persistent data, but are outside the class of $I(1)$ models introduced in Section 2. DGP7 is the Brownian sheet with $y_l = \int_{\mathbb{R}^2} \mathbf{1}[0 \leq r \leq s_l] dW(r)$. DGP8 and DGP9 generate data from the spatial autoregression (SAR) model $\mathbf{Y} = \rho\mathbf{W}\mathbf{Y} + \mathbf{U}$, where $\mathbf{U} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ and $\mathbf{W}$ is the adjacency matrix for the commuting zones with row sums normalized to unity. DGP8 uses $\rho = 0.99$ and DGP9 uses $\rho = 0.999$.

Remark 5.1. The models generate data with different degrees of spatial persistence. One way to gauge their relative persistence is by comparing their rejection frequencies for the spatial unit root ($I(1)$) and stationarity ($I(0)$) tests introduced in Section 4. These are shown in panel (a) of Table 2. By design, the rejection frequency for Lévy-Brownian motion for the $I(1)$ test is 5% and similarly for the $J_{c_{0.03}}$ process for the $I(0)$ test. Using this gauge, the other $I(1)$ process are more persistent than Lévy-Brownian motion (with smaller $I(1)$ and larger $I(0)$ rejection frequencies) as is the SAR model with $\rho = 0.999$.

Remark 5.2. There is a small literature on unit roots and spurious regression in SAR models, initiated by Fingleton (1999) and summarized in Rossi and Lieberman (2023). With the row sums of $\mathbf{W}$ normalized to unity, the unit root SAR model is not well defined (hence our use of $\rho = 0.99$ and 0.999 in DGP8 and DGP9). Lee and Yu (2009, 2013) study asymptotic properties of the row normalized SAR model with a SAR coefficient that converges to unity. For particular forms of the matrix $\mathbf{W}$, they find that this model does not induce spurious regression effects of the type encountered in time series: OLS coefficients remain asymptotically normal, the regression R^2 converges in probability to zero, and t-statistics do not diverge. These results are markedly different from our findings based on a continuous parameter model $Y(\cdot)$ of a spatial $I(1)$ process. More in line with our findings, Fingleton (1999) generates data from a version of the SAR model that is well-defined with a unit SAR coefficient and presents Monte Carlo results suggesting spurious regression phenomena. Our DGP8-DGP9 designs are similar to his. Using a related model, Rossi and Lieberman (2023) derive non-standard large-sample distributions for the estimated SAR coefficient for particular specifications of the SAR weight matrix, and suggest that model-specific non-standard results will hold more generally.

5.3 Simulation Results

Panels (b) and (c) of Table 2 summarize the null rejection frequency of nominal 5% level tests for each method and each DGP. Panel (b) also shows the expected length of confidence intervals. There are three clear takeaways from the table. First, spatial differencing used in conjunction with C-SCPC inference significantly reduces the size distortions from strong spatial persistence, particularly using a relatively large isotropic bandwidth ($b = 0.08$) or a small number of clusters ($m = 60$). Second, inference using traditional clustered standard errors results in much larger size distortions than C-SCPC inference. Third, LBM-GLS (with C-SCPC inference) controls size as well or better than the other methods and produces confidence intervals with the smallest average length. These results, along with the observation that LBM-GLS does not require the choice of a bandwidth or cluster size, suggests that it dominates the other methods considered here.

5.4 Regressions in Chetty et al. (2014)

We now return to Table 1. As noted in Section 4.7 the first three columns of the table suggest substantial spatial correlation in many of the variables. The final columns summarize results

Table 2: Summary of Simulation Experiments

	Lévy-BM	$I(1)_{c_{0.01}}$	$I(1)_{c_{0.03}}$	$I(1)_{\text{Matérn}}$	$J_{c_{0.03}}$	$J_{c_{0.50}}$	Br. Sheet	$\text{SAR}_{0.99}$	$\text{SAR}_{0.999}$	Avg
Test	(a) Rejection Frequency for $I(1)$ and $I(0)$ Tests									
LFUR ($I(1)$ null)	0.05	0.03	0.02	0.03	0.81	0.13	0.13	0.14	0.02	
LFST ($I(0)$ null)	0.82	0.86	0.89	0.87	0.05	0.67	0.49	0.60	0.89	
Method	(b) C-SCPC SEs: Rejection Frequency and *Average Length of Confidence Interval*									Avg
OLS	0.28	0.30	0.33	0.30	0.04	0.18	0.40	0.17	0.39	0.26
	0.88	*1.01*	*1.05*	*1.04*	*0.44*	*0.85*	*0.76*	*0.98*	*1.08*	*0.90*
NN-Difference	0.06	0.14	0.19	0.16	0.05	0.06	0.20	0.09	0.22	0.13
	0.25	*0.47*	*0.58*	*0.57*	*0.23*	*0.24*	*0.30*	*0.40*	*0.58*	*0.40*
Iso-Difference ($b = 0.04$)	0.05	0.12	0.18	0.14	0.04	0.05	0.18	0.07	0.19	0.11
	0.22	*0.36*	*0.46*	*0.47*	*0.20*	*0.21*	*0.31*	*0.40*	*0.61*	*0.36*
Iso-Difference: ($b = 0.08$)	0.04	0.06	0.10	0.08	0.03	0.04	0.16	0.06	0.14	0.08
	0.34	*0.54*	*0.61*	*0.61*	*0.28*	*0.32*	*0.41*	*0.60*	*0.83*	*0.50*
Fixed Effects (State)	0.15	0.20	0.24	0.22	0.08	0.12	0.25	0.15	0.26	0.19
	0.44	*0.66*	*0.73*	*0.70*	*0.28*	*0.41*	*0.55*	*0.60*	*0.79*	*0.57*
Fixed Effects ($m = 60$)	0.05	0.09	0.13	0.10	0.03	0.04	0.12	0.05	0.10	0.08
	0.44	*0.73*	*0.78*	*0.76*	*0.30*	*0.40*	*0.55*	*0.60*	*0.86*	*0.60*
Fixed Effects ($m = 240$)	0.05	0.11	0.15	0.12	0.04	0.04	0.14	0.05	0.19	0.10
	0.32	*0.58*	*0.70*	*0.67*	*0.27*	*0.31*	*0.42*	*0.50*	*0.73*	*0.50*
LBM-GLS	0.03	0.07	0.10	0.07	0.03	0.03	0.13	0.06	0.17	0.08
	0.20	*0.33*	*0.44*	*0.46*	*0.20*	*0.20*	*0.29*	*0.40*	*0.64*	*0.35*
Method	(c) Clustered SEs: Rejection Frequency									Avg
OLS (State)	0.58	0.61	0.64	0.62	0.18	0.49	0.61	0.46	0.69	0.54
Fixed Effects: State	0.23	0.31	0.38	0.34	0.12	0.19	0.35	0.20	0.36	0.28
Fixed Effects ($m = 60$)	0.20	0.33	0.42	0.37	0.10	0.15	0.35	0.21	0.38	0.28
Fixed Effects ($m = 240$)	0.13	0.43	0.53	0.49	0.08	0.11	0.31	0.22	0.41	0.30

Notes: Panel (a) shows the rejection frequency for nominal 5% $I(1)$ and $I(0)$ tests developed in Section 4. Panel (b) shows rejection frequencies for nominal 5% tests of the null that $\beta = 0$ using the regression estimated using the transformation shown in the first column and using C-SCPC standard errors and critical values. Also shown (in italics) is the average length of the (non-size adjusted) nominal 95% confidence interval for β. Panel (c) shows the rejection frequency for nominal 5% tests using clustered standard errors and normal critical values. The data generating processes are defined in the text.

from the regression of the Absolute Mobility Index (the first variable in the table) onto each of the other variables. These regressions were reported in Figure VII of Chetty et al. (2014).[11] The first set of results are for regressions using the levels of the variables, and the second set uses the LBM-GLS transformed variables.

We highlight three results. First, the residuals from the levels regressions are highly spatially correlated: the $I(1)$ null is not rejected at the 10% level for any of the regressions. Second, the LBM-GLS estimates of β and the regression R^2 tend to be smaller in magnitude than in the levels regression. Third, while the OLS and LBM-GLS results differ, the substantive conclusions made in Chetty et al. (2014) about the correlation of the various socioeconomic factors with intergenerational income mobility largely continue to hold after accounting for the strong spatial correlation in the variables.

6 Concluding Remarks

Applied researchers are well aware of the pitfalls of conducting inference with persistent time series data. Variables are routinely tested for the presence of a unit root, and often differenced to avoid spurious regression effects.

This paper demonstrates that inference with highly persistent spatial data is equally fraught: HAC corrections for spatial dependence and standard clustering fail in the presence of strong correlations, leading to spurious significance between independent spatial variables. We have provided tools to detect such strong spatial persistence, akin to time series unit root and stationarity tests.

We have also suggested ways of restoring valid regression inference by suitably transforming the spatial variables, combined with spatial HAR corrections to accommodate any residual weak correlations. The theory here is much less complete, however. It would be especially desirable for future research to obtain a good theoretical understanding of the most promising of these transformations, namely the GLS approach using the canonical spatial unit root model as a baseline.

[11] The results in Table 1 differ slightly from the results in Chetty et al. because Table 1 only uses data from the 48 contiguous U.S. states.

A Proofs

Proof of Lemma 1: By the corollary on page 48 of Adler (2010), the result holds if for some $m > 2d$, $\mathbb{E}\left[(Y(s) - Y(r))^{2m}\right] \leq C|s-r|^m$ for some C. Let $m > 2d$ and apply Condition 2 to obtain

$$\mathbb{E}[(Y(s) - Y(r))^{2m}] \leq C_m \left(\int_{\mathbb{R}^d} (h(u,s) - h(u,r))^2 du \right)^m = C_m |s-r|^m$$

where the equality follows from the representation (5) of L.

For the corresponding result about spatial local-to-unity processes, we similarly have with $Y_c(s) = \int_{\mathbb{R}^d} h_c(r,s)B(r)dr$

$$\mathbb{E}[(Y_c(s) - Y_c(r))^{2m}] \leq C_m \left(\int_{\mathbb{R}^d} (h_c(u,s) - h_c(u,r))^2 du \right)^m = C_m \mathbb{E}[(J_c(s) - J_c(r))^2]^m \tag{21}$$

where the last equality follows from the representation (8) of J_c, and $\mathbb{E}[(J_c(s) - J_c(r))^2] = (1 - \exp(-c|s-r|))/c \leq |s-r|$. $\square$

Proof of Theorem 2: Consider first the claim for the convergence for the LTU process (9). From

$$\int_{\mathbb{R}^d} h_c(r,0)^2 dr = (2c)^{-1} = \lambda^d \int_{\mathbb{R}^d} h_c(\lambda r, 0)^2 dr = \lambda^{(1+d)/2} \frac{\kappa_{c,d}^2}{\kappa_{\lambda c,d}^2} \int_{\mathbb{R}^d} h_{\lambda c}(r,0)^2 dr = \lambda^{(1+d)/2} \frac{\kappa_{c,d}^2}{\kappa_{\lambda c,d}^2} (2c\lambda)^{-1}$$

for all $\lambda > 0$ it follows that $\kappa_{\lambda c,d} = \lambda^{(d-1)/4} \kappa_{c,d}$. Thus, the LTU process can be written as

$$Y_n^0(s) = \lambda_n^{-d/2} \int_{\mathbb{R}^d} h_c(\lambda_n^{-1} r, s) B(r) dr, \quad s \in \mathcal{S}^0. \tag{22}$$

We show convergence of finite dimensional distributions and tightness of the process Y_n^0. The latter follows by Theorem 23.7 of Kallenberg (2021) from (21) and

$$\mathbb{E}[Y_c^0(0)^2] \leq C_2 \lambda_n^{-d} \int_{\mathbb{R}^d} h_c(\lambda_n^{-1} r, s)^2 dr = C_2 \int_{\mathbb{R}^d} h_c(r,s)^2 dr = C_2/(2c)$$

where the inequality invokes Condition 2. For the former, consider the $p \times 1$ vector $(Y_n^0(t_1), \ldots, Y_n^0(t_p))$ for arbitrary $t_1, \ldots, t_p \in \mathcal{S}^0$. By the Cramér-Wold device, it suffices to establish the convergence $X_n = \sum_{j=1}^p v_j Y_n^0(t_j) \Rightarrow \sum_{j=1}^p v_j \omega J_c(t_j)$ for $(v_1, \ldots, v_p) \in \mathbb{R}^p$. Let $f_v(r) = \sum_{j=1}^p v_j h_c(r, t_j)$, so that from (8), $\sum_{j=1}^p v_j J_c(t_j) \sim \mathcal{N}(0, \int_{\mathbb{R}^d} f_v(r)^2 dr)$ and from (22), $X_n = \lambda_n^{-d/2} \int_{\mathbb{R}^d} f_v(\lambda_n^{-1} r) B(r) dr$.

For $\varepsilon > 0$, define $f_v^\varepsilon(r) = f_v(r) \mathbf{1}[|r| < 1/\varepsilon] \prod_{j=1}^p \mathbf{1}[|t_j - r| > \varepsilon]$ and let $X_n^\varepsilon =$

$\lambda_n^{-d/2}\int_{\mathbb{R}^d} f_v^\varepsilon(\lambda_n^{-1}r)B(r)dr$. From Condition 2 we find

$$\begin{aligned}
\mathbb{E}[(X_n^\varepsilon - X_n)^2] &= \lambda_n^{-d}\mathbb{E}\left[\left(\int_{\mathbb{R}^d}(f_v(\lambda_n^{-1}r) - f_v^\varepsilon(\lambda_n^{-1}r))B(r)dr\right)^2\right] \\
&\leq C_2\lambda_n^{-d}\int_{\mathbb{R}^d}(f_v(\lambda_n^{-1}r) - f_v^\varepsilon(\lambda_n^{-1}r))^2 dr \\
&= C_2\int_{\mathbb{R}^d}(f_v(r) - f_v^\varepsilon(r))^2 dr.
\end{aligned}$$

Since $\int_{\mathbb{R}^d}(f_v(r) - f_v^\varepsilon(r))^2 dr \leq 2\int_{\mathbb{R}^d} f_v(r)^2 dr < \infty$, and $f_v^\varepsilon(r) \leq f_v(r)$ for all r, it follows from the dominated convergence theorem that this quantity can be made arbitrarily small by picking ε small enough. Furthermore

$$\begin{aligned}
\mathbb{E}[(X_n^\varepsilon)^2] &= \lambda_n^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_v^\varepsilon(\lambda_n^{-1}r)f_v^\varepsilon(\lambda_n^{-1}s)\sigma_B(s-r)drds \\
&= \int_{\mathbb{R}^d}\sigma_B(s)\int_{\mathbb{R}^d} f_v^\varepsilon(r)f_v^\varepsilon(r+\lambda_n^{-1}s)drds \to \int_{\mathbb{R}^d}\sigma_B(s)ds\int_{\mathbb{R}^d} f_v^\varepsilon(r)^2 dr
\end{aligned}$$

by dominated convergence, since by Cauchy-Schwarz, $|\int_{\mathbb{R}^d} f_v^\varepsilon(r)f_v^\varepsilon(r+\lambda_n^{-1}s)dr| \leq \int_{\mathbb{R}^d} f_v^\varepsilon(r)^2 dr < \infty$ and $\int_{\mathbb{R}^d}|\sigma_B(s)|ds < \infty$.

Finally, note that f_v^ε is bounded and $f_v^\varepsilon(\lambda_n^{-1}r) = 0$ for $|r| > \lambda_n/\varepsilon$. Thus, using Condition 3, $X_n^\varepsilon \Rightarrow \mathcal{N}\left(0, \int_{\mathbb{R}^d}\sigma_B(s)ds\int_{\mathbb{R}^d} f_v^\varepsilon(r)^2 dr\right)$. The result for the LTU process (9) now follows since mean square convergence implies convergence in distribution, and $\varepsilon > 0$ was arbitrary.

For the convergence in Theorem 2, note that from (7), $Y_n^0(s) = \lambda_n^{-d/2}\int_{\mathbb{R}^d} h(\lambda_n^{-1}r, s)B(r)dr$, and $Y_n^0(0) = 0$, so the result follows from the same steps. $\square$

Proof of Theorem 3: The results follow straightforwardly from the CMT if we can show that $\lambda_n^{-1/2}n^{-1}\sum_{l=1}^n y_l \Rightarrow \int Y^0(s)dG(s)$, $\lambda_n^{-1/2}n^{-1}\sum_{l=1}^n x_l \Rightarrow \int X^0(s)dG(s)$, $\lambda_n^{-1}n^{-1}\sum_{l=1}^n x_l y_l \Rightarrow \int X^0(s)Y^0(s)dG(s)$ and $\lambda_n^{-1}n^{-1}\sum_{l=1}^n x_l x_l' \Rightarrow \int X^0(s)X^0(s)'dG(s)$.

Consider the convergence $\lambda_n^{-1}n^{-1}\sum_{l=1}^n x_l y_l \Rightarrow \int X^0(s)Y^0(s)dG(s)$. By the Skorohod almost sure representation theorem (see, for instance, Theorem 11.7.2 of Dudley (2002)), there exist random elements $(Y_n^*(\cdot), X_n^*(\cdot))$ such that $\sup_{s\in\mathcal{S}^0}|(Y_n^*(s) - Y^*(s), X_n^*(s) - X^*(s))| \overset{a.s.}{\to} 0$, $(Y^*(\cdot), X^*(\cdot)) \sim (Y^0(\cdot), X^0(\cdot))$ and $(Y_n^0(\cdot), X_n^0(\cdot)) \sim (Y_n^*(\cdot), X_n^*(\cdot))$ for $n = 1, 2, \ldots$. Thus it suffices to show the claim for $\int X_n^*(s)Y_n^*(s)dG_n(s) = n^{-1}\sum_{l=1}^n X_n^*(s_l)Y_n^*(s_l) \sim n^{-1}\sum_{l=1}^n X_n^0(s_l)Y_n^0(s_l) = \lambda_n^{-1}n^{-1}\sum_{l=1}^n x_l y_l$. We have

$$\left|\int (X_n^*(s)Y_n^*(s) - X^*(s)Y^*(s))dG_n(s)\right| \leq \sup_{s\in\mathcal{S}^0}|(Y_n^*(s) - Y^*(s), X_n^*(s) - X^*(s))| \overset{a.s.}{\to} 0$$

so it suffices to show the claim for $\int X^*(s)Y^*(s)dG_n(s)$. Now almost all realizations of the

$\mathbb{R}^p \mapsto \mathbb{R}$ function $s \mapsto X^*(s)Y^*(s)$ on $\mathcal{S}^0$ are continuous and bounded. For any such realization, $\int X^*(s)Y^*(s)dG_n(s) \to \int X^*(s)Y^*(s)dG(s)$ by the definition of convergence in distribution. Thus $\int X^*(s)Y^*(s)dG_n(s) \overset{a.s.}{\to} \int X^*(s)Y^*(s)dG(s)$. But almost sure convergence implies convergence in distribution, so the desired result follows. The argument for the other terms is analogous. □

The following Lemma is used in the proof of Theorem 4.

Lemma 5. *Let $\mathcal{B}_\delta(r) = \{s : |s-r| \leq \delta\} \subset \mathbb{R}^d$ be a ball of radius δ with center r. Under Condition 1, for any $\delta > 0$, $\limsup_{n\to\infty} \sup_{r\in\mathcal{S}^0} G_n(\mathcal{B}_\delta(r)) \leq \sup_{r\in\mathcal{S}^0} G(\mathcal{B}_\delta(r))$, where $G_n(A)$ and $G(A)$ are the measures that are assigned to the Borel set $A \subset \mathbb{R}^d$ by the distributions G_n and G, respectively.*

Proof. Suppose otherwise. Then there exists $\varepsilon > 0$ and a sequence r_n such that

$$\limsup_{n\to\infty} \sup_{r\in\mathcal{S}^0} G_n(\mathcal{B}_\delta(r)) = \lim_{n\to\infty} G_n(\mathcal{B}_\delta(r_n)) \geq \sup_{r\in\mathcal{S}^0} G(\mathcal{B}_\delta(r)) + \varepsilon.$$

Since G is a continuous distribution, there exists $\delta' > \delta$ such that $\sup_{r\in\mathcal{S}^0} G(\mathcal{B}_{\delta'}(r)) \leq \sup_{r\in\mathcal{S}^0} G(\mathcal{B}_\delta(r)) + \varepsilon/2$. Since $\mathcal{S}^0$ is compact, $r_n \to r_0$ along some subsequence. Along that subsequence, for all n large enough so that $|r_n - r_0| < \delta' - \delta$, we have

$$G_n(\mathcal{B}_\delta(r_n)) \leq G_n(\mathcal{B}_{\delta'}(r_0)) \to G(\mathcal{B}_{\delta'}(r_0)) \leq \sup_{r\in\mathcal{S}^0} G(\mathcal{B}_\delta(r)) + \varepsilon/2$$

yielding the desired contradiction.

□

Proof of Theorem 4: From Theorem 3 and the CMT, $H\hat{\beta} \Rightarrow H\Xi_{\tilde{x}\tilde{x}}^{-1}\Xi_{\tilde{x}\tilde{y}}$ with the r.h.s. non-zero with probability one. Thus $H\hat{\beta} = O_p(1)$ (and not $H\hat{\beta} = o_p(1)$). The result hence follows if we can show that $||S_{\tilde{x}\tilde{x}}^{-1}\hat{\Omega}_n S_{\tilde{x}\tilde{x}}^{-1}|| = o_p(n)$ (since this implies that the smallest eigenvalue of $n(HS_{\tilde{x}\tilde{x}}^{-1}\hat{\Omega}_n S_{\tilde{x}\tilde{x}}^{-1}H')^{-1}$ diverges).

Since $\lambda_n^{-1}S_{\tilde{x}\tilde{x}} \Rightarrow \Xi_{\tilde{x}\tilde{x}}$ and $\Xi_{\tilde{x}\tilde{x}}$ is full rank with probability one, it suffices to show that $n^{-1}\lambda_n^{-2}||\hat{\Omega}_n|| \overset{p}{\to} 0$.

Let $\tilde{Y}_n^0(\cdot) = Y_n^0(\cdot) - \int Y_n^0(s)dG_n(s)$, $\tilde{X}_n^0(\cdot) = \tilde{X}_n^0(\cdot) - \int X_n^0(s)dG_n(s)$ and $e_n^0(\cdot) = (\tilde{Y}_n^0(\cdot) - \hat{\beta}\tilde{X}_n^0(\cdot))\tilde{X}_n^0(\cdot)$, so that $e_l = \lambda_n e_n^0(\lambda_n^{-1}s_l)$. By (11), Theorem 3 and the CMT, $e_n^0(\cdot) \Rightarrow e^0(\cdot) = (\tilde{Y}(\cdot) - \tilde{X}(\cdot)'\Xi_{\tilde{x}\tilde{x}}^{-1}\Xi_{\tilde{x}\tilde{y}})\tilde{X}(\cdot)$, so that $\sup_l |e_n^0(s_l)| \Rightarrow \sup_{s\in\mathcal{S}^0} |e^0(s)|$, and therefore $\lambda_n^{-1}\sup_{l\leq n}|e_l| = O_p(1)$. Consider first the HAC estimator. We have

$$\lambda_n^{-2}n^{-2}\left\|\sum_{l,\ell=1}^{n} \kappa(b_n(s_l - s_\ell))e_l e_\ell'\right\| \leq \lambda_n^{-2}(\sup_{l\leq n}|e_l|)^2 \cdot n^{-2}\sum_{l,\ell=1}^{n}|\kappa(b_n(s_l - s_\ell))|$$

and with $b_n^0 = \lambda_n b_n$ and $s_l^0 = \lambda_n^{-1} s_l$, $\sum_{l,\ell=1}^n |\kappa(b_n(s_l - s_\ell))| = \sum_{l,\ell=1}^n |\kappa(b_n^0(s_l^0 - s_\ell^0))|$ and

$$\sum_{l,\ell=1}^n |\kappa(b_n^0(s_l^0 - s_\ell^0))| \leq \bar\kappa \sum_{l,\ell=1}^n \mathbf{1}[|s_l^0 - s_\ell^0| \leq (b_n^0)^{-1/2}] + \sum_{l,\ell=1}^n \mathbf{1}[|s_l^0 - s_\ell^0| > (b_n^0)^{-1/2}]|\kappa(b_n^0(s_l^0 - s_\ell^0))|.$$

Now

$$n^{-2} \sum_{l,\ell=1}^n \mathbf{1}[|s_l^0 - s_\ell^0| > (b_n^0)^{-1/2}]|\kappa(b_n^0(s_l^0 - s_\ell^0))| \leq \sup_{|a| \geq \sqrt{b_n^0}} |\kappa(a)| \xrightarrow{p} 0$$

by (14) and $1/b_n^0 = o_p(1)$. Furthermore, since G is continuous, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\sup_{r \in \mathcal{S}^0} G(\mathcal{B}_\delta(r)) \leq \varepsilon$ in the notation of Lemma 5. Note that

$$n^{-2} \sum_{l,\ell=1}^n \mathbf{1}[|s_l^0 - s_\ell^0| \leq (b_n^0)^{-1/2}] \leq \sup_{r \in \mathcal{S}^0} G_n(\mathcal{B}_{(b_n^0)^{-1/2}}(r)) \leq \sup_{r \in \mathcal{S}^0} G_n(\mathcal{B}_\delta(r)) + \mathbb{P}((b_n^0)^{-1/2} > \delta).$$

Since by assumption, $1/b_n^0 = o_p(1)$, we have $\mathbb{P}((b_n^0)^{-1/2} > \delta) \to 0$, and by Lemma 5, $\limsup_{n\to\infty} \sup_{r \in \mathcal{S}^0} G_n(\mathcal{B}_\delta(r)) \leq \varepsilon$. But $\varepsilon > 0$ was arbitrary, and the result follows.

For the cluster estimator, we similarly have $\lambda_n^{-2} n^{-2} \left\| \sum_{j=1}^{n_C} \left(\sum_{l \in C_j}^n e_l \right) \left(\sum_{l \in C_j}^n e_l \right)' \right\| \leq \lambda_n^{-2} (\sup_{l \leq n} |e_l|)^2 \cdot n^{-2} \sum_{j=1}^{n_C} |C_j|^2$ and $\max_{1 \leq j \leq n_C} |C_j|/n \to 0$ implies $n^{-2} \sum_{j=1}^{n_C} |C_j|^2 \to 0$, as shown in equation (4) of Hansen and Lee (2019). □

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Supplementary Appendix to

Spatial Unit Roots

by Ulrich K. Müller and Mark W. Watson

This appendix provides supplemental material. Section S.1 develops the formal results for the asymptotic validity of the tests developed in Sections 4.2-4.5 sketched in Section 4.6. Section S.2 provides details on the technique used to generate Figures 2 and 3.

S.1 Asymptotic Validity of Tests of Degree of Persistence of Section 4

We first establish a result about the convergence of eigenvectors. The following lemma is similar to Lemma 6 of Müller and Watson (2022) (also see Rosasco (2010)). The two differences are the replacement of the i.i.d. assumption for s_l^0 by $G_n \Rightarrow G$, and that it allows for a larger class of projections. The latter generalization is necessary for the analysis of residual-based tests discussed in greater detail below.

We will use the following notation: let $k : \mathcal{S}^0 \times \mathcal{S}^0 \mapsto \mathbb{R}$ be a continuous positive definite kernel (not necessarily equal to the covariance kernel of Lévy-Brownian Motion), and let $\mathbf{\Sigma}_n$ be the $n \times n$ matrix with l, ℓth element equal to $k(s_l^0, s_\ell^0)$. Let $\mathcal{L}_G^2$ be the Hilbert space of function $\mathcal{S}^0 \mapsto \mathbb{R}$ with inner product $\langle f_1, f_2 \rangle = \int f_1(s) f_2(s) dG(s)$. Define $L_k : \mathcal{L}_G^2 \mapsto \mathcal{L}_G^2$ as the linear operator $L_k(f)(s) = \int f(r)k(r,s)dG(r)$, and $L_{k,n} = \int f(r)k(r,s)dG_n(r)$.

Lemma S.1. *Suppose the $p \times 1$ vector x_l is such that $x_l = \psi(s_l^0)$ for $l = 1, \ldots, n$ for some continuous function $\psi : \mathcal{S}^0 \mapsto \mathbb{R}^p$, and $\int \psi(s)\psi(s)' dG_n(s) = H_n \to H$ for some positive definite matrix H. Let M and M_n be the projection operators*

$$\begin{aligned} M_n(f)(s) &= f(s) - \int \psi(r)' f(r) dG_n(r) H_n^{-1} \psi(s) \\ M(f)(s) &= f(s) - \int \psi(r)' f(r) dG(r) H^{-1} \psi(s). \end{aligned}$$

Let $\hat{k}_n$, and $\bar{k}$ be the kernels corresponding to the linear operators $M_n L_{k,n} M_n$ and $M L_k M$, respectively, so that the (l, ℓ) element of $\mathbf{M}_X \mathbf{\Sigma}_n \mathbf{M}_X$ is given by $\hat{k}_n(s_l^0, s_\ell^0)$. Let

$$\bar{k}(s,r) = \sum_{i=1}^{\infty} \bar{\nu}_i \bar{\varphi}_i(s) \bar{\varphi}_i(r)$$

with $\int \bar{\varphi}_i(s)\bar{\varphi}_j(s)dG(s) = \mathbf{1}[i=j]$, $\bar{\nu}_i \geq \bar{\nu}_{i+1} \geq 0$ *be the spectral decomposition of* $\bar{k}$*. Define*

$$\hat{\varphi}_i(\cdot) = n^{-1}\hat{\nu}_i^{-1}\sum_{l=1}^{n} r_{i,l}\hat{k}_n(\cdot, s_l^0),$$

where $(\hat{\nu}_i, (r_{i,1}, \ldots, r_{i,n})')$ *is the ith eigenvalue/eigenvector pair of* $\mathbf{M}_X\mathbf{\Sigma}_n\mathbf{M}_X$*. If* $\bar{\nu}_1 > \bar{\nu}_2 > \ldots > \bar{\nu}_q > \bar{\nu}_{q+1}$ *and Condition 1 holds, then for any* $q \geq 1$*,* $\sup_{s\in\mathcal{S}^0, 1\leq i\leq q}|\hat{\varphi}_i(s) - \bar{\varphi}_i(s)| \to 0$ *and* $\max_{1\leq i\leq q}|\hat{\nu}_i - \bar{\nu}_i| \to 0$.

Proof. Set $k_0(s,r) = \bar{k}(s,r) + \psi(s)'H^{-1}\psi(r)$ and define the associated operators $L(f)(s) = \int f(r)k_0(r,s)dG(r)$, $L_n(f)(s) = \int f(r)k_0(r,s)dG_n(r)$, $\bar{L} = MLM$, $\bar{L}_n = ML_nM$ and $\hat{L}_n = M_nL_nM_n$. Note that $\bar{L} = ML_kM$ and $\hat{L}_n = M_nL_{k,n}M_n$. Let $\mathcal{H} \subset \mathcal{L}_G^2$ be the Reproducing Kernel Hilbert Space of functions $f : \mathcal{S}^0 \mapsto \mathbb{R}$ with kernel k_0 and inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ satisfying $\langle f, k_0(\cdot, r)\rangle_{\mathcal{H}} = f(r)$ and associated norm $||f||_{\mathcal{H}}$. By Theorem 2.16 in Saitoh and Sawano (2016), $\mathcal{H}$ contains all functions of the form $a'\psi$ for $a \in \mathbb{R}^p$, so $\sup_{|a|=1}||a'\psi||_{\mathcal{H}} < \infty$. Now proceed as in the proof of Lemma 6 of Müller and Watson (2022) to argue that $\sup_{r\in\mathcal{S}^0}|f(r)| \leq \sqrt{\sup_{s\in\mathcal{S}^0}k_0(s,s)} \cdot ||f||_{\mathcal{H}}$, and

$$||Mf||_{\mathcal{H}} = ||f - \int \psi(r)'f(r)dG(r)H^{-1}\psi||_{\mathcal{H}} \leq ||f||_{\mathcal{H}} + \sup_{r\in\mathcal{S}^0}|f(r)| \cdot \sup_{r\in\mathcal{S}^0}|H^{-1}\psi(r)| \cdot \sup_{|a|=1}||a'\psi||_{\mathcal{H}}$$

so $M : \mathcal{H} \mapsto \mathcal{H}$ is a bounded operator. By the same argument, so is M_n.

From $\langle f, k_0(\cdot, r)\rangle_{\mathcal{H}} = f(r)$, we further obtain

$$\int \psi(r)f(r)(dG_n(r) - dG(r)) = \left\langle f, \int \psi(r)k_0(\cdot, r)(dG_n(r) - dG(r))\right\rangle_{\mathcal{H}} \tag{S.1}$$

and for each component ψ_i of ψ, $i = 1, \ldots, p$,

$$\begin{aligned} & \left\|\int \psi_i(r)k_0(\cdot, r)(dG_n(r) - dG(r))\right\|_{\mathcal{H}}^2 \qquad \text{(S.2)} \\ = & \int\int \psi_i(s)k_0(s,r)\psi_i(r)(dG_n(s) - dG(s))(dG_n(r) - dG(r)) \\ = & \mathbb{E}[\psi_i(S_n)k_0(S_n, R_n)\psi_i(R_n) - \psi_i(S_n)k_0(S, R_n)\psi_i(R) \\ & \qquad -\psi_i(S)k_0(S_n, R)\psi_i(R_n) + \psi_i(S)k_0(S, R)\psi_i(R)] \to 0 \end{aligned}$$

where (S_n, R_n) is a sequence of $\mathbb{R}^{2d}$ random variables with distribution $G_n \times G_n$ converging to (S, R) with distribution $G \times G$. The convergence then follows since the $\mathbb{R}^{2d} \mapsto \mathbb{R}$ function $(s, r) \mapsto$

$\psi_i(s)k_0(s,r)\psi_i(r)$ is continuous and bounded. Thus, by (S.1), (S.2) and Cauchy-Schwarz,

$$\sup_{||f||_{\mathcal{H}}\leq 1}\left|\int \psi(r)f(r)(dG_n(r)-dG(r))\right|\to 0.$$

From $H_n^{-1} \to H^{-1}$ and $|\int \psi(r)f(r)dG_n(r)| \leq \sup_{r\in\mathcal{S}^0}|f(r)| \cdot \sup_{r\in\mathcal{S}^0}|\psi(r)| \leq \sup_{r\in\mathcal{S}^0}|\psi(r)|\sqrt{\sup_{s\in S}k_0(s,s)}\cdot||f||_{\mathcal{H}}$, we conclude that with $\Delta_n(f) = H_n^{-1}\int\psi(r)f(r)dG_n(r) - H^{-1}\int\psi(r)f(r)dG(r)$, $\sup_{||f||_{\mathcal{H}}\leq 1}|\Delta_n(f)|\to 0$. Thus

$$\sup_{||f||_{\mathcal{H}}\leq 1}||(M_n-M)f||_{\mathcal{H}} = \left\|\Delta_n(f)'\psi\right\|_{\mathcal{H}} \leq \sup_{||f||_{\mathcal{H}}\leq 1}|\Delta_n(f)|\cdot\sup_{|a|=1}||a'\psi||_{\mathcal{H}}\to 0.$$

The only remaining piece of the proof is to show that $||L_n - L||_{HS}^2\to 0$ under the assumption of $G_n\Rightarrow G$, where for any Hilbert-Schmidt operator $A:\mathcal{H}\mapsto\mathcal{H}$, $||A||_{HS}^2 = \sum_{j\geq 1}\langle Ae_j, Ae_j\rangle_{\mathcal{H}}$ for an orthonormal base e_j. One choice for e_j are the eigenfunctions scaled by the square root of the eigenvalues of the spectral decomposition of k_0, so that $k_0(r,s) = \sum_{j=1}^{\infty}e_j(r)e_j(s)$; see the discussion in the proof of Lemma 6 in Müller and Watson (2022). We find

$$\begin{aligned}
||L_n - L||_{HS}^2 &= \sum_{j\geq 1}\left\langle \int e_j(s)k_0(s,\cdot)(dG_n(s)-dG(s)), \int e_j(s)k_0(s,\cdot)(dG_n(s)-dG(s))\right\rangle_{\mathcal{H}} \\
&= \int\int\left(\sum_{j\geq 1}e_j(s)e_j(r)\right)k_0(s,r)(dG_n(s)-dG(s))(dG_n(r)-dG(r)) \\
&= \int\int k_0(s,r)^2(dG_n(r)-dG(r))(dG_n(r)-dG(r)) \\
&= \mathbb{E}[k_0(S_n,R_n)^2 - k_0(S,R_n)^2 - k_0(S_n,R)^2 + k_0(S,R)^2]\to 0
\end{aligned}$$

where the change of the order of integration and summation is justified by Fubini's Theorem, and the convergence follows, since the $\mathbb{R}^{2d}\mapsto\mathbb{R}$ function $(s,r)\mapsto k_0(s,r)^2$ is bounded and continuous.

□

Specializing Lemma S.1 to $p=1$ and $\psi=1$, note that the l,ith element of the matrix of eigenvectors $\mathbf{R}$ is given by $\hat{\varphi}_i(s_l)$. Thus Lemma S.1 shows that the weights in the weighted average $\mathbf{Z}=\mathbf{R}'\mathbf{Y}$ converge to continuous functions, so the FCLT of Theorem 2 and the continuous mapping theorem (CMT) yield $\lambda_n^{-1/2}n^{-1}\mathbf{Z}_n\Rightarrow\mathcal{N}(0,\omega^2\operatorname{diag}(\bar{\nu}_1,\ldots,\bar{\nu}_q))$. Since the LFUR statistic is scale invariant, the scale parameters $\lambda_n^{-1/2}n^{-1}$ and ω^2 vanish, and the critical value computed from the canonical model converges to the asymptotically correct critical value for generic spatial $I(1)$ processes.

By the same arguments, the LFST test is large sample valid under the general local-to-unity model (9) with $c\geq c_{0.03}$. A more subtle question asks whether it also remains valid under generic weak dependence, defined as $y_l = B(s_l) = B(\lambda_n s_l^0)$, with $\lambda_n\to\infty$ and B a weakly dependent random field as in Section 2. The CLT in Lahiri (2003) shows that under such generic weak dependence (and under the assumption that $s_l^0 \sim G$ is i.i.d.), a suitably scaled version of $\mathbf{Z}$ becomes Gaussian, but not necessarily with covariance matrix proportional to $\mathbf{I}_q$. In the spatial case, the effect of weak dependence on the covariance of smoothly weighted averages is generically more subtle than a multiplication by the scalar long-run standard deviation. Still, the LFST test remains valid, since for every n, its critical value is chosen to be valid for all $c \geq c_{0.03}$, so it is also valid under all sequences of $c_n \to \infty$, including those that induce the different possible limits identified by Lahiri's (2003) CLT. This result is summarized in the following theorem.

Theorem S.2. *If $y_l = B(\lambda_n s_l^0)$ and $\lambda_n \to \infty$ with $\lambda_n^d/n \to a \in [0, \infty)$, then under the assumptions of Lahiri's CLT in his Theorem 3.2,* $\limsup_{n\to\infty} \mathbb{P}(\mathrm{LFST} \geq \mathrm{cv}_n^{\mathrm{LFST}}) \leq \alpha$.

Proof. By Lemmas 3 and 12 in Müller and Watson (2022), we have

$$\lambda_n^{d/2} n^{-1} \mathbf{Z}_n \Rightarrow \mathcal{N}\left(\mathbf{0}, a\sigma_B(0) \int \bar{\boldsymbol{\varphi}}(s)\bar{\boldsymbol{\varphi}}(s)' dG(s) + \omega^2 \int \bar{\boldsymbol{\varphi}}(s)\bar{\boldsymbol{\varphi}}(s)' g(s) dG(s)\right) \tag{S.3}$$

where $\bar{\boldsymbol{\varphi}} = (\bar{\varphi}_1, \ldots, \bar{\varphi}_q)$, $\omega^2 = \int_{\mathbb{R}^d} \sigma_B(s) ds$ and g is the density of the distribution G. Since the LFST statistic is scale invariant, its limiting distribution under (S.3) only depends on the properties of B through the ratio $\chi = a\sigma_B(0)/\omega^2 \in [0, \infty)$. We need to show that $\liminf_{n\to\infty} \mathrm{cv}_n^{\mathrm{LFST}}$ is at least as large as the $1 - \alpha$ quantile, say $\mathrm{cv}_\chi^{\mathrm{LFST}}$, of the (continuous) limit distribution of LFST for this value of χ.

Note that for $B = J_c$, $\sigma_B(0)/\omega^2 = K_d c^{1+d}$ for some $K_d > 0$. For $a > 0$, let c_* be such $K_d c_*^{1+d} = \chi/a$, and let $c_* = 1$ otherwise. For all n sufficiently large so that $\lambda_n c_* \geq c_{0.03}$, $\mathrm{cv}^{\mathrm{LFST}}$ is such that the LFST test controls size under $B = J_{c^*}$. But since $B = J_{c_*}$ satisfies the assumptions of Lahiri (2003), this model induces the same limit (S.3), so its $1 - a$ quantile converges to $\mathrm{cv}_\chi^{\mathrm{LFST}}$, and the result follows.

□

We now turn to the validity of residual based tests. We first need a further generalization of the convergence of eigenvectors that accommodates projections off converging sequences of functions.

Lemma S.3. *Assume the conditions of Lemma S.1 hold. Suppose $\tilde{x}_l = \psi_n(s_l^0)$, where the continuous functions $\psi_n : \mathcal{S}^0 \mapsto \mathbb{R}^p$ are such that $\sup_{s\in\mathcal{S}^0} |\psi_n(s) - \psi(s)| \to 0$, for some continuous function ψ. Define the the projection operator $\tilde{M}_n : \mathcal{L}_G^2 \mapsto \mathcal{L}_G^2$ as*

$$\tilde{M}_n(f)(s) = f(s) - \int \psi_n(r)' f(r) dG_n(r) H_n^{-1} \psi_n(s),$$

and let $\tilde{k}_n$ be the kernel corresponding to the linear operator $\tilde{M}_n L_{k,n} \tilde{M}_n$, so that the (l, ℓ) element of $\mathbf{M}_{\tilde{X}} \boldsymbol{\Sigma}_n \mathbf{M}_{\tilde{X}}$ is given by $\tilde{k}_n(s_l^0, s_\ell^0)$. Let $(\tilde{\nu}_i, (\tilde{r}_{i,1}, \ldots, \tilde{r}_{i,n})')$ be the ith eigenvalue/eigenvector pair *of* $\mathbf{M}_{\tilde{X}}\mathbf{\Sigma}_n\mathbf{M}_{\tilde{X}}$*, and define* $\tilde{\varphi}_i(\cdot) = n^{-1}\tilde{\nu}_i^{-1}\sum_{l=1}^n \tilde{r}_{i,l}\tilde{k}_n(\cdot, s_l^0)$*. Then* $\sup_{s\in\mathcal{S}^0, 1\leq i\leq q}|\tilde{\varphi}_i(s) - \bar{\varphi}_i(s)| \to 0$ *and* $\max_{1\leq i\leq q}|\tilde{\nu}_i - \bar{\nu}_i| \to 0$.

Proof. From standard arguments, we obtain $\int \psi_n(s)\psi_n(s)'dG_n(s) \to H$ and $\int \psi(s)\psi_n(s)'dG_n(s) \to H$. Thus, $||\mathbf{M}_{\tilde{X}} - \mathbf{M}_X|| \to 0$, and by a direct calculation, $\sup_{s,r\in\mathcal{S}^0}|\tilde{k}_n(r,s) - \hat{k}_n(r,s)| \to 0$, and $\sup_{s,r\in\mathcal{S}^0}|\hat{k}_n(r,s) - \bar{k}(r,s)| \to 0$ and thus $\sup_{s,r\in\mathcal{S}^0}|\tilde{k}_n(r,s) - \bar{k}(r,s)| \to 0$. Furthermore, proceeding as in the proof of Lemma S.1 shows that $||\mathbf{\Sigma}_n||$ converges to $\bar{\nu}_1$, the largest eigenvalue of the integral operator with kernel $\bar{k}$, so $||\mathbf{\Sigma}_n|| = O(1)$. Thus also $||\mathbf{M}_{\tilde{X}}\mathbf{\Sigma}_n\mathbf{M}_{\tilde{X}} - \mathbf{M}_X\mathbf{\Sigma}_n\mathbf{M}_X|| \to 0$, and from Weyl's inequality, $\max_{1\leq i\leq q}|\tilde{\nu}_i - \hat{\nu}_i| \to 0$. Since also $\max_{1\leq i\leq q}|\hat{\nu}_i - \bar{\nu}_i| \to 0$ from Lemma S.1, we can conclude that

$$\sup_{s\in\mathcal{S}^0}|(\tilde{\nu}_i^{-1} - \hat{\nu}_i^{-1})n^{-1}\sum_{l=1}^n r_{i,l}\hat{k}_n(s, s_l^0)| \leq |\tilde{\nu}_i^{-1} - \hat{\nu}_i^{-1}| \cdot \sup_{s\in\mathcal{S}^0}|\hat{\varphi}_i(s)| \cdot \sup_{s,r\in\mathcal{S}^0}|\hat{k}_n(r,s)| \to 0$$

where the inequality uses $r_{i,l} = \hat{\varphi}_i(s_l^0)$, and the convergence follows from the above results and $\sup_{s\in\mathcal{S}^0}|\hat{\varphi}_i(s)| \to \sup_{s\in\mathcal{S}^0}|\varphi_i(s)| < \infty$ from Lemma S.1. Also,

$$\sup_{s\in\mathcal{S}^0}|n^{-1}\sum_{l=1}^n r_{i,l}(\tilde{k}_n(s, s_l^0) - \hat{k}_n(s, s_l^0))| \leq \sup_{s\in\mathcal{S}^0}|\hat{\varphi}_i(s)| \cdot \sup_{r,s\in\mathcal{S}^0}|\tilde{k}_n(r,s) - \hat{k}(r,s)| \to 0.$$

Finally, since $\max_{1\leq i\leq q}|\tilde{\nu}_i - \bar{\nu}_i| \to 0$ and $\bar{\nu}_1 > \bar{\nu}_2 > \ldots > \bar{\nu}_q > \bar{\nu}_{q+1}$, we can apply Corollary 1 of Yu, Wang and Samworth (2015) and conclude that $n^{-1}\sum_{l=1}^n(\tilde{r}_{i,l} - r_{i,l})^2 \to 0$ for $i = 1, \ldots, q$. Applying Cauchy-Schwarz then yields

$$\sup_{s\in\mathcal{S}^0}|n^{-1}\sum_{l=1}^n(\tilde{r}_{i,l} - r_{i,l})\tilde{k}_n(s, s_l^0)|^2 \leq n^{-1}\sum_{l=1}^n(\tilde{r}_{i,l} - r_{i,l})^2 \cdot \sup_{s\in\mathcal{S}^0} n^{-1}\sum_{l=1}^n \tilde{k}_n(s, s_l^0)^2 \to 0$$

where the convergence follows from $n^{-1}\sum_{l=1}^n \tilde{k}_n(s, s_l^0)^2 \leq 2\sup_{r,s\in\mathcal{S}^0}|\bar{k}(r,s)|^2 + 2\sup_{s,r\in\mathcal{S}^0}|\tilde{k}_n(r,s) - \bar{k}(r,s)|^2 = O(1)$.

□

Theorem S.4. *Suppose* $y_l = x_l'\beta + u_l$, $(x_l', u_l) = \lambda_n^{1/2}(X_n^0(s_l^0)', U_n^0(s_l^0)) \in \mathbb{R}^p \times \mathbb{R}$ *with* $(X_n^0(\cdot), U_n^0(\cdot))$ *satisfying*

$$\begin{bmatrix} U_n^0(\cdot) \\ X_n^0(\cdot) \end{bmatrix} = \begin{bmatrix} \lambda_n^{-1/2}U(\lambda_n\cdot) \\ \lambda_n^{-1/2}X(\lambda_n\cdot) \end{bmatrix} \Rightarrow \begin{bmatrix} U^0(\cdot) \\ X^0(\cdot) \end{bmatrix} \tag{S.4}$$

on $\mathcal{S}^0$*. Let* $\mathbf{R}^X$ *be the* $n \times p$ *matrix of* q *eigenvectors of* $\mathbf{M}_X\mathbf{\Sigma}_n\mathbf{M}_X$ *corresponding to the largest eigenvalues. Suppose for almost every realization of* X^0*, the largest* $q+1$ *eigenvalues of the kernel* $k_{X^0} : \mathcal{S}^0 \times \mathcal{S}^0 \mapsto \mathbb{R}$ *corresponding to the linear operator* $M_{X^0}L_kM_{X^0}$ *with* $M_{X^0}(f)(s) = f(s) -$

$X^0(s)\left(\int X^0(r)X^0(r)'dG(r)\right)^{-1}\int X^0(r)'f(r)dG(r)$ *are distinct. If also Condition 1 holds, then*

$$\lambda_n^{-1/2}\mathbf{R}_X'\mathbf{Y} \Rightarrow \omega\int \boldsymbol{\varphi}_{X^0}(s)U^0(s)dG(s) \tag{S.5}$$

where $\boldsymbol{\varphi}_{X^0}(\cdot)$ *are the* q *eigenfunctions of* k_{X^0} *corresponding to the largest eigenvalues.*

Furthermore, let $\tilde{U}_n^0$ *be independent of* (X_n^0, U_n^0)*, and suppose* $\tilde{U}_n^0$ *satisfies* $\tilde{U}_n^0(\cdot) \Rightarrow \tilde{U}^0(\cdot)$ *with* $\tilde{U}^0 \sim U^0$*. Let* $\mathrm{cv}_n(X_n^0)$ *be the* $1-\alpha$ *quantile of the conditional distribution of* $\phi(\mathbf{R}^{X\prime}\tilde{\mathbf{U}})$ *given* $\mathbf{R}^X$ *for some continuous function* $\phi : \mathbb{R}^q \mapsto \mathbb{R}$ *satisfying* $\phi(ax) = \phi(x)$ *for all* $a \neq 0$ *and* $x \in \mathbb{R}^q$*, and* $\tilde{\mathbf{U}} = (\tilde{U}_n^0(s_1^0), \ldots, \tilde{U}_n^0(s_n^0))'$*. Suppose that (i)* X^0 *is independent of* U^0*, (ii) for almost all realizations of* X^0 *the conditional distribution of* $\phi(\int \boldsymbol{\varphi}_{X^0}(s)U^0(s)dG(s))$ *is continuous. Then* $\mathbb{P}(\phi(\mathbf{R}^{X\prime}\mathbf{Y}) > \mathrm{cv}_n(X_n^0)) \to \alpha$*.*

Proof. For conciseness, we note the dependence of $\mathbf{R}_X$ and $\mathbf{Y}$ on n. We will show that $(\phi(\mathbf{R}_{X,n}'\mathbf{Y}_n), \mathrm{cv}_n(X_n^0)) \Rightarrow (\phi(\int \boldsymbol{\varphi}_X(s)U^0(s)dG(s)), q_{1-\alpha}^{\phi}(X^0))$ with $q_{1-\alpha}^{\phi}(X^0)$ the $1-\alpha$ quantile of $\phi(\int \boldsymbol{\varphi}_X(s)U^0(s)dG(s))$ conditional on X^0. The result then follows from the CMT applied to $\mathbf{1}[\phi(\mathbf{R}_{X,n}'\mathbf{Y}_n) > \mathrm{cv}_n(X_n^0)]$, and taking expectations.

Apply the almost sure representation theorem to argue that there exists a probability space $(\Omega_0, \mathfrak{F}_0, P_0)$ and associated random processes X^*, U^* and X_n^*, U_n^*, $n \geq 1$ such that $(X_n^*, U_n^*) \sim (X_n^0, U_n^0)$, $(X^*, U^*) \sim (X^0, U^0)$ and $\sup_{s\in S^0}|X_n^*(s) - X^*(s)| \overset{a.s.}{\to} 0$, $\sup_{s\in S^0}|U_n^*(s) - U^*(s)| \overset{a.s.}{\to} 0$. Using the same arguments as in the proof of Theorem 3, and a realization by realization application of Lemma S.3, then yields

$$\lambda_n^{-1/2}\mathbf{R}_{X,n}^{*\prime}\mathbf{Y}_n^* \to \omega\int \boldsymbol{\varphi}_{X^*}(s)U^*(s)dG(s) \sim \omega\int \boldsymbol{\varphi}_{X^*}(s)U^0(s)dG(s) \tag{S.6}$$

where $(\mathbf{R}_{X,n}^*, \mathbf{Y}_n^*)$ are defined analogously to $(\mathbf{R}_{X,n}, \mathbf{Y}_n)$ on $(\Omega_0, \mathfrak{F}_0, P_0)$, and $(\mathbf{R}_{X,n}^*, \mathbf{Y}_n^*) \sim (\mathbf{R}_{X,n}, \mathbf{Y}_n)$ for all $n \geq 0$ by construction, so (S.5) holds.

The further result now follows if we can show that also $\mathrm{cv}_n(X_n^*) \overset{a.s.}{\to} q_{1-\alpha}^{\phi}(X^*)$, since almost sure convergence implies convergence in distribution. To that end, note there exists a separate probability space $(\Omega_1, \mathfrak{F}_1, P_1)$ with associated sequences of random process $\tilde{U}^*$ and $\tilde{U}_n^*$ and such that $\tilde{U}_n^* \sim \tilde{U}_n^0$, $\tilde{U}^* \sim \tilde{U}^0 \sim U^0$ and $\sup_{s\in\mathcal{S}^0}|\tilde{U}_n^*(s) - \tilde{U}^*(s)| \overset{a.s.}{\to} 0$. Form the product space $(\Omega_0 \times \Omega_1, \mathfrak{F}_0 \otimes \mathfrak{F}_1, P_0 \times P_1)$, so that on this new space, $(X^*, \{X_n^*\}_{n=1}^{\infty})$ is independent of $(\tilde{U}^*, \{\tilde{U}_n^*\}_{n=1}^{\infty})$ by construction. Use the same arguments as for (S.6) obtain that for P_0-almost all $\omega_0 \in \Omega_0$ and P_1-almost all $\omega_1 \in \Omega_1$, in obvious notation,

$$\lambda_n^{-1/2}\mathbf{R}_{X,n}^{*\prime}\tilde{\mathbf{U}}_n^* \to \int \boldsymbol{\varphi}_{X^*}(s)\tilde{U}^*(s)dG(s)$$

jointly with (S.6). But almost sure convergence implies convergence in distribution, and $\tilde{U}^* \sim U^0$, so for P_0-almost all $\omega_0 \in \Omega_0$, the distribution of $\lambda_n^{-1/2}\mathbf{R}_{X,n}^{*\prime}\tilde{\mathbf{U}}_n^*$ induced by P_1 converges to the conditional distribution of $\int \boldsymbol{\varphi}_{X^*}(s)U^0(s)dG(s)$ given X^*. Since ϕ is continuous and the conditional distribution is assumed continuous, this implies that for all such ω_0, $\mathrm{cv}_n(X_n^0) \overset{a.s.}{\to} q_{1-\alpha}^{\phi}(X^*)$. Thus $(\phi(\mathbf{R}'_{X,n}\mathbf{Y}_n), \mathrm{cv}_n(X_n^0)) \sim (\phi(\mathbf{R}^{*\prime}_{X,n}\mathbf{Y}_n^*), \mathrm{cv}_n(X_n^*)) \overset{a.s.}{\to} (\phi(\int \boldsymbol{\varphi}_{X^*}(s)U^*(s)dG(s)), q_{1-\alpha}^{\phi}(X^*)) \sim (\phi(\int \boldsymbol{\varphi}_{X^0}(s)U^0(s)dG(s)), q_{1-\alpha}^{\phi}(X^0))$, and the result follows, because almost sure convergence implies convergence in distribution.

□

The theorem justifies the conditional use of a critical value for the test statistic $\phi(\mathbf{R}'_X\mathbf{Y})$ that is equal to the $1-\alpha$ quantile of $\phi(\mathbf{R}'_X\tilde{\mathbf{U}})$ conditional on $\mathbf{R}_X$, for some (pseudo-) random variable draws of $\tilde{u}_l = \tilde{U}(s_l^0)$ that induce the same limiting process as the actual regression errors u_l. Since ϕ is assumed scale invariant, the scaling of $\tilde{u}_l$ is immaterial in this construction. This formally justifies the critical value construction for the Engle and Granger (1987)-type test of coinegration of Remark 4.2.

S.2 Generation of Figures 2-3

For the left panel of Figure 2 and Figure 3, we approximate the non-stationary processes by stationary ones with a very small degree of mean reversion. In particular, with $f_0(\omega) = 1$, let $\tilde{f}_i(\omega) = f_i(\omega)/(c^2 + |\omega|^2)^{3/2}$ with $c = 0.1$ for the three processes Y_i, $i = 0, 1, 2$ of Figures 2 and 3. These spectral densities are isotropic, so the covariance functions satisfy $E[Y_i(r)Y_i(s)] = \sigma_i(|r-s|)$ with

$$\sigma_i(x) = \int_0^\infty J_0(\omega x) f_i(\omega) d\omega$$

where J_0 is the Bessel function function of the first kind with zero parameter (cf. equation (1.2.6) in Ivanov and Leonenko (1989)). We approximate $\sigma_i(\cdot)$ numerically on the interval $[0, 1]$, and then use Stein's (2002) technique to generate the figures via the fast Fourier transform on a grid of 700×700 points.

Additional References

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