Supplementary Appendix to

Spatial Unit Roots

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This appendix provides supplemental material. Section S.1 provides details on the technique used to generate Figures 2 and 4. Section S.2 contains proofs of all formal results in Sections 4-5. Section

S.1 Generation of Figures 2-4

For the left panel of Figure 2 and Figure 4, we approximate the non-stationary processes by stationary ones with a very small degree of mean reversion. In particular, with $f_0(\omega) = 1$, let $\tilde{f}_i(\omega) = f_i(\omega)/(c^2 + |\omega|^2)^{3/2}$ with $c = 0.1$ for the three processes Y_i, $i = 0, 1, 2$ of Figures 2 and 4. These spectral densities are isotropic, so the covariance functions satisfy $E[Y_i(r)Y_i(s)] = \sigma_i(|r - s|)$ with

$$\sigma_i(x) = \int_0^\infty J_0(\omega x) f_i(\omega) d\omega$$

where J_0 is the Bessel function function of the first kind with zero parameter (cf. equation (1.2.6) in Ivanov and Leonenko (1989)). We approximate $\sigma_i(\cdot)$ numerically on the interval $[0, 1]$, and then use Stein's (2002) technique to generate the figures via the fast Fourier transform on a grid of 700×700 points.

The eigenfunctions of Figure 3 are approximated via (22) using 1000 locations $\{s_l^0\}_{l=1}^{1000}$ drawn at random within the contiguous U.S.

S.2 Proofs of Results from Sections 4 and 5

Proof of Theorem 5: Clearly,

$$\hat{\gamma} = \frac{\int_{\mathcal{I}_b} \int Y_n^0(s)\kappa_b(|s - r|)(Y_n^0(r) - Y_n^0(s))dG_n(r)dG_n(s)}{\int_{\mathcal{I}_b} Y_n^0(s)^2 dG_n(s)} \tag{S.1}$$

and proceeding as in the proof of Theorem 3 shows that it suffices to show the claim with $Y_n^0(s)$ replaced by $Y^*(s) = \omega J_c(s)$ in (S.1). Denote the resulting expression by $\hat{\gamma}^*$, we have

$$\hat{\gamma}^* = \frac{\mathbb{E}\left[\mathbf{1}[S_n \in \mathcal{I}_b] Y^*(S_n)\kappa_b(|S_n - R_n|)(Y^*(R_n) - Y^*(S_n))|Y^*\right]}{\mathbb{E}\left[\mathbf{1}[S_n \in \mathcal{I}_b] Y^*(S_n)^2|Y^*\right]}$$

$$
\begin{aligned}
&\overset{a.s.}{\to} \frac{\mathbb{E}\left[\mathbf{1}[S\in\mathcal{I}_b]Y^*(S)\kappa_b(|S-R|)(Y^*(R)-Y^*(S))|Y^*\right]}{\mathbb{E}\left[\mathbf{1}[S\in\mathcal{I}_b]Y^*(S)^2|Y^*\right]} \\
= & \frac{\int_{\mathcal{I}_b}\int J_c(s)\kappa_b(|s-r|)(J_c(r)-J_c(s))dG(r)dG(s)}{\int_{\mathcal{I}_b} J_c(s)^2dG(s)}
\end{aligned}
$$

where (S_n,R_n) is a sequence of $\mathbb{R}^{2d}$ random variables with distribution $G_n\times G_n$ converging to (S,R) with distribution $G\times G$, and the convergence follows, since for almost all realizations of Y^*, the $\mathbb{R}^{2d}\mapsto\mathbb{R}$ function $(s,r)\mapsto\mathbf{1}[s\in\mathcal{I}_b]Y^*(s)\kappa_b(|s-r|)(Y^*(r)-Y^*(s))$ and the $\mathbb{R}^d\mapsto\mathbb{R}$ function $s\mapsto\mathbf{1}[s\in\mathcal{I}_b]Y^*(s)^2$ is bounded with a discontinuity set of Lebesgue measure zero. $\square$

Proof of Theorem 6: We first show the result for L in place of J_c. In the proof, C denotes a sufficiently large constant, not necessarily the same in each instance it is used.

As a first step, we show that replacing $L(s)$ by $L(s)-\hat{m}$ induces a $o_p(1)$ difference, where the convergences throughout the proof are with respect to $b\to 0$. By Cauchy-Schwarz, the second moment of the difference is bounded above by

$$
\begin{aligned}
&\mathbb{E}\left[\left(b^{-d-1}\hat{m}\int_{\mathcal{I}_b}\int\kappa_b(|s-r|)(L(r)-L(s))dG(r)dG(s)\right)^2\right] \\
\leq\ & \mathbb{E}[\hat{m}^2]\mathbb{E}\left[\left(b^{-d-1}\int_{\mathcal{I}_b}\int\kappa_b(|s-r|)(L(r)-L(s))dG(r)dG(s)\right)^2\right].
\end{aligned}
$$

Consider first $d=1$. The support $\mathcal{S}^0$ of G then consists of a countable number of disjoint intervals, and it suffices to show that the integral over each of those intervals is $o_p(1)$. Take one such interval $[l,u]\subset\mathbb{R}$. We have

$$
\int_{l+b}^{u-b}\int_l^u\kappa_b(|s-r|)(L(r)-L(s))dG(r)dG(s)=\int_l^u h_b(r)L(r)dG(r)
$$

with $h_b(r)=\int_l^u(\mathbf{1}[l+b\leq s\leq u-b]\kappa_b(|s-r|)-\mathbf{1}[l+b\leq r\leq u-b]\kappa_b(|s-r|))dG(s)$. By inspection, for all small enough b, $h_b(r)=0$ for $r\in[l+2b,u-2b]$, $\sup_{r\in[l,u]}|h_b(r)|\leq Cb$, $\int_l^{l+2b}h_b(r)dr=\int_{u-2b}^u h_b(r)dr=0$, so that $\int_l^{l+2b}h_b(r)g(r)dr=b\int_0^2 h_b(br)g(l+br)dr=O(b^3)$ from a first order Taylor expansion of $g(\cdot)$ around $g(l)$, and similarly, $\int_{u-2b}^u h_b(r)dG(r)=O(b^3)$. Thus

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_l^u h_b(r)L(r)dG(r)\right)^2\right] &= \int_l^u\int_l^u h_b(r)h_b(s)\min(r,s)dG(r)dG(s) \\
&= \int_l^{l+2b}\int_l^{l+2b}h_b(r)h_b(s)(\min(r,s)-l)dG(r)dG(s) \\
&\quad+\int_{u-2b}^u\int_{u-2b}^u h_b(r)h_b(s)(\min(r,s)-u)dG(r)dG(s)+O(b^6)
\end{aligned}
$$

$$= O(b^5)$$

so the desired result follows.

For $d > 1$,

$$\begin{aligned} D_b^2 &= \mathbb{E}\left[\left(b^{-d-1}\int_{\mathcal{I}_b}\int \kappa_b(|s-r|)(L(r)-L(s))dG(r)dG(s)\right)^2\right] \\ &= \mathbb{E}\left[\left(b^{-1}\int_{\mathcal{I}_b}\int \kappa_0(|r|)(L(s+br)-L(s))g(s+br)drdG(s)\right)^2\right] \\ &= \int_{\mathcal{I}_b}\int_{\mathcal{I}_b}\int\int b^{-2}\kappa_0(|r|)\kappa_0(|u|)\zeta_b(s,r,t,u)g(s+br)g(t+bu)dr\cdot du\cdot dG(s)dG(t) \end{aligned}$$

with

$$\begin{aligned} 2\zeta_b(s,r,t,u) &= 2\mathbb{E}[(L(s+br)-L(s))(L(t+bu)-L(t))] \\ &= |br+s-t|+|bu+s-t|-|br+bu+s-t|-|s-t|. \end{aligned}$$

Now split the integral over $dG(s)$ and $dG(t)$ into a piece $\mathcal{R}_b^0 = \{s,t:\ s,t\in\mathcal{I}_b,\ |s-t|<2b\}\subset \mathcal{I}_b\times\mathcal{I}_b$ and $\mathcal{R}_b^1 = (\mathcal{I}_b\times\mathcal{I}_b)\backslash\mathcal{R}_b^0$. For the integral over $\mathcal{R}_b^0$, note that for $|s-t|<2b$, $|\zeta_b(s,r,t,u)|<Cb$. At the same time, the area of integration for $\mathcal{R}_b^0$ is of order b^d. So with g and κ_0 bounded, the integral over $\mathcal{R}_b^0$ is of order $b^{d-1}\to 0$, and makes a vanishing contribution to D_b^2.

For any $\omega, v\in\mathbb{R}^d$ and $x\in\mathbb{R}$ such that $\omega+xv\neq 0$, we have

$$\begin{aligned} \frac{\partial}{\partial x}|\omega+xv| &= \frac{(\omega+xv)'v}{|\omega+xv|} \\ \frac{\partial^2}{\partial x^2}|\omega+xv| &= -\frac{((\omega+xv)'v)^2}{|\omega+xv|^3}+\frac{v'v}{|\omega+xv|} \\ \frac{\partial^3}{\partial x^3}|\omega+xv| &= 3\frac{((\omega+xv)'v)^3}{|\omega+xv|^5}-3\frac{((\omega+xv)'v)v'v}{|\omega+xv|^3}. \end{aligned}$$

For the integral over $\mathcal{R}_b^1$ where $|s-t|\geq 2b$, apply a second order Taylor expansion to $\zeta_b(s,r,t,u)g(s+br)g(t+bu)$ around $b=0$. Since $\zeta_0(s,r,t,u)=\partial\zeta_b(s,r,t,u)/\partial b|_{b=0}=0$, we find

$$\zeta_b(s,r,t,u)g(s+br)g(t+bu) = \tfrac{1}{2}b^2 g(s)g(t)\left(\frac{(s-t)'r(s-t)'u}{|s-t|^3}-\frac{r'u}{|s-t|}\right)+\frac{b^3}{|s-t|^2}\Psi_b(s,r,t,u)$$

where here and below Ψ_b denote uniformly bounded functions, that is,

$\sup_{b>0,s,t\in\mathcal{I}_b,|u|\le 1,|r|\le 1}|\Psi_b(s,r,t,u)|<\infty$. By symmetry, for all $|s-t|>2b$

$$\int\int \kappa_0(|r|)\kappa_0(|u|)\left(\frac{(s-t)'r(s-t)'u}{|s-t|^3}-\frac{r'u}{|s-t|}\right)dudr=0.$$

Furthermore,

$$\begin{aligned}\int_{\mathcal{I}_b}\int_{\mathcal{I}_b}\min\left(\frac{b^3}{|s-t|^2},\tfrac{1}{2}b\right)dG(s)dG(t) &\le C\int_{|s|<C}\min\left(\frac{b^3}{|s|^2},b\right)ds\\ &= C\int_0^C x^{d-1}\min\left(\frac{b^3}{x^2},b\right)dx=O(b^3\log(b)) \qquad \text{(S.2)}\end{aligned}$$

so $D_b^2\to 0$.

Given this first result, it is without loss of generality to assume that $\mathcal{S}^0$ does not contain the origin. Let $Q_b = b^{-1}\int_{\mathcal{I}_b}\int\kappa_0(|r|)(L(s+br)-L(s))g(s+br)drdG(s)$. We will show that Q_b converges in mean square. We have

$$\mathbb{E}\left[Q_b\right]=\tfrac{1}{2}b^{-1}\int_{\mathcal{I}_b}\int\kappa_0(|r|)(|s+br|-|s|-b|r|)g(s+br)drdG(s).$$

By a fist order Taylor expansion, for $|s|\ge 2b$,

$$(|s+br|-|s|-b|r|)g(s+br)=bg(s)\left(\frac{s'r}{|s|}-|r|\right)+b^2\Psi_b(s,r)$$

and $\mathbb{E}\left[Q_b\right]\to-\frac{1}{2}\int|r|\kappa_0(|r|)dr\cdot\int g(s)^2ds$ follows from $\int(s'r)\kappa_0(|r|)dr=0$.

Note that for (X_1,X_2,X_3,X_4) mean-zero multivariate normal with covariances $\sigma_{ij}=\mathbb{E}[X_iX_j]$, $\mathbb{E}[(X_1X_2-\sigma_{12})(X_3X_4-\sigma_{34})]=\sigma_{14}\sigma_{23}+\sigma_{13}\sigma_{24}$. We have

$$\begin{aligned}\zeta_b^0(s,t) &= 2\mathbb{E}[L(s)L(t)]=|s|+|t|-|s+t|\\ \zeta_b^1(s,r,t) &= 2\mathbb{E}[(L(s+br)-L(s))L(t)]=|br+s|-|s|+|s-t|-|br+s-t|\\ \zeta_b^1(t,u,s) &= 2\mathbb{E}[(L(t+bu)-L(t))L(s)].\end{aligned}$$

Thus,

$$\begin{aligned}4\,\mathrm{Var}[Q_b] &= 4\mathbb{E}\left[(Q_b-\mathbb{E}\left[Q_b\right])^2\right]\\ &= \int_{\mathcal{I}_b}\int_{\mathcal{I}_b}\int\int b^{-2}\kappa_0(|r|)\kappa_0(|u|)[\zeta_b^0(s,t)\zeta_b(s,r,t,u)g(s+br)g(t+bu)\\ &\qquad +\zeta_b^1(s,r,t)\zeta_b^1(t,u,s)g(s+br)g(t+bu)]dr\cdot du\cdot dG(s)dG(t)\end{aligned}$$

Split the integral again into integrals over $\mathcal{R}_b^0$ and $\mathcal{R}_b^1$. For the integral over $\mathcal{R}_b^0$, note that for $|s-t|<2b$, $|\zeta_b^0(s,t)\zeta_b(s,r,t,u)|<Cb^2$ and $|\zeta_b^1(s,r,t)\zeta_b^1(t,u,s)|<Cb^2$ uniformly. At the same time, the area of integration for $\mathcal{R}_b^0$ is of order b^d, so the integral over $\mathcal{R}_b^0$ is of order $b^d \to 0$, and makes a vanishing contribution to $\text{Var}[Q_b]$.

For the integral over $\mathcal{R}_b^1$, the term involving $\zeta_b^0(s,t)\zeta_b(s,r,t,u)$ is negligible as shown above, since $\sup_{s,t\in\mathcal{I}_b}\zeta_b^0(s,t)<\infty$. For the remaining term, apply a second order Taylor expansion to $\zeta_b^1(s,r,t)\zeta_b^1(t,u,s)g(s+br)g(t+bu)$

$$\begin{aligned}\zeta_b^1(s,r,t)\zeta_b^1(t,u,s)g(s+br)g(t+bu)\\ = \tfrac{1}{2}b^2g(s)g(t)\left(\frac{s'r}{|s|}-\frac{(s-t)'r}{|s-t|}\right)\left(\frac{t'u}{|t|}-\frac{(t-s)'u}{|s-t|}\right)+\frac{b^3}{|s-t|^2}\Psi_b^1(s,r,t,u)\end{aligned}$$

since $\zeta_0^1(s,r,t)=\zeta_0^1(t,u,s)=0$. By symmetry, for all $|s-t|>2b$,

$$\int \kappa_0(|r|)\left(\frac{s'r}{|s|}-\frac{(s-t)'r}{|s-t|}\right)dr=0$$

so using (S.2) we conclude $\text{Var}[Q_b]\to 0$.

Finally, the result for J_c follows, since the measure of $(J_c-J_c(0))$ is absolutely continuous with respect to the measure of L, and $J_c(0)$ has finite second moment. □

Lemma 7 is a special case of the following more general result applied with $p=1$ and $\psi(s)=1$. We will use the following notation: let $k:\mathcal{S}^0\times\mathcal{S}^0\mapsto\mathbb{R}$ be a continuous positive definite kernel (not necessarily equal to the covariance kernel of Lévy-Brownian Motion), and let $\mathbf{\Sigma}_n$ be the $n\times n$ matrix with l,ℓth element equal to $k(s_l^0,s_\ell^0)$. Let $\mathcal{L}_G^2$ be the Hilbert space of function $\mathcal{S}^0\mapsto\mathbb{R}$ with inner product $\langle f_1,f_2\rangle=\int f_1(s)f_2(s)dG(s)$. Define $L_k:\mathcal{L}_G^2\mapsto\mathcal{L}_G^2$ as the linear operator $L_k(f)(s)=\int f(r)k(r,s)dG(r)$, and $L_{k,n}=\int f(r)k(r,s)dG_n(r)$.

Lemma S.1. *Suppose the $p\times 1$ vector x_l is such that $x_l=\psi(s_l^0)$ for $l=1,\ldots,n$ for some continuous function $\psi:\mathcal{S}^0\mapsto\mathbb{R}^p$, and $\int\psi(s)\psi(s)'dG_n(s)=H_n\to H$ for some positive definite matrix H. Let M and M_n be the projection operators $M_n(f)(s)=f(s)-\int\psi(r)'f(r)dG_n(r)H_n^{-1}\psi(s)$ and $M(f)(s)=f(s)-\int\psi(r)'f(r)dG(r)H^{-1}\psi(s)$. Let $\hat{k}_n$, and $\bar{k}$ be the kernels corresponding to the linear operators $M_nL_{k,n}M_n$ and ML_kM, respectively, so that the (l,ℓ) element of $\mathbf{M}_X\mathbf{\Sigma}_{n,L}\mathbf{M}_X$ is given by $\hat{k}_n(s_l^0,s_\ell^0)$. Let $\bar{k}(s,r)=\sum_{i=1}^\infty\bar{\nu}_i\bar{\varphi}_i(s)\bar{\varphi}_i(r)$ with $\int\bar{\varphi}_i(s)\bar{\varphi}_j(s)dG(s)=\mathbf{1}[i=j]$, $\bar{\nu}_i\geq\bar{\nu}_{i+1}\geq 0$ be the spectral decomposition of $\bar{k}$. Define $\hat{\varphi}_i(\cdot)=n^{-1}\hat{\nu}_i^{-1}\sum_{l=1}^n r_{i,l}\hat{k}_n(\cdot,s_l^0)$, where $(\hat{\nu}_i,(r_{i,1},\ldots,r_{i,n})')$ is the ith eigenvalue/eigenvector pair of $\mathbf{M}_X\mathbf{\Sigma}_n\mathbf{M}_X$. If $\bar{\nu}_1>\bar{\nu}_2>\ldots>\bar{\nu}_q>\bar{\nu}_{q+1}$ and Condition 1 holds, then for any $q\geq 1$, $\sup_{s\in\mathcal{S}^0,1\leq i\leq q}|\hat{\varphi}_i(s)-\bar{\varphi}_i(s)|\to 0$ and $\max_{1\leq i\leq q}|\hat{\nu}_i-\bar{\nu}_i|\to 0$.*

Proof. The proof follows from the same arguments as the proof of Lemma 6 in Müller and Watson (2022a). The two differences are (i) the generalization of the demeaning by the more general projection of ψ; and (ii) the replacement of the i.i.d. assumption for s_l^0 by $G_n \Rightarrow G$.

Set $k_0(s,r) = \bar{k}(s,r) + \psi(s)'H^{-1}\psi(r)$ and define the associated operators $L(f)(s) = \int f(r)k_0(r,s)dG(r)$, $L_n(f)(s) = \int f(r)k_0(r,s)dG_n(r)$, $\bar{L} = MLM$, $\bar{L}_n = ML_nM$ and $\hat{L}_n = M_nL_nM_n$. Note that $\bar{L} = ML_kM$ and $\hat{L}_n = M_nL_{k,n}M_n$. Let $\mathcal{H} \subset \mathcal{L}_G^2$ be the Reproducing Kernel Hilbert Space of functions $f : \mathcal{S}^0 \mapsto \mathbb{R}$ with kernel k_0 and inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ satisfying $\langle f, k_0(\cdot,r)\rangle_{\mathcal{H}} = f(r)$ and associated norm $||f||_{\mathcal{H}}$. By Theorem 2.16 in Saitoh and Sawano (2016), $\mathcal{H}$ contains all functions of the form $a'\psi$ for $a \in \mathbb{R}^p$, so $\sup_{|a|=1} ||a'\psi||_{\mathcal{H}} < \infty$. Now proceed as in the proof of Lemma 6 of Müller and Watson (2022a) to argue that $\sup_{r\in\mathcal{S}^0} |f(r)| \leq \sqrt{\sup_{s\in\mathcal{S}^0} k_0(s,s)} \cdot ||f||_{\mathcal{H}}$, and

$$||Mf||_{\mathcal{H}} = ||f - \int \psi(r)'f(r)dG(r)H^{-1}\psi||_{\mathcal{H}} \leq ||f||_{\mathcal{H}} + \sup_{r\in\mathcal{S}^0} |f(r)| \cdot \sup_{r\in\mathcal{S}^0} |H^{-1}\psi(r)| \cdot \sup_{|a|=1} ||a'\psi||_{\mathcal{H}}$$

so $M : \mathcal{H} \mapsto \mathcal{H}$ is a bounded operator. By the same argument, so is M_n.

From $\langle f, k_0(\cdot,r)\rangle_{\mathcal{H}} = f(r)$, we further obtain

$$\int \psi(r)f(r)(dG_n(r) - dG(r)) = \left\langle f, \int \psi(r)k_0(\cdot,r)(dG_n(r) - dG(r))\right\rangle_{\mathcal{H}} \tag{S.3}$$

and for each component ψ_i of ψ, $i = 1,\ldots,p$,

$$\begin{aligned} & \left\|\int \psi_i(r)k_0(\cdot,r)(dG_n(r) - dG(r))\right\|_{\mathcal{H}}^2 \qquad \text{(S.4)} \\ = \; & \int\int \psi_i(s)k_0(s,r)\psi_i(r)(dG_n(s) - dG(s))(dG_n(r) - dG(r)) \\ = \; & \mathbb{E}[\psi_i(S_n)k_0(S_n,R_n)\psi_i(R_n) - \psi_i(S_n)k_0(S,R_n)\psi_i(R) \\ & \qquad -\psi_i(S)k_0(S_n,R)\psi_i(R_n) + \psi_i(S)k_0(S,R)\psi_i(R)] \\ \to \; & 0 \end{aligned}$$

where (S_n, R_n) is a sequence of $\mathbb{R}^{2d}$ random variables with distribution $G_n \times G_n$ converging to (S,R) with distribution $G \times G$. The convergence then follows since the $\mathbb{R}^{2d} \mapsto \mathbb{R}$ function $(s,r) \mapsto \psi_i(s)k_0(s,r)\psi_i(r)$ is continuous and bounded. Thus, by (S.3), (S.4) and Cauchy-Schwarz,

$$\sup_{||f||_{\mathcal{H}}\leq 1} \left|\int \psi(r)f(r)(dG_n(r) - dG(r))\right| \to 0.$$

From $H_n^{-1} \to H^{-1}$ and $|\int \psi(r)f(r)dG_n(r)| \leq \sup_{r\in\mathcal{S}^0} |f(r)| \cdot \sup_{r\in\mathcal{S}^0} |\psi(r)| \leq \sup_{r\in\mathcal{S}^0} |\psi(r)|\sqrt{\sup_{s\in S} k_0(s,s)} \cdot ||f||_{\mathcal{H}}$, we conclude that with $\Delta_n(f) = H_n^{-1}\int \psi(r)f(r)dG_n(r) -$

$H^{-1}\int\psi(r)f(r)dG(r)$, $\sup_{||f||_{\mathcal{H}}\leq 1}|\Delta_n(f)|\to 0$. Thus

$$\sup_{||f||_{\mathcal{H}}\leq 1}||(M_n-M)f||_{\mathcal{H}}=\left\|\Delta_n(f)'\psi\right\|_{\mathcal{H}}\leq\sup_{||f||_{\mathcal{H}}\leq 1}|\Delta_n(f)|\cdot\sup_{|a|=1}||a'\psi||_{\mathcal{H}}\to 0.$$

The only remaining piece of the proof is to show that $||L_n-L||_{HS}^2\to 0$ under the assumption of $G_n\Rightarrow G$, where for any Hilbert-Schmidt operator $A:\mathcal{H}\mapsto\mathcal{H}$, $||A||_{HS}^2=\sum_{j\geq 1}\langle Ae_j,Ae_j\rangle_{\mathcal{H}}$ for an orthonormal base e_j. One choice for e_j are the eigenfunctions scaled by the square root of the eigenvalues of the spectral decomposition of k_0, so that $k_0(r,s)=\sum_{j=1}^{\infty}e_j(r)e_j(s)$; see the discussion in the proof of Lemma 6 in Müller and Watson (2022a). We find

$$\begin{aligned}||L_n-L||_{HS}^2 &= \sum_{j\geq 1}\left\langle\int e_j(s)k_0(s,\cdot)(dG_n(s)-dG(s)),\int e_j(s)k_0(s,\cdot)(dG_n(s)-dG(s))\right\rangle_{\mathcal{H}}\\ &= \int\int\left(\sum_{j\geq 1}e_j(s)e_j(r)\right)k_0(s,r)(dG_n(s)-dG(s))(dG_n(r)-dG(r))\\ &= \int\int k_0(s,r)^2(dG_n(r)-dG(r))(dG_n(r)-dG(r))\\ &= \mathbb{E}[k_0(S_n,R_n)^2-k_0(S,R_n)^2-k_0(S_n,R)^2+k_0(S,R)^2]\to 0\end{aligned}$$

where the change of the order of integration and summation is justified by Fubini's Theorem, and the convergence follows, since the $\mathbb{R}^{2d}\mapsto\mathbb{R}$ function $(s,r)\mapsto k_0(s,r)^2$ is bounded and continuous. □

Lemma S.2. *Assume the conditions of Lemma S.1 hold. Suppose $\tilde{x}_l=\psi_n(s_l^0)$, where the continuous functions $\psi_n:\mathcal{S}^0\mapsto\mathbb{R}^p$ are such that $\sup_{s\in\mathcal{S}^0}|\psi_n(s)-\psi(s)|\to 0$, for some continuous function ψ. Define the the projection operator $\tilde{M}_n:\mathcal{L}_G^2\mapsto\mathcal{L}_G^2$ as $\tilde{M}_n(f)(s)=f(s)-\int\psi_n(r)'f(r)dG_n(r)H_n^{-1}\psi_n(s)$, and let $\tilde{k}_n$ be the kernel corresponding to the linear operator $\tilde{M}_nL_{k,n}\tilde{M}_n$, so that the (l,ℓ) element of $\mathbf{M}_{\tilde{X}}\mathbf{\Sigma}_n\mathbf{M}_{\tilde{X}}$ is given by $\tilde{k}_n(s_l^0,s_\ell^0)$. Let $(\tilde{\nu}_i,(\tilde{r}_{i,1},\ldots,\tilde{r}_{i,n})')$ be the ith eigenvalue/eigenvector pair of $\mathbf{M}_{\tilde{X}}\mathbf{\Sigma}_n\mathbf{M}_{\tilde{X}}$, and define $\tilde{\varphi}_i(\cdot)=n^{-1}\tilde{\nu}_i^{-1}\sum_{l=1}^n\tilde{r}_{i,l}\tilde{k}_n(\cdot,s_l^0)$. Then $\sup_{s\in\mathcal{S}^0,1\leq i\leq q}|\tilde{\varphi}_i(s)-\bar{\varphi}_i(s)|\to 0$ and $\max_{1\leq i\leq q}|\tilde{\nu}_i-\bar{\nu}_i|\to 0$.*

Proof. From standard arguments, we obtain $\int\psi_n(s)\psi_n(s)'dG_n(s)\to H$ and $\int\psi(s)\psi_n(s)'dG_n(s)\to H$. Thus, $||\mathbf{M}_{\tilde{X}}-\mathbf{M}_X||\to 0$, and by a direct calculation, $\sup_{s,r\in\mathcal{S}^0}|\tilde{k}_n(r,s)-\hat{k}_n(r,s)|\to 0$, and $\sup_{s,r\in\mathcal{S}^0}|\hat{k}_n(r,s)-\bar{k}(r,s)|\to 0$ and thus $\sup_{s,r\in\mathcal{S}^0}|\tilde{k}_n(r,s)-\bar{k}(r,s)|\to 0$. Furthermore, proceeding as in the proof of Lemma S.1 shows that $||\mathbf{\Sigma}_n||$ converges to $\bar{\nu}_1$, the largest eigenvalue of the integral operator with kernel $\bar{k}$, so $||\mathbf{\Sigma}_n||=O(1)$. Thus also $||\mathbf{M}_{\tilde{X}}\mathbf{\Sigma}_n\mathbf{M}_{\tilde{X}}-\mathbf{M}_X\mathbf{\Sigma}_n\mathbf{M}_X||\to 0$, and from Weyl's inequality, $\max_{1\leq i\leq q}|\tilde{\nu}_i-\hat{\nu}_i|\to 0$. Since also $\max_{1\leq i\leq q}|\hat{\nu}_i-\bar{\nu}_i|\to 0$ from Lemma S.1, we can conclude that

$$\sup_{s\in\mathcal{S}^0}|(\tilde{\nu}_i^{-1}-\hat{\nu}_i^{-1})n^{-1}\sum_{l=1}^n r_{i,l}\hat{k}_n(s,s_l^0)|\leq|\tilde{\nu}_i^{-1}-\hat{\nu}_i^{-1}|\cdot\sup_{s\in\mathcal{S}^0}|\hat{\varphi}_i(s)|\cdot\sup_{s,r\in\mathcal{S}^0}|\hat{k}_n(r,s)|\to 0$$

where the inequality uses $r_{i,l} = \hat{\varphi}_i(s_l^0)$, and the convergence follows from the above results and $\sup_{s\in\mathcal{S}^0} |\hat{\varphi}_i(s)| \to \sup_{s\in\mathcal{S}^0} |\varphi_i(s)| < \infty$ from Lemma S.1. Also,

$$\sup_{s\in\mathcal{S}^0} |n^{-1} \sum_{l=1}^n r_{i,l}(\tilde{k}_n(s,s_l^0) - \hat{k}_n(s,s_l^0))| \leq \sup_{s\in\mathcal{S}^0} |\hat{\varphi}_i(s)| \cdot \sup_{r,s\in\mathcal{S}^0} |\tilde{k}_n(r,s) - \hat{k}(r,s)| \to 0.$$

Finally, since $\max_{1\leq i\leq q} |\tilde{\nu}_i - \bar{\nu}_i| \to 0$ and $\bar{\nu}_1 > \bar{\nu}_2 > \ldots > \bar{\nu}_q > \bar{\nu}_{q+1}$, we can apply Corollary 1 of Yu, Wang and Samworth (2015) and conclude that $n^{-1}\sum_{l=1}^n (\tilde{r}_{i,l} - r_{i,l})^2 \to 0$ for $i = 1, \ldots, q$. Applying Cauchy-Schwarz then yields

$$\sup_{s\in\mathcal{S}^0} |n^{-1} \sum_{l=1}^n (\tilde{r}_{i,l} - r_{i,l})\tilde{k}_n(s,s_l^0)|^2 \leq n^{-1} \sum_{l=1}^n (\tilde{r}_{i,l} - r_{i,l})^2 \cdot \sup_{s\in\mathcal{S}^0} n^{-1} \sum_{l=1}^n \tilde{k}_n(s,s_l^0)^2 \to 0$$

where the convergence follows from $n^{-1}\sum_{l=1}^n \tilde{k}_n(s,s_l^0)^2 \leq 2\sup_{r,s\in\mathcal{S}^0} |\bar{k}(r,s)|^2 + 2\sup_{s,r\in\mathcal{S}^0} |\tilde{k}_n(r,s) - \bar{k}(r,s)|^2 = O(1)$. □

Theorem S.3. *Suppose $y_l = x_l'\beta + u_l$, $(x_l', u_l) = \lambda_n^{1/2}(X_n^0(s_l^0)', U_n^0(s_l^0)) \in \mathbb{R}^p \times \mathbb{R}$ with $(X_n^0(\cdot), U_n^0(\cdot))$ satisfying (27), but X^0 is not necessarily independent of U^0. Let $\mathbf{R}_n^X$ be the $n \times p$ matrix of q eigenvectors of $\mathbf{M}_X\boldsymbol{\Sigma}_L\mathbf{M}_X$ corresponding to the largest eigenvalues. Suppose for almost every realization of X^0, the largest $q+1$ eigenvalues of the kernel $k_{X^0} : \mathcal{S}^0 \times \mathcal{S}^0 \mapsto \mathbb{R}$ corresponding to the linear operator $M_{X^0}L_kM_{X^0}$ with $M_{X^0}(f)(s) = f(s) - X^0(s)\left(\int X^0(r)X^0(r)'dG(r)\right)^{-1}\int X^0(r)'f(r)dG(r)$ are distinct. If also Condition 1 holds, then*

$$\lambda_n^{-1/2}\mathbf{R}_n^{X\prime}\mathbf{Y}_n \Rightarrow \omega \int \boldsymbol{\varphi}_{X^0}(s)U^0(s)dG(s) \tag{S.5}$$

where $\boldsymbol{\varphi}_{X^0}(\cdot)$ are the q eigenfunctions of k_{X^0} corresponding to the largest eigenvalues.

Furthermore, let $\tilde{U}_n^0$ be independent of (X_n^0, U_n^0), and suppose $\tilde{U}_n^0$ satisfies $\tilde{U}_n^0(\cdot) \Rightarrow \tilde{U}^0(\cdot)$ with $\tilde{U}^0 \sim U^0$. Let $\mathrm{cv}_n(X_n^0)$ be the $1-\alpha$ quantile of the conditional distribution of $\phi(\mathbf{R}_n^{X\prime}\tilde{\mathbf{U}}_n)$ given $\mathbf{R}_n^X$ for some continuous function $\phi : \mathbb{R}^q \mapsto \mathbb{R}$ satisfying $\phi(ax) = \phi(x)$ for all $a \neq 0$ and $x \in \mathbb{R}^q$. Suppose that (i) X^0 is independent of U^0, (ii) for almost all realizations of X^0 the conditional distribution of $\phi(\int \boldsymbol{\varphi}_{X^0}(s)U^0(s)dG(s))$ is continuous. Then $\mathbb{P}(\phi(\mathbf{R}_n^{X\prime}\mathbf{Y}_n) > \mathrm{cv}_n(X_n^0)) \to \alpha$.

Proof. We will show that $(\phi(\mathbf{R}_n^{X\prime}\mathbf{Y}_n), \mathrm{cv}_n(X_n^0)) \Rightarrow (\phi(\int \boldsymbol{\varphi}_X(s)U^0(s)dG(s)), q_{1-\alpha}^{\phi}(X^0))$ with $q_{1-\alpha}^{\phi}(X^0)$ the $1-\alpha$ quantile of $\phi(\int \boldsymbol{\varphi}_X(s)U^0(s)dG(s))$ conditional on X^0. The result then follows from the CMT applied to $\mathbf{1}[\phi(\mathbf{R}_n^{X\prime}\mathbf{Y}_n) > \mathrm{cv}_n(X_n^0)]$, and taking expectations.

Apply the almost sure representation theorem to argue that there exists a probability space $(\Omega_0, \mathfrak{F}_0, P_0)$ and associated random processes X^*, U^* and X_n^*, U_n^*, $n \geq 1$ such that $(X_n^*, U_n^*) \sim (X_n^0, U_n^0)$, $(X^*, U^*) \sim (X^0, U^0)$ and $\sup_{s\in\mathcal{S}^0} |X_n^*(s) - X^*(s)| \stackrel{a.s.}{\to} 0$, $\sup_{s\in\mathcal{S}^0} |U_n^*(s) - U^*(s)| \stackrel{a.s.}{\to} 0$. Using the same arguments as in the proof of Theorem 3, and a realization by realization application of Lemma S.2, then yields

$$\lambda_n^{-1/2}\mathbf{R}_n^{X^*\prime}\mathbf{Y}_n^* \stackrel{a.s.}{\to} \omega\int \boldsymbol{\varphi}_{X^*}(s)U^*(s)dG(s) \sim \omega\int \boldsymbol{\varphi}_{X^*}(s)U^0(s)dG(s) \tag{S.6}$$

where $(\mathbf{R}_n^{X^*}, \mathbf{Y}_n^*)$ are defined analogously to $(\mathbf{R}_n^X, \mathbf{Y}_n)$ on $(\Omega_0, \mathfrak{F}_0, P_0)$, and $(\mathbf{R}_n^{X^*}, \mathbf{Y}_n^*) \sim (\mathbf{R}_n^X, \mathbf{Y}_n)$ by construction, so (S.5) holds.

The further result now follows if we can show that also $\text{cv}_n(X_n^*) \stackrel{a.s.}{\to} q_{1-\alpha}^{\phi}(X^*)$, since almost sure convergence implies convergence in distribution. To that end, note there exists a separate probability space $(\Omega_1, \mathfrak{F}_1, P_1)$ with associated sequences of random process $\tilde{U}^*$ and $\tilde{U}_n^*$ and such that $\tilde{U}_n^* \sim \tilde{U}_n^0$, $\tilde{U}^* \sim \tilde{U}^0 \sim U^0$ and $\sup_{s\in S^0} |\tilde{U}_n^*(s) - \tilde{U}^*(s)| \stackrel{a.s.}{\to} 0$. Form the product space $(\Omega_0 \times \Omega_1, \mathfrak{F}_0 \otimes \mathfrak{F}_1, P_0 \times P_1)$, so that on this new space, $(X^*, \{X_n^*\}_{n=1}^\infty)$ is independent of $(\tilde{U}^*, \{\tilde{U}_n^*\}_{n=1}^\infty)$ by construction. Use the same arguments as for (S.6) obtain that for P_0-almost all $\omega_0 \in \Omega_0$ and P_1-almost all $\omega_1 \in \Omega_1$, in obvious notation,

$$\lambda_n^{-1/2}\mathbf{R}_n^{X^*\prime}\tilde{\mathbf{U}}_n^* \to \int \boldsymbol{\varphi}_{X^*}(s)\tilde{U}^*(s)dG(s)$$

jointly with (S.6). But almost sure convergence implies convergence in distribution, and $\tilde{U}^* \sim U^0$, so for P_0-almost all $\omega_0 \in \Omega_0$, the distribution of $\lambda_n^{-1/2}\mathbf{R}_n^{X^*\prime}\tilde{\mathbf{U}}_n^*$ induced by P_1 converges to the conditional distribution of $\int \boldsymbol{\varphi}_{X^*}(s)U^0(s)dG(s)$ given X^*. Since ϕ is continuous and the conditional distribution is assumed continuous, this implies that for all such ω_0, $\text{cv}_n(X_n^0) \stackrel{a.s.}{\to} q_{1-\alpha}^{\phi}(X^*)$. Thus $(\phi(\mathbf{R}_n^{X\prime}\mathbf{Y}_n), \text{cv}_n(X_n^0)) \sim (\phi(\mathbf{R}_n^{X^*\prime}\mathbf{Y}_n^*), \text{cv}_n(X_n^*)) \stackrel{a.s.}{\to} (\phi(\int \boldsymbol{\varphi}_{X^*}(s)U^*(s)dG(s)), q_{1-\alpha}^{\phi}(X^*)) \sim (\phi(\int \boldsymbol{\varphi}_{X^0}(s)U^0(s)dG(s)), q_{1-\alpha}^{\phi}(X^0))$, and the result follows, because almost sure convergence implies convergence in distribution. □

In applications, the theorem justifies use of a critical value for the test statistic $\phi(\mathbf{R}_n^{X\prime}\mathbf{Y}_n)$ that is equal to the $1-\alpha$ quantile of $\phi(\mathbf{R}_n^{X\prime}\tilde{\mathbf{U}}_n)$ conditional on $\mathbf{R}_n^X$, for some (pseudo-) random variable draws of $\tilde{u}_l = \tilde{\mathbf{U}}_n(s_l^0)$ that induce the same limiting process as the actual regression errors u_l. Since ϕ is assumed scale invariant, the scaling of $\tilde{u}_l$ is immaterial in this construction.

Proof of Theorem 8:

By Lemmas 3 and 12 in Müller and Watson (2022a), we have

$$\lambda_n^{d/2}n^{-1}\mathbf{Z}_n \Rightarrow \mathcal{N}\left(\mathbf{0}, a\sigma_B(0)\int \bar{\boldsymbol{\varphi}}(s)\bar{\boldsymbol{\varphi}}(s)'dG(s) + \omega^2\int \bar{\boldsymbol{\varphi}}(s)\bar{\boldsymbol{\varphi}}(s)'g(s)dG(s)\right) \tag{S.7}$$

where $\bar{\boldsymbol{\varphi}} = (\bar{\varphi}_1, \ldots, \bar{\varphi}_q)$, $\omega^2 = \int_{\mathbb{R}^d} \sigma_B(s)ds$ and g is the density of the distribution G. Since the LFST_n statistic is scale invariant, its limiting distribution under (S.7) only depends on the properties of B through the ratio $\chi = a\sigma_B(0)/\omega^2 \in [0, \infty)$. We need to show that $\liminf_{n\to\infty} \text{cv}_n^{\text{LFST}}$ is at least as large as the $1-\alpha$ quantile, say $\text{cv}_\chi^{\text{LFST}}$, of the (continuous) asymptotic distribution of LFST_n for this value of χ.

Note that for $B = J_c$, $\sigma_B(0)/\omega^2 = K_d c^{1+d}$ for some $K_d > 0$. For $a > 0$, let c_* be such $K_d c_*^{1+d} = \chi/a$, and let $c_* = 1$ otherwise. For all n sufficiently large so that $\lambda_n c_* \geq c_{0.03}$, $\mathrm{cv}_n^{\mathrm{LFST}}$ is such that the LFST_n test controls size under $B = J_{c^*}$. But since $B = J_{c_*}$ satisfies the assumptions of Lahiri (2003), this model induces the same limit (S.7), so its $1 - a$ quantile converges to $\mathrm{cv}_\chi^{\mathrm{LFST}}$, and the result follows. $\square$

S.3 Detailed Monte Carlo Results

The following tables summarize the distributions of the null rejection probability and average length of confidence intervals for each method and DGP across the 96 spatial designs described in Section 6.

Entries show the median across spatial locations and the values in parentheses are 5^{th} and 95^{th} percentiles.

Method: OLS (C-SCPC)

Null Rejection Probability: k = 1

DGP	
Levy-BM	0.227 (0.202,0.267)
I(1) c=0.01	0.243 (0.217,0.276)
I(1) c=0.03	0.271 (0.243,0.312)
I(1) Matern	0.249 (0.227,0.284)
J c=0.03	0.035 (0.032,0.040)
J c = 0.50	0.145 (0.131,0.168)
Br. Sheet	0.254 (0.218,0.302)

Null Rejection Probability: k = 5

DGP	
Levy-BM	0.196 (0.183,0.213)
I(1) c=0.01	0.198 (0.185,0.211)
I(1) c=0.03	0.225 (0.210,0.243)
I(1) Matern	0.202 (0.189,0.218)
J c=0.03	0.038 (0.033,0.042)
J c = 0.50	0.145 (0.132,0.156)
Br. Sheet	0.233 (0.205,0.259)

Average Length: k = 1

DGP	
Levy-BM	1.133 (1.071,1.204)
I(1) c=0.01	1.338 (1.262,1.437)
I(1) c=0.03	1.419 (1.346,1.495)
I(1) Matern	1.385 (1.325,1.453)
J c=0.03	0.497 (0.488,0.507)
J c = 0.50	1.030 (0.995,1.095)
Br. Sheet	1.071 (1.003,1.146)

Average Length: k = 5

DGP	
Levy-BM	0.854 (0.828,0.884)
I(1) c=0.01	1.101 (1.060,1.141)
I(1) c=0.03	1.181 (1.130,1.244)
I(1) Matern	1.168 (1.101,1.219)
J c=0.03	0.484 (0.478,0.489)
J c = 0.50	0.833 (0.807,0.869)
Br. Sheet	0.801 (0.750,0.858)

Method: Isotropic difference (C-SCPC)

Null Rejection Probability: k = 1

DGP	b =0.030	b =0.060	b =0.090	b =0.120	b =0.150
Levy-BM	0.020 (0.016,0.024)	0.022 (0.017,0.027)	0.028 (0.023,0.035)	0.034 (0.027,0.044)	0.040 (0.033,0.055)
I(1) c=0.01	0.056 (0.046,0.065)	0.045 (0.041,0.051)	0.045 (0.039,0.056)	0.049 (0.042,0.065)	0.056 (0.045,0.074)
I(1) c=0.03	0.097 (0.080,0.112)	0.079 (0.069,0.089)	0.071 (0.062,0.083)	0.072 (0.060,0.093)	0.076 (0.063,0.105)
I(1) Matern	0.079 (0.067,0.089)	0.065 (0.057,0.073)	0.059 (0.054,0.065)	0.060 (0.053,0.073)	0.065 (0.056,0.086)
J c=0.03	0.019 (0.015,0.024)	0.021 (0.017,0.026)	0.026 (0.021,0.031)	0.029 (0.024,0.035)	0.033 (0.027,0.038)
J c = 0.50	0.020 (0.016,0.024)	0.022 (0.018,0.028)	0.027 (0.022,0.034)	0.033 (0.026,0.045)	0.038 (0.031,0.055)
Br. Sheet	0.042 (0.033,0.066)	0.067 (0.050,0.117)	0.092 (0.071,0.153)	0.109 (0.086,0.175)	0.120 (0.096,0.185)

Null Rejection Probability: k = 5

DGP	b =0.030	b =0.060	b =0.090	b =0.120	b =0.150
Levy-BM	0.023 (0.019,0.028)	0.024 (0.020,0.030)	0.029 (0.025,0.038)	0.035 (0.029,0.049)	0.042 (0.032,0.058)
I(1) c=0.01	0.059 (0.050,0.069)	0.047 (0.042,0.052)	0.045 (0.039,0.053)	0.048 (0.042,0.064)	0.053 (0.045,0.076)
I(1) c=0.03	0.096 (0.082,0.105)	0.077 (0.069,0.088)	0.068 (0.062,0.075)	0.067 (0.060,0.081)	0.071 (0.062,0.092)
I(1) Matern	0.080 (0.069,0.089)	0.064 (0.057,0.072)	0.058 (0.051,0.065)	0.058 (0.050,0.071)	0.063 (0.053,0.081)
J c=0.03	0.022 (0.017,0.025)	0.023 (0.019,0.028)	0.026 (0.022,0.032)	0.030 (0.025,0.037)	0.032 (0.028,0.040)
J c = 0.50	0.022 (0.019,0.026)	0.024 (0.019,0.028)	0.028 (0.023,0.036)	0.033 (0.028,0.045)	0.039 (0.032,0.056)
Br. Sheet	0.047 (0.037,0.079)	0.072 (0.055,0.131)	0.090 (0.072,0.162)	0.108 (0.086,0.175)	0.120 (0.097,0.182)

Average Length: k = 1

DGP	b =0.030	b =0.060	b =0.090	b =0.120	b =0.150
Levy-BM	0.465 (0.410,0.533)	0.415 (0.384,0.454)	0.428 (0.400,0.483)	0.473 (0.433,0.563)	0.531 (0.475,0.625)
I(1) c=0.01	0.705 (0.640,0.783)	0.636 (0.588,0.686)	0.644 (0.599,0.720)	0.701 (0.634,0.809)	0.762 (0.690,0.893)
I(1) c=0.03	0.824 (0.772,0.932)	0.736 (0.683,0.822)	0.729 (0.680,0.791)	0.770 (0.712,0.858)	0.838 (0.770,0.947)
I(1) Matern	0.843 (0.779,0.928)	0.746 (0.699,0.837)	0.733 (0.689,0.803)	0.764 (0.723,0.868)	0.819 (0.766,0.958)
J c=0.03	0.465 (0.405,0.517)	0.403 (0.377,0.435)	0.404 (0.374,0.436)	0.418 (0.389,0.463)	0.436 (0.405,0.483)
J c = 0.50	0.462 (0.417,0.541)	0.411 (0.382,0.441)	0.426 (0.399,0.472)	0.467 (0.431,0.552)	0.518 (0.473,0.620)
Br. Sheet	0.536 (0.478,0.595)	0.498 (0.468,0.543)	0.517 (0.486,0.569)	0.542 (0.510,0.610)	0.575 (0.543,0.661)

Average Length: k = 5

DGP	b =0.030	b =0.060	b =0.090	b =0.120	b =0.150
Levy-BM	0.449 (0.405,0.502)	0.402 (0.376,0.435)	0.425 (0.394,0.472)	0.468 (0.427,0.534)	0.514 (0.471,0.600)
I(1) c=0.01	0.661 (0.607,0.711)	0.606 (0.570,0.656)	0.633 (0.591,0.706)	0.691 (0.641,0.797)	0.756 (0.703,0.868)
I(1) c=0.03	0.779 (0.716,0.817)	0.705 (0.658,0.745)	0.717 (0.676,0.785)	0.774 (0.723,0.885)	0.844 (0.775,0.965)
I(1) Matern	0.786 (0.738,0.859)	0.721 (0.684,0.772)	0.728 (0.690,0.785)	0.777 (0.728,0.868)	0.839 (0.778,0.942)
J c=0.03	0.456 (0.408,0.506)	0.393 (0.372,0.425)	0.397 (0.374,0.429)	0.415 (0.387,0.450)	0.433 (0.403,0.471)
J c = 0.50	0.449 (0.408,0.495)	0.403 (0.377,0.430)	0.422 (0.391,0.476)	0.464 (0.430,0.542)	0.512 (0.471,0.605)
Br. Sheet	0.506 (0.464,0.562)	0.480 (0.452,0.517)	0.498 (0.464,0.535)	0.527 (0.489,0.576)	0.557 (0.519,0.624)

Method: Cluster fixed-effects (clustered standard error)

Null Rejection Probability: k = 1

DGP	m = 30	m = 60	m = 120	m = 240
Levy-BM	0.168 (0.155,0.178)	0.139 (0.130,0.148)	0.105 (0.098,0.111)	0.076 (0.072,0.082)
I(1) c=0.01	0.263 (0.239,0.277)	0.281 (0.263,0.296)	0.285 (0.261,0.310)	0.238 (0.217,0.257)
I(1) c=0.03	0.350 (0.331,0.367)	0.390 (0.363,0.412)	0.412 (0.391,0.435)	0.369 (0.336,0.391)
I(1) Matern	0.305 (0.284,0.322)	0.339 (0.318,0.360)	0.364 (0.337,0.390)	0.326 (0.296,0.347)
J c=0.03	0.092 (0.086,0.097)	0.080 (0.076,0.085)	0.070 (0.066,0.075)	0.066 (0.061,0.070)
J c = 0.50	0.140 (0.132,0.149)	0.117 (0.109,0.124)	0.093 (0.087,0.100)	0.075 (0.070,0.081)
Br. Sheet	0.282 (0.243,0.339)	0.258 (0.219,0.310)	0.213 (0.185,0.262)	0.133 (0.116,0.162)

Null Rejection Probability: k = 5

DGP	m = 30	m = 60	m = 120	m = 240
Levy-BM	0.175 (0.164,0.185)	0.142 (0.130,0.151)	0.109 (0.101,0.116)	0.083 (0.078,0.088)
I(1) c=0.01	0.271 (0.255,0.287)	0.283 (0.265,0.297)	0.284 (0.268,0.298)	0.243 (0.230,0.265)
I(1) c=0.03	0.348 (0.326,0.366)	0.378 (0.356,0.399)	0.398 (0.374,0.414)	0.356 (0.338,0.375)
I(1) Matern	0.311 (0.288,0.328)	0.338 (0.315,0.355)	0.363 (0.339,0.378)	0.327 (0.306,0.342)
J c=0.03	0.097 (0.092,0.104)	0.084 (0.079,0.090)	0.074 (0.071,0.079)	0.072 (0.068,0.076)
J c = 0.50	0.149 (0.142,0.160)	0.123 (0.115,0.133)	0.098 (0.092,0.105)	0.079 (0.074,0.084)
Br. Sheet	0.295 (0.256,0.340)	0.266 (0.231,0.312)	0.221 (0.188,0.285)	0.141 (0.124,0.178)

Average Length: k = 1

DGP	m = 30	m = 60	m = 120	m = 240
Levy-BM	0.353 (0.342,0.364)	0.307 (0.299,0.317)	0.294 (0.288,0.301)	0.355 (0.347,0.364)
I(1) c=0.01	0.474 (0.458,0.495)	0.412 (0.396,0.431)	0.382 (0.365,0.408)	0.442 (0.420,0.474)
I(1) c=0.03	0.501 (0.480,0.520)	0.424 (0.406,0.448)	0.389 (0.363,0.410)	0.441 (0.415,0.469)
I(1) Matern	0.497 (0.481,0.515)	0.430 (0.407,0.453)	0.392 (0.369,0.414)	0.450 (0.422,0.478)
J c=0.03	0.275 (0.271,0.280)	0.264 (0.260,0.269)	0.272 (0.267,0.276)	0.342 (0.337,0.348)
J c = 0.50	0.343 (0.335,0.352)	0.302 (0.295,0.312)	0.291 (0.287,0.297)	0.354 (0.347,0.361)
Br. Sheet	0.376 (0.358,0.402)	0.329 (0.312,0.351)	0.314 (0.303,0.328)	0.380 (0.361,0.398)

Average Length: k = 5

DGP	m = 30	m = 60	m = 120	m = 240
Levy-BM	0.334 (0.327,0.344)	0.297 (0.289,0.307)	0.288 (0.283,0.295)	0.349 (0.343,0.356)
I(1) c=0.01	0.448 (0.438,0.463)	0.395 (0.386,0.411)	0.371 (0.358,0.383)	0.424 (0.407,0.444)
I(1) c=0.03	0.476 (0.463,0.492)	0.411 (0.399,0.426)	0.382 (0.369,0.395)	0.431 (0.412,0.450)
I(1) Matern	0.475 (0.463,0.491)	0.414 (0.400,0.429)	0.384 (0.366,0.398)	0.433 (0.413,0.454)
J c=0.03	0.269 (0.264,0.274)	0.260 (0.255,0.265)	0.269 (0.264,0.273)	0.338 (0.333,0.344)
J c = 0.50	0.327 (0.320,0.336)	0.293 (0.287,0.300)	0.286 (0.282,0.292)	0.347 (0.342,0.352)
Br. Sheet	0.347 (0.337,0.362)	0.313 (0.303,0.326)	0.305 (0.294,0.316)	0.370 (0.355,0.381)

Method: Cluster fixed-effects (C-SCPC)

Null Rejection Probability: k = 1

DGP	m = 30	m = 60	m = 120	m = 240
Levy-BM	0.053 (0.045,0.062)	0.056 (0.044,0.073)	0.056 (0.046,0.065)	0.047 (0.040,0.061)
I(1) c=0.01	0.084 (0.076,0.094)	0.097 (0.080,0.129)	0.112 (0.091,0.132)	0.102 (0.081,0.133)
I(1) c=0.03	0.122 (0.112,0.134)	0.132 (0.116,0.175)	0.157 (0.134,0.183)	0.150 (0.126,0.173)
I(1) Matern	0.098 (0.089,0.112)	0.114 (0.097,0.149)	0.134 (0.118,0.166)	0.127 (0.107,0.150)
J c=0.03	0.030 (0.026,0.035)	0.034 (0.029,0.044)	0.041 (0.034,0.049)	0.042 (0.035,0.049)
J c = 0.50	0.043 (0.039,0.051)	0.048 (0.039,0.064)	0.050 (0.041,0.062)	0.046 (0.040,0.059)
Br. Sheet	0.104 (0.082,0.145)	0.106 (0.080,0.150)	0.105 (0.081,0.150)	0.076 (0.058,0.103)

Null Rejection Probability: k = 5

DGP	m = 30	m = 60	m = 120	m = 240
Levy-BM	0.053 (0.048,0.061)	0.056 (0.046,0.073)	0.060 (0.048,0.076)	0.051 (0.041,0.063)
I(1) c=0.01	0.080 (0.073,0.090)	0.098 (0.078,0.139)	0.122 (0.104,0.147)	0.116 (0.097,0.136)
I(1) c=0.03	0.107 (0.096,0.118)	0.129 (0.107,0.178)	0.177 (0.153,0.202)	0.165 (0.146,0.188)
I(1) Matern	0.090 (0.081,0.103)	0.102 (0.088,0.157)	0.149 (0.129,0.179)	0.144 (0.130,0.163)
J c=0.03	0.030 (0.026,0.035)	0.036 (0.030,0.047)	0.043 (0.034,0.054)	0.046 (0.040,0.058)
J c = 0.50	0.044 (0.038,0.051)	0.050 (0.039,0.067)	0.055 (0.047,0.067)	0.050 (0.042,0.059)
Br. Sheet	0.106 (0.090,0.145)	0.115 (0.085,0.163)	0.109 (0.086,0.152)	0.083 (0.064,0.114)

Average Length: k = 1

DGP	m = 30	m = 60	m = 120	m = 240
Levy-BM	0.550 (0.530,0.572)	0.447 (0.420,0.468)	0.393 (0.373,0.411)	0.453 (0.431,0.471)
I(1) c=0.01	0.809 (0.774,0.841)	0.697 (0.648,0.744)	0.627 (0.588,0.664)	0.683 (0.632,0.723)
I(1) c=0.03	0.907 (0.876,0.942)	0.813 (0.745,0.854)	0.737 (0.683,0.775)	0.773 (0.721,0.820)
I(1) Matern	0.878 (0.848,0.916)	0.773 (0.712,0.822)	0.708 (0.647,0.747)	0.763 (0.716,0.806)
J c=0.03	0.405 (0.392,0.418)	0.370 (0.348,0.382)	0.352 (0.337,0.365)	0.433 (0.409,0.458)
J c = 0.50	0.527 (0.510,0.546)	0.433 (0.411,0.451)	0.386 (0.368,0.401)	0.449 (0.424,0.466)
Br. Sheet	0.620 (0.576,0.675)	0.514 (0.476,0.559)	0.455 (0.419,0.485)	0.502 (0.454,0.532)

Average Length: k = 5

DGP	m = 30	m = 60	m = 120	m = 240
Levy-BM	0.530 (0.513,0.548)	0.433 (0.410,0.455)	0.379 (0.363,0.397)	0.444 (0.424,0.467)
I(1) c=0.01	0.795 (0.765,0.820)	0.680 (0.595,0.722)	0.586 (0.552,0.615)	0.626 (0.596,0.672)
I(1) c=0.03	0.901 (0.872,0.922)	0.778 (0.687,0.824)	0.659 (0.622,0.692)	0.702 (0.651,0.741)
I(1) Matern	0.878 (0.842,0.906)	0.780 (0.661,0.814)	0.655 (0.618,0.685)	0.699 (0.665,0.731)
J c=0.03	0.401 (0.390,0.412)	0.363 (0.344,0.378)	0.346 (0.330,0.362)	0.427 (0.404,0.444)
J c = 0.50	0.513 (0.500,0.527)	0.420 (0.394,0.445)	0.375 (0.356,0.390)	0.439 (0.421,0.459)
Br. Sheet	0.583 (0.556,0.619)	0.493 (0.454,0.532)	0.435 (0.408,0.461)	0.483 (0.450,0.516)

Method: LBM-GLS

Null Rejection Probability: k = 1

DGP	
Levy-BM	0.053 (0.049,0.057)
I(1) c=0.01	0.256 (0.244,0.267)
I(1) c=0.03	0.392 (0.374,0.412)
I(1) Matern	0.379 (0.359,0.396)
J c=0.03	0.058 (0.055,0.062)
J c = 0.50	0.053 (0.050,0.056)
Br. Sheet	0.234 (0.204,0.298)

Null Rejection Probability: k = 5

DGP	
Levy-BM	0.054 (0.051,0.058)
I(1) c=0.01	0.257 (0.243,0.268)
I(1) c=0.03	0.392 (0.377,0.408)
I(1) Matern	0.380 (0.363,0.400)
J c=0.03	0.060 (0.056,0.063)
J c = 0.50	0.054 (0.051,0.057)
Br. Sheet	0.234 (0.206,0.300)

Average Length: k = 1

DGP	
Levy-BM	0.195 (0.195,0.195)
I(1) c=0.01	0.212 (0.209,0.215)
I(1) c=0.03	0.224 (0.219,0.231)
I(1) Matern	0.222 (0.215,0.229)
J c=0.03	0.196 (0.195,0.196)
J c = 0.50	0.195 (0.195,0.196)
Br. Sheet	0.208 (0.199,0.213)

Average Length: k = 5

DGP	
Levy-BM	0.195 (0.195,0.195)
I(1) c=0.01	0.212 (0.208,0.214)
I(1) c=0.03	0.224 (0.218,0.229)
I(1) Matern	0.223 (0.218,0.228)
J c=0.03	0.196 (0.195,0.196)
J c = 0.50	0.195 (0.195,0.195)
Br. Sheet	0.208 (0.199,0.212)

Method: LBM-GLS (C-SCPC)

Null Rejection Probability: k = 1

DGP	
Levy-BM	0.030 (0.027,0.035)
I(1) c=0.01	0.049 (0.043,0.055)
I(1) c=0.03	0.069 (0.060,0.076)
I(1) Matern	0.059 (0.051,0.066)
J c=0.03	0.029 (0.025,0.033)
J c = 0.50	0.030 (0.027,0.035)
Br. Sheet	0.088 (0.072,0.125)

Null Rejection Probability: k = 5

DGP	
Levy-BM	0.031 (0.027,0.035)
I(1) c=0.01	0.050 (0.043,0.056)
I(1) c=0.03	0.069 (0.061,0.078)
I(1) Matern	0.059 (0.052,0.067)
J c=0.03	0.029 (0.025,0.033)
J c = 0.50	0.030 (0.027,0.034)
Br. Sheet	0.085 (0.072,0.132)

Average Length: k = 1

DGP	
Levy-BM	0.254 (0.251,0.257)
I(1) c=0.01	0.419 (0.408,0.430)
I(1) c=0.03	0.541 (0.524,0.559)
I(1) Matern	0.545 (0.523,0.562)
J c=0.03	0.264 (0.260,0.266)
J c = 0.50	0.255 (0.252,0.258)
Br. Sheet	0.333 (0.319,0.349)

Average Length: k = 5

DGP	
Levy-BM	0.256 (0.253,0.258)
I(1) c=0.01	0.419 (0.408,0.430)
I(1) c=0.03	0.536 (0.517,0.553)
I(1) Matern	0.547 (0.528,0.565)
J c=0.03	0.266 (0.262,0.268)
J c = 0.50	0.257 (0.253,0.259)
Br. Sheet	0.335 (0.320,0.347)

Method: Low-pass eigenvector

Null Rejection Probability: k = 1

DGP	q = 10	q = 20	q = 50
Levy-BM	0.050 (0.046,0.054)	0.050 (0.047,0.054)	0.050 (0.046,0.053)
I(1) c=0.01	0.051 (0.047,0.054)	0.052 (0.049,0.056)	0.064 (0.060,0.068)
I(1) c=0.03	0.053 (0.050,0.057)	0.063 (0.058,0.067)	0.105 (0.099,0.110)
I(1) Matern	0.051 (0.047,0.055)	0.055 (0.052,0.059)	0.082 (0.077,0.087)
J c=0.03	0.100 (0.093,0.107)	0.094 (0.088,0.099)	0.078 (0.074,0.083)
J c = 0.50	0.056 (0.052,0.060)	0.054 (0.050,0.059)	0.052 (0.048,0.055)
Br. Sheet	0.128 (0.095,0.171)	0.160 (0.120,0.209)	0.210 (0.170,0.272)

Null Rejection Probability: k = 5

DGP	q = 10	q = 20	q = 50
Levy-BM	0.050 (0.046,0.054)	0.050 (0.046,0.054)	0.050 (0.048,0.054)
I(1) c=0.01	0.050 (0.047,0.054)	0.051 (0.048,0.056)	0.062 (0.059,0.066)
I(1) c=0.03	0.052 (0.048,0.055)	0.060 (0.057,0.063)	0.101 (0.096,0.107)
I(1) Matern	0.050 (0.046,0.053)	0.054 (0.050,0.058)	0.080 (0.074,0.085)
J c=0.03	0.095 (0.089,0.100)	0.095 (0.088,0.099)	0.079 (0.075,0.083)
J c = 0.50	0.054 (0.050,0.057)	0.054 (0.050,0.057)	0.052 (0.048,0.055)
Br. Sheet	0.104 (0.080,0.135)	0.147 (0.119,0.180)	0.201 (0.168,0.243)

Average Length: k = 1

DGP	q = 10	q = 20	q = 50
Levy-BM	1.507 (1.499,1.515)	0.960 (0.957,0.963)	0.574 (0.573,0.575)
I(1) c=0.01	1.508 (1.500,1.515)	0.960 (0.956,0.964)	0.574 (0.573,0.575)
I(1) c=0.03	1.507 (1.500,1.516)	0.960 (0.957,0.964)	0.574 (0.573,0.576)
I(1) Matern	1.508 (1.499,1.518)	0.961 (0.956,0.964)	0.574 (0.572,0.576)
J c=0.03	1.509 (1.496,1.517)	0.960 (0.956,0.964)	0.574 (0.573,0.576)
J c = 0.50	1.508 (1.499,1.513)	0.960 (0.957,0.963)	0.574 (0.573,0.575)
Br. Sheet	1.507 (1.498,1.518)	0.959 (0.956,0.967)	0.574 (0.572,0.576)

Average Length: k = 5

DGP	q = 10	q = 20	q = 50
Levy-BM	2.299 (2.279,2.317)	1.101 (1.095,1.106)	0.601 (0.599,0.602)
I(1) c=0.01	2.297 (2.280,2.313)	1.100 (1.096,1.105)	0.600 (0.599,0.602)
I(1) c=0.03	2.297 (2.276,2.317)	1.100 (1.095,1.105)	0.600 (0.599,0.602)
I(1) Matern	2.298 (2.283,2.316)	1.101 (1.095,1.105)	0.600 (0.599,0.602)
J c=0.03	2.297 (2.274,2.318)	1.100 (1.095,1.104)	0.600 (0.599,0.602)
J c = 0.50	2.300 (2.282,2.320)	1.101 (1.096,1.106)	0.600 (0.599,0.602)
Br. Sheet	2.300 (2.283,2.322)	1.101 (1.095,1.106)	0.600 (0.598,0.602)

Method: High-pass eigenvector (C-SCPC)

Null Rejection Probability: k = 1

DGP	q = 5	q = 10	q = 20	q = 50	q = 100
Levy-BM	0.129 (0.117,0.139)	0.095 (0.087,0.103)	0.070 (0.063,0.078)	0.050 (0.045,0.056)	0.042 (0.037,0.046)
I(1) c=0.01	0.174 (0.160,0.184)	0.141 (0.132,0.152)	0.118 (0.106,0.128)	0.090 (0.081,0.099)	0.069 (0.061,0.078)
I(1) c=0.03	0.215 (0.205,0.234)	0.183 (0.168,0.197)	0.150 (0.137,0.167)	0.111 (0.096,0.128)	0.081 (0.071,0.097)
I(1) Matern	0.193 (0.180,0.206)	0.165 (0.152,0.180)	0.146 (0.131,0.159)	0.118 (0.106,0.136)	0.097 (0.077,0.121)
J c=0.03	0.050 (0.045,0.054)	0.051 (0.046,0.056)	0.050 (0.045,0.055)	0.045 (0.040,0.049)	0.040 (0.035,0.044)
J c = 0.50	0.120 (0.112,0.133)	0.093 (0.086,0.099)	0.070 (0.064,0.076)	0.050 (0.045,0.055)	0.041 (0.037,0.047)
Br. Sheet	0.213 (0.186,0.270)	0.192 (0.163,0.246)	0.167 (0.141,0.221)	0.132 (0.113,0.174)	0.099 (0.084,0.136)

Null Rejection Probability: k = 5

DGP	q = 5	q = 10	q = 20	q = 50	q = 100
Levy-BM	0.125 (0.116,0.134)	0.093 (0.087,0.101)	0.070 (0.065,0.078)	0.051 (0.045,0.057)	0.041 (0.037,0.046)
I(1) c=0.01	0.161 (0.151,0.170)	0.135 (0.125,0.147)	0.114 (0.106,0.126)	0.089 (0.078,0.102)	0.068 (0.061,0.078)
I(1) c=0.03	0.200 (0.187,0.212)	0.173 (0.161,0.184)	0.144 (0.133,0.158)	0.108 (0.094,0.126)	0.082 (0.070,0.097)
I(1) Matern	0.179 (0.167,0.188)	0.157 (0.147,0.168)	0.139 (0.129,0.153)	0.118 (0.104,0.134)	0.095 (0.080,0.113)
J c=0.03	0.051 (0.046,0.054)	0.051 (0.048,0.056)	0.051 (0.046,0.054)	0.045 (0.042,0.051)	0.040 (0.036,0.044)
J c = 0.50	0.117 (0.108,0.128)	0.090 (0.085,0.096)	0.069 (0.063,0.074)	0.050 (0.045,0.057)	0.041 (0.037,0.045)
Br. Sheet	0.203 (0.182,0.249)	0.183 (0.161,0.232)	0.160 (0.140,0.214)	0.129 (0.108,0.174)	0.100 (0.083,0.138)

Average Length: k = 1

DGP	q = 5	q = 10	q = 20	q = 50	q = 100
Levy-BM	0.565 (0.552,0.578)	0.467 (0.459,0.476)	0.391 (0.382,0.399)	0.328 (0.322,0.335)	0.320 (0.314,0.325)
I(1) c=0.01	0.744 (0.720,0.770)	0.647 (0.625,0.671)	0.558 (0.541,0.576)	0.464 (0.450,0.479)	0.428 (0.414,0.441)
I(1) c=0.03	0.789 (0.755,0.825)	0.690 (0.656,0.715)	0.587 (0.567,0.617)	0.489 (0.468,0.510)	0.449 (0.427,0.466)
I(1) Matern	0.788 (0.759,0.820)	0.690 (0.666,0.720)	0.607 (0.581,0.628)	0.521 (0.492,0.542)	0.501 (0.466,0.522)
J c=0.03	0.419 (0.412,0.425)	0.388 (0.381,0.394)	0.353 (0.349,0.359)	0.318 (0.314,0.324)	0.317 (0.313,0.322)
J c = 0.50	0.558 (0.542,0.575)	0.465 (0.455,0.475)	0.389 (0.383,0.399)	0.329 (0.322,0.334)	0.320 (0.314,0.325)
Br. Sheet	0.592 (0.571,0.614)	0.523 (0.498,0.551)	0.472 (0.449,0.498)	0.423 (0.401,0.447)	0.409 (0.394,0.427)

Average Length: k = 5

DGP	q = 5	q = 10	q = 20	q = 50	q = 100
Levy-BM	0.535 (0.524,0.549)	0.456 (0.447,0.467)	0.386 (0.380,0.394)	0.329 (0.322,0.334)	0.322 (0.317,0.327)
I(1) c=0.01	0.735 (0.711,0.755)	0.647 (0.623,0.665)	0.557 (0.542,0.576)	0.465 (0.448,0.480)	0.428 (0.413,0.443)
I(1) c=0.03	0.792 (0.764,0.820)	0.693 (0.667,0.719)	0.594 (0.573,0.618)	0.491 (0.471,0.511)	0.448 (0.430,0.467)
I(1) Matern	0.786 (0.765,0.812)	0.697 (0.677,0.725)	0.613 (0.594,0.634)	0.526 (0.498,0.548)	0.501 (0.477,0.525)
J c=0.03	0.412 (0.406,0.418)	0.383 (0.378,0.389)	0.352 (0.346,0.357)	0.318 (0.314,0.324)	0.319 (0.314,0.325)
J c = 0.50	0.533 (0.520,0.545)	0.455 (0.446,0.462)	0.387 (0.380,0.393)	0.329 (0.323,0.335)	0.322 (0.316,0.327)
Br. Sheet	0.551 (0.529,0.575)	0.498 (0.476,0.514)	0.454 (0.436,0.471)	0.413 (0.398,0.430)	0.404 (0.390,0.417)

Method: Ibragimov-Müller

Null Rejection Probability: k = 1

DGP	m = 10	m = 20	m = 50
Levy-BM	0.105 (0.090,0.117)	0.105 (0.096,0.114)	0.080 (0.072,0.087)
I(1) c=0.01	0.125 (0.110,0.137)	0.144 (0.131,0.157)	0.154 (0.137,0.168)
I(1) c=0.03	0.152 (0.130,0.166)	0.193 (0.174,0.207)	0.235 (0.211,0.254)
I(1) Matern	0.134 (0.115,0.147)	0.163 (0.149,0.175)	0.193 (0.179,0.206)
J c=0.03	0.062 (0.058,0.067)	0.062 (0.056,0.067)	0.053 (0.047,0.058)
J c = 0.50	0.088 (0.081,0.095)	0.087 (0.082,0.094)	0.070 (0.063,0.077)
Br. Sheet	0.182 (0.132,0.223)	0.198 (0.156,0.234)	0.158 (0.131,0.193)

Null Rejection Probability: k = 5

DGP	m = 10	m = 20	m = 50
Levy-BM	0.084 (0.077,0.091)	0.076 (0.068,0.082)	0.048 (0.043,0.052)
I(1) c=0.01	0.091 (0.082,0.098)	0.091 (0.081,0.098)	0.062 (0.057,0.069)
I(1) c=0.03	0.104 (0.092,0.114)	0.114 (0.102,0.121)	0.080 (0.072,0.087)
I(1) Matern	0.092 (0.082,0.101)	0.098 (0.088,0.105)	0.072 (0.064,0.079)
J c=0.03	0.060 (0.055,0.064)	0.055 (0.049,0.060)	0.043 (0.039,0.047)
J c = 0.50	0.075 (0.070,0.081)	0.068 (0.060,0.072)	0.045 (0.042,0.051)
Br. Sheet	0.142 (0.115,0.169)	0.135 (0.106,0.159)	0.070 (0.063,0.085)

Average Length: k = 1

DGP	m = 10	m = 20	m = 50
Levy-BM	0.587 (0.575,0.599)	0.442 (0.432,0.470)	0.418 (0.379,0.479)
I(1) c=0.01	0.871 (0.851,0.899)	0.696 (0.680,0.722)	0.627 (0.583,0.707)
I(1) c=0.03	1.004 (0.976,1.046)	0.798 (0.780,0.824)	0.701 (0.656,0.789)
I(1) Matern	0.964 (0.942,0.988)	0.782 (0.762,0.807)	0.709 (0.664,0.768)
J c=0.03	0.365 (0.357,0.376)	0.330 (0.320,0.345)	0.375 (0.329,0.438)
J c = 0.50	0.550 (0.537,0.564)	0.428 (0.417,0.442)	0.407 (0.376,0.455)
Br. Sheet	0.609 (0.590,0.643)	0.468 (0.454,0.488)	0.435 (0.397,0.473)

Average Length: k = 5

DGP	m = 10	m = 20	m = 50
Levy-BM	0.480 (0.472,0.491)	0.411 (0.393,0.481)	0.461 (0.385,0.539)
I(1) c=0.01	0.755 (0.740,0.770)	0.647 (0.614,0.727)	0.652 (0.552,0.767)
I(1) c=0.03	0.867 (0.851,0.886)	0.730 (0.708,0.805)	0.668 (0.577,0.821)
I(1) Matern	0.883 (0.863,0.910)	0.783 (0.761,0.873)	0.780 (0.617,0.935)
J c=0.03	0.359 (0.349,0.372)	0.357 (0.338,0.448)	0.454 (0.359,0.527)
J c = 0.50	0.467 (0.457,0.477)	0.404 (0.383,0.466)	0.478 (0.396,0.571)
Br. Sheet	0.503 (0.495,0.522)	0.435 (0.413,0.531)	0.487 (0.390,0.571)

R^2 values in OLS regression

k = 1

DGP	
Levy-BM	0.137 (0.125,0.162)
I(1) c=0.01	0.179 (0.166,0.204)
I(1) c=0.03	0.208 (0.197,0.238)
I(1) Matern	0.192 (0.179,0.215)
J c=0.03	0.010 (0.010,0.011)
J c = 0.50	0.085 (0.079,0.099)
Br. Sheet	0.139 (0.117,0.161)

k = 5

DGP	
Levy-BM	0.434 (0.419,0.471)
I(1) c=0.01	0.561 (0.548,0.592)
I(1) c=0.03	0.638 (0.626,0.664)
I(1) Matern	0.595 (0.584,0.625)
J c=0.03	0.049 (0.047,0.050)
J c = 0.50	0.314 (0.298,0.354)
Br. Sheet	0.443 (0.404,0.471)

Additional References

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