TESTING FOR COMMON TRENDS

JAMES H. STOCK and MARK W. WATSON

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Testing for Common Trends:

Technical Appendix

by

James H. Stock

Kennedy School of Government
Harvard University
Cambridge, MA 02138

and

Mark W. Watson
Department of Economics
Northwestern University
Evanston, IL 60201

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1. Summary

This technical report contains the statements and proofs of the lemmas and theorems in Stock and Watson (1988).

Cointegrated multiple time series share one or more common trends. In Stock and Watson (1988), we develop two tests for the number of common stochastic trends (i.e. for the order of cointegration) in a multiple time series with and without drift. Both tests involve the roots of the OLS coefficient matrix obtained by regressing the series onto its first lag. Critical values for the tests are tabulated, and their power is examined in a Monte Carlo study.

Economic time series are often modeled as having a unit root in their autoregressive representation, or (equivalently) as containing a stochastic trend. But both casual observation and economic theory suggest that many series might contain the same stochastic trends, so that they are cointegrated. If each of n series is integrated of order one but can be jointly characterized by k < n stochastic trends, then the vector representation of these series will have k unit roots and n-k distinct stationary linear combinations. Our tests can be viewed as tests of the number of common trends, the number of autoregressive unit roots, or the number of linearly independent cointegrating vectors.

Both of the proposed tests are asymptotically similar but differ in their treatment of the nuisance parameters of the process. The first test (qF) is developed under the assumption that certain components of the process have a finite order VAR representation, and the nuisance parameters are handled by
estimating this VAR. The second test \( q_c \) entails computing the eigenvalues of a "corrected" sample first order autocorrelation matrix, where the correction is essentially a sum of the autocovariance matrices.

Previous researchers have found that U.S. postwar interest rates, taken individually, appear to be integrated of order one. In addition, the theory of the term structure (equating the expected future spot rate to the implicit forward rate, plus a stationary risk premium) implies that yields on similar assets of different maturities will be cointegrated. Applying these tests to postwar U.S. data on the Federal Funds rate and the three- and twelve-month Treasury Bill rates, we find support for this prediction: the three interest rates are found to be cointegrated, possessing a single common trend.

2. Theorems and Lemmas

This section contains the statements of the theorems and lemmas concerning the proposed tests of the number of common trends. For definitions and discussion, see Stock and Watson (1988).

**Theorem 3.1.** Suppose that \( D_{BD} \), that \( W_t \) is generated by (3.1) with \( W_0 - \gamma = 0 \), that \( \Phi(L) = R_2 \Pi(L) R_2^{-1} \), and that \( \max_{k \leq 4} E(\eta_{1,t}^4) \leq \mu_4 < \infty \). Then

(i) \( T(\phi_{f,t} - I_k) \Rightarrow R_2 \Psi_{f,k}^{-1} R_2^{-1} \)

(ii) \( T(\lambda_{f,t} - \gamma) \Rightarrow \lambda_* \)

(iii) \( T(\vert \lambda_{f,t} - \gamma \vert) \Rightarrow \text{Re}(\lambda_* ) \).
Lemma 4.1. If \( \max_1 E(\nu^{1}_{1t}) \leq \mu_{1t} < \infty \) and \( \beta_1 - \beta_2 = 0 \) in (2.7), then

\[ T(\Phi - I_k) - [\hat{C}(1)\psi'_t\hat{C}(1)' + M'][\hat{C}(1)\Gamma_{nt}\hat{C}(1)']^{-1} E 0 \]

where \( \psi'_{nt} = T^{-1} \sum \xi_{t-1}\nu'_t, \Gamma_{nt} = T^{-2} \sum \xi_{t-1}\xi'_{t}, \) and

\[ M = \sum_{j=0}^{\infty} (\hat{C}_j^* - \hat{C}_j)\hat{C}_j + \hat{C}(1)\hat{C}(1)' = E \sum_{j=1}^{\infty} u_{t-j}u'_{t}. \]

Lemma 4.2. Let \( \Omega \) be a \( k \times k \) matrix such that \( \Omega' - \hat{C}(1)\hat{C}(1)' \). Then, under the conditions of Lemma 4.1,

(i) \[ T(\Phi - I_k) \rightarrow \Omega\psi'_{nt}\Gamma^{-1}_{nt}\Omega^{-1} \]

(ii) \[ T(\hat{\lambda}_c - \hat{\lambda}) \rightarrow \lambda_* \]

Theorem 4.1. Suppose that \( \mathcal{B}^RD \) and \( \mathcal{R}^RMR_2 \). Then under the assumptions of Lemma 4.1,

(i) \[ T(\Phi - I_k) \rightarrow R_2\Omega\psi'_{nt}\Gamma^{-1}_{nt}\Omega^{-1}R_2^{-1} \]

(ii) \[ T(\hat{\lambda}_c - \hat{\lambda}) \rightarrow \lambda_* \].
Theorem 5.1. Suppose that $\mathcal{E}^{RD}$, that $\mathcal{W}_{k}$ is generated by (3.1), that $\hat{\Omega}(L)^{-1}R_{2}\Pi(L)R_{2}^{-1}$, and that $\max_{1}E(\eta_{1t}^{4})\leq\mu_{4}<\infty$. Then

a. if $\gamma = 0$ and $W_{0}$ is an arbitrary constant,

(a.i) \[ T(\hat{\phi}_{k}^{s} - I_{k}) \Rightarrow R_{2}\hat{\psi}_{k}^{s}(T_{k}^{s})^{-1}R_{2}^{-1} \]

(a.ii) \[ T(\phi_{k}^{s} - I_{k}) \Rightarrow \lambda_{k}^{s} \]

(a.iii) \[ T(|\phi_{k}^{s}| - I_{k}) \Rightarrow Re(\lambda_{k}^{s}) \]

b. if $\gamma$ and $W_{0}$ are arbitrary constants,

(b.i) \[ T(\hat{\phi}_{k}^{s} - I_{k}) \Rightarrow R_{2}\hat{\psi}_{k}^{s}(T_{k}^{s})^{-1}R_{2}^{-1} \]

(b.ii) \[ T(\phi_{k}^{s} - I_{k}) \Rightarrow \lambda_{k}^{s} \]

(b.iii) \[ T(|\phi_{k}^{s}| - I_{k}) \Rightarrow Re(\lambda_{k}^{s}) \]

Lemma 5.1. If $\max_{1}E(\eta_{1t}^{4})\leq\mu_{4}<\infty$ and

(i) if $\beta_{2} = 0$ in (2.7) and $\beta_{1}$ is an arbitrary constant, then

\[ T(\hat{\phi}_{k}^{s} - I_{k}) \Rightarrow [\hat{\phi}(1)\hat{\psi}_{k}^{s}(1), + \hat{M}][\hat{\phi}(1)\hat{\psi}_{k}^{s}(1),]^{-1} \Rightarrow 0 \]

(ii) if $\beta_{1}$ and $\beta_{2}$ in (2.7) are arbitrary constants, then

\[ T(\hat{\phi}_{k}^{s} - I_{k}) \Rightarrow [\hat{\phi}(1)\hat{\psi}_{k}^{s}(1), + \hat{M}][\hat{\phi}(1)\hat{\psi}_{k}^{s}(1),]^{-1} \Rightarrow 0 \]

where $\hat{\psi}_{k}^{s} = \sum_{i=0}^{\infty} \xi_{i}^{s}$, $\sum_{i=0}^{\infty} \xi_{i}^{s}$, and $\sum_{i=0}^{\infty} \xi_{i}^{s}$, where $\xi_{i}^{s} = \xi_{i}^{s}T_{0}$ and $\xi_{i}^{s} = \xi_{i}^{s}T_{1}$, $\theta_{0}$ and $\theta_{2}$, where $\theta_{0} = -T_{0}^{-3/2}T_{0}^{-1}T_{i}^{-1}a_{i}^{t}$, $i = 0, 1, 2$, with $a_{0} = 1$, $a_{1} = 4 - 6(t/T)$, and $a_{2} = 6 + 12(t/T)$, and where $M = \sum_{i=1}^{\infty} \sum_{j=1}^{u_{t}} u_{t}^{j}$. 

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Theorem 5.2. If $\mathcal{B}^{RD}$ and $\mathcal{N}^{R_2 MR_2'}$, then under the assumptions of Lemma 5.1,
a. if $\beta_2 = 0$ in (2.7) and $\beta_1$ is an arbitrary constant, then

(a.i) $T(\Phi^\mu_{\gamma}(\mathcal{I}^\mu_k)\mathcal{N}^{-1}_{\gamma}R^\mu_{\gamma})$

(a.ii) $T(\lambda^\mu_{\gamma}) \Rightarrow \lambda^\mu_{\gamma}$

(a.iii) $T(|\lambda^\mu_{\gamma}| - t) \Rightarrow \text{Re}(\lambda^\mu_{\gamma})$

b. if $\beta_1$ and $\beta_2$ in (2.7) are arbitrary constants, then

(b.i) $T(\Phi^\mu_{\gamma}(\mathcal{I}^\mu_k)\mathcal{N}^{-1}_{\gamma}R^\mu_{\gamma})$

(b.ii) $T(\lambda^\mu_{\gamma}) \Rightarrow \lambda^\mu_{\gamma}$

(b.iii) $T(|\lambda^\mu_{\gamma}| - t) \Rightarrow \text{Re}(\lambda^\mu_{\gamma})$

3. Proofs

Proof of Theorem 3.1.

(i) We first show that $T[\Phi^\mu_{\gamma}(\mathcal{I}^\mu_k)\mathcal{N}^{-1}_{\gamma}R^\mu_{\gamma}] E_0$. From the definitions of $\Phi^\mu_{\gamma}$ and $\Phi^\mu_{\gamma}$ and the (almost sure) invertibility of $\Gamma_k$, this follows if

$$ T^{-1}\sum_{t-1}^{t} \Delta^\mu_{\gamma} - R_2T^{-1}\sum_{t-1}^{t} \Delta^\mu_{\gamma}R^\mu_{\gamma} R_2 E_0 $$

$$ T^{-2}\sum_{t-1}^{t} \Delta^\mu_{\gamma} - R_2T^{-2}\sum_{t-1}^{t} \Delta^\mu_{\gamma}R^\mu_{\gamma} R_2 E_0 $$

To show (A.1a), since $\xi_{\gamma}=\pi(L)\xi_{\gamma}$ and $\xi_{\gamma}=\pi(L)\xi_{\gamma}$,

$$ T^{-1}\sum_{t-1}^{t} \Delta^\mu_{\gamma} - R_2T^{-1}\sum_{t-1}^{t} \Delta^\mu_{\gamma}R^\mu_{\gamma} R_2 $$

$$ - \sum_{j=1}^{p} \sum_{k=1}^{\pi} \Delta^\mu_{\gamma} \xi_{\gamma} D_{i} T^{-1}\sum_{t-p+2}^{T} X_{t-1} \Delta^\mu_{\gamma} R^\mu_{\gamma} R_2 $$

$$ - R_2\sum_{j=1}^{p} \sum_{k=1}^{\pi} \Delta^\mu_{\gamma} \xi_{\gamma} D_{i} T^{-1}\sum_{t-p+2}^{T} X_{t-1} \Delta^\mu_{\gamma} R^\mu_{\gamma} R_2 $$

(A.2)

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For fixed \((i,j)\), under the stated conditions it is straightforward to show that
\[
T^{-1}\Sigma_{t-p}^{T} X_{t-i-1} \Delta X_{t-j} = \mathbb{O}(1) \quad \text{for fixed } (i,j).
\]
By assumption, \(\mathcal{D}_{s}^{RD}\) and \(\hat{N}_{1}s_{k}\mathcal{D}_{s}^{R_2^R}\mathcal{D}_{s}^{R_2^L}\mathcal{S}_{k}^{RD}\), so that \(\hat{N}_{1}s_{k}\mathcal{D}_{s}^{R_2^R}\mathcal{S}_{k}^{RD}\). Since \(s_{k}R_{2} = s_{k}R_{2}\), the difference on the right hand side of (A.2) thus converges in probability to zero, proving (A.1a). The proof of (A.1b) is analogous, using the fact that
\[
T^{-2}\sum_{t-i}^{T} X_{t-j} = \mathbb{O}(1) \quad \text{for fixed } (i,j).
\]
Since, from (3.2), \(T[\hat{\phi}_{k} - I_{k}] \rightarrow \psi_{k}^{'} \Psi_{k}^{-1}\), it follows that \(T[\hat{\phi}_{k} - I_{k}] \rightarrow R_{k}^{2} \Psi_{k}^{'} \Psi_{k}^{-1} R_{k}^{-1}\).

(ii) Let \(\lambda_{k}^{\dagger}\) denote the vector of ordered eigenvalues of \(T[\hat{\phi}_{k} - I_{k}]\). It follows from (i) and the continuity of the eigenvalues as functions of the elements of \(T[\hat{\phi}_{k} - I_{k}]\) that \(\lambda_{k}^{\dagger} \rightarrow \lambda_{k}\). But \(T^{k} \text{det}[\hat{\phi}_{k} - \lambda I_{k}] = \text{det}[T(\hat{\phi}_{k} - I_{k}) - T(\lambda - 1)I_{k}]\), so \(\lambda_{k}^{\dagger} = T(\lambda_{k} - i)\), from which it follows that \(T(\lambda_{k} - i) \rightarrow \lambda_{k}\).

(iii) Write \(T(\lambda_{k} - i) = a_{j} + ib_{j}\), where \(i = \sqrt{-1}\). By (ii), \(a_{j}\) and \(b_{j}\) are \(\mathbb{O}(1)\) random variables. Now
\[
T[|\lambda_{k}| - 1] = T[(1+a_{j}+ib_{j})/|T| - 1] = T[((1+a_{j}/T)^{2} + b_{j}^{2}/T^{2})^{\frac{k}{2}} - 1]
- T[(1+a_{j}/T) + \mathbb{O}(T^{-2}) - 1] = a_{j} + \mathbb{O}(1)
\]
so that \(T(|\lambda_{k}| - i) \rightarrow \text{Re}(T(\lambda_{k} - i))\). Also, from part (ii) of this Lemma, \(\text{Re}(T(\lambda_{k} - i)) \rightarrow \text{Re}(\lambda_{k})\). Thus \(T(|\lambda_{k}| - 1) \rightarrow \text{Re}(\lambda_{k})\), the desired result.

Proof of Lemma 4.1.

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First write $T(\Phi-I_k)$ as

$$T(\Phi-I_k) = U_T^TV_T^{-1}$$  \hspace{1cm} (A.3)

where $U_T = T^{-2}T_w^{-1}L_{L-1}$ and $V_T = T^{-2}T_w^{-1}L_{L-1}$. Using (2.7) with $\beta_1 = \beta_2 = 0$, $V_T$ can be written,

$$V_T = T^{-2}T_w^{-1}L_{L-1}[(\hat{c}(1)\xi_t + \hat{c}^*(L)\nu_t) \hat{c}(1)\xi_t + \hat{c}^*(L)\nu_t]' .$$

Under the stated conditions, Stock's (1987) Theorem 1 applies directly, and

$$V_T = \hat{c}(1)\Gamma_n \hat{c}(1)' = o_p(1).$$  \hspace{1cm} (A.4)

where $\Gamma_n$ is defined in the statement of the Theorem. Using Chan and Wei (1988, Theorem 2.4), $\Gamma_n \xrightarrow{p} \int_0^1 B_n(t)B_n(t)'dt$, where $B_n(t)$ is a $n$-dimensional Wiener process and $\xrightarrow{p}$ denotes convergence on $C[0,1]^n$. Since $\hat{c}(1)$ has full row rank under the null, $[\hat{c}(1)\Gamma_n \hat{c}(1)']$ is almost surely invertible in the limit.

Turning to $U_T$, write $TU_T = T^{-1}\sum_t \Delta w_t \Delta w_t' - T^{-1}\sum_t \Delta w_t \Delta w_t'$. Now

$$T^{-1}\sum_t \Delta w_t \Delta w_t' \xrightarrow{P} \sum_{j=0}^\infty \hat{c}_j \hat{c}_j'$$  \hspace{1cm} (A.5)

from (2.5). Using (2.5) and (2.7),

$$T^{-1}\sum_t \Delta w_t \Delta w_t' = \hat{c}(1)T^{-1}\sum_t [\hat{c}(L)\nu_t]' + T^{-1}\sum_t [\hat{c}^*(L)\nu_t]' \hat{c}(L)\nu_t]' .$$  \hspace{1cm} (A.6)
The second term in (A.6) converges to a constant matrix:

\[
T^{-1} \sum \{ C^* (L) \nu \_L \} \{ C(L) \nu \_L \}' = \sum_{j=0}^{\infty} C^*_j C_j'.
\]  

(A.7a)

The first term in (A.6) can be treated using the decomposition in Stock's (1987) Theorem 1, yielding

\[
\hat{C}(1)T^{-1} \sum \epsilon_t \{ \hat{C}(L) \nu \_L \}' = \hat{C}(1) [\Psi_{nT+I_n} \hat{C}(1)', \mathbb{E} = 0.
\]

(A.7b)

Combining (A.5), (A.6) and (A.7),

\[
TU_T = \{ \hat{C}(1) \Psi_{nT+I_n} \hat{C}(1)' - \sum_{j=0}^{\infty} C^*_j C_j' + \sum_{j=0}^{\infty} C^*_j C_j' \} \mathbb{E} = 0.
\]

(A.8)

Combining (A.3), (A.4) and (A.8) yields the desired result with

\[M = \{ \hat{C}(1) \hat{C}(1)' + \sum_{j=0}^{\infty} (C^*_j \cdot C_j) \hat{C}_j' \} \].

The second expression for M in the Theorem obtains by direct calculation.

Proof of Lemma 4.2.

(i) Using Lemma 4.1 (i) and the definitions of \(U_T\) and \(V_T\) given in its proof,

\[
T(\hat{\Phi}_c - I_k) = (TU_T - M)' V_T^{-1}
\]

\[
= \{ \hat{C}(1) \Psi_{nT} \hat{C}(1)' - M \} '[\hat{C}(1) \Gamma_{nT} \hat{C}(1)']^{-1} + o_p(1).
\]

(A.9)

From Theorem 2.4 of Chan and Wei (1988), \((\Psi_{nT}, \Gamma_{nT}) \Rightarrow (\Psi_n, \Gamma_n)\). Letting \(\Omega' = \hat{C}(1) \hat{C}(1)'\) (where \(\Omega\) is \(k \times k\), \(\{\hat{C}(1) \Gamma_n(t)\}\) has the same distribution as
Thus \((\hat{c}(1) \Psi_n \hat{c}(1)', \hat{c}(1) \Gamma_n \hat{c}(1)')\) has the same distribution as 
\((\Omega_\Psi \Omega', \Omega_\Gamma \Omega')\). Since \(\hat{c}(1)\) has full row rank by construction, \((\hat{c}(1) \Gamma_n \hat{c}(1)')^{-1}\) exists almost surely. It follows that 
\(T(\Phi_{c-I}) \rightarrow (\Omega_\Psi \Omega')'(\Omega_\Gamma \Omega')^{-1} = \Omega_\Psi \Gamma_n \Omega^{-1}\).

\(\text{(ii) The proof of Theorem 3.1 (ii) applies directly. } \square\)

Proof of Theorem 4.1.

(i) Let \(\hat{U}_T = T^{-2}\sum_t \hat{Q}_{t-1} \hat{Q}'_t\) and \(\hat{V}_T = T^{-2}\sum_t \hat{Q}_{t-1} \hat{Q}'_t\). Then:

\[ T[\hat{\Phi}_{c-I}] = (T\hat{U}_T - \hat{A}) \hat{V}^{-1}_T. \] \hspace{1cm} (A.10)

Comparing (A.9) and (A.10), one finds that the result (i) follows if

\begin{align*}
T\hat{U}_T - R_2 T U_T R'_2 & \equiv 0 & \text{(A.11a)} \\
T\hat{V}_T - R_2 V_T R'_2 & \equiv 0. & \text{(A.11b)}
\end{align*}

To show (A.11a), use \(\hat{W}_t = S_k D X_t\), \(\hat{Q}_t = S_k D X_t\), and \(S_k = R_2 S_k\) to write

\[ T\hat{U}_T - R_2 T U_T R'_2 = S_k DT^{-1}\sum_t D X_t' P_s D X_t' - S_k R D T^{-1} \sum_t D X_t' D'R' S_k'. \]

Since \(DTD\) by assumption and since \(T^{-1} \sum_t D X_t' D X_t = O_p(1)\), (A.11a) follows. The proof of (A.11b) is similar. Thus, from Lemma 4.2 (i),

\[ T[\hat{\Phi}_{c-I}] = [R_2 \hat{C}(1) n T \hat{C}(1)' R'_2'] [R_2 \hat{C}(1) \Gamma_n \hat{C}(1)' R'_2']^{-1} + o_p(1). \]
The argument used to prove Lemma 4.2 (i) implies that

\[ T[\Phi_c \cdot I_k] = [R_2 \Omega \Gamma_k \Omega' R_2'][R_2 \Omega \Gamma_k \Omega' R_2']^{-1} = R_2 \Omega \Gamma_k \Omega^{-1} R_2^{-1}. \]

(ii) This result is an immediate consequence of (i) since \( \Omega_{k'}^{-1} \) and \( R_2 \Omega \Gamma_k \Omega^{-1} R_2^{-1} \) are similar matrices.

It is convenient to prove Lemma 5.1 before proving Theorem 5.1.

Proof of Lemma 5.1.

We prove (ii) first. Let \( a_{1t} = 4 - 6(t/T), a_{2t} = -6 + 12(t/T), \) and \( \omega_t = \beta_3 \xi_t + \beta_4 (1)_t \nu_t \) (so that from (2.7) \( Y_t = \beta_1 + \beta_2 t + \omega_t \)). Using the definition of \( \omega_t \) and Chebyshev’s inequality one obtains

\[
\begin{bmatrix}
T^{-1}\xi_i (\beta_1 - \beta_1)
\end{bmatrix} = \begin{bmatrix}
T^{-3/2} \sum a_{1t} \omega_t
\end{bmatrix} + o_p(1) = \begin{bmatrix}
\beta_3 \xi_{1t} T
\end{bmatrix} + o_p(1) \quad (A.12)
\]

where \( \theta_{1t} = T^{-3/2} \sum a_{1t} \xi_t, i=1,2. \)

Turning to the moment matrices comprising \( \Phi_c \), write

\[ Y_t^T \omega_t = (\beta_1 - \beta_1) - (\beta_2 - \beta_2) t. \] Since \( W_t = S_k Y_t^T, \)

\[ T^{-2} \sum w_t w_t^T = S_k T^{-2} \sum [\omega_t - (\beta_1 - \beta_1) - (\beta_2 - \beta_2) t] [\omega_t - (\beta_1 - \beta_1) - (\beta_2 - \beta_2) t]' S_k. \quad (A.13) \]
Using (A.12), Chebyshev's inequality, and the bound on the fourth moment of $\nu_\tau$, direct calculation of each of the nine terms in (A.13) shows that

$$
T^{-2\sum W^T_t} = c(1)[(\eta_{0T} - \theta_0 T^{1/2}) + \theta_{1T} T^{1/2} + \theta_{2T} T^{3/2})
+ \theta_{1T} T^{1/2} + \theta_{2T} T^{3/2}]^\gamma(1)' + o_p(1)
$$

(A.14)

where $\theta_0 T^{-3/2} \sum \xi_t$ and $\theta_3 T^{-3/2} \sum (t/T) \xi_t$. This can be rewritten to give the desired result,

$$
T^{-2\sum W^T_t} \xi_t^T = c(1) \eta_{0T} T^{1/2} \xi_t^T, \quad P \to 0
$$

(A.15)

where $\eta_{0T} = T^{-3/2} \sum \xi_t^T$, where $\xi_t^T$ is defined in the statement of the Lemma.

To obtain a limiting representation in terms of functionals of Wiener processes, note that $\eta_{0T} \to \int_0^1 a_i(s) B(s) ds = \theta_1$, $i = 0, \ldots, 3$, by the continuous mapping theorem, where $a_0(s) = 1$, $a_1(s) = 4s$, $a_2(s) = 6s$, and $a_3(s) = s$. Thus the terms in (A.14) converges to their counterparts expressed as $\theta_1$ and $\eta_{0T}$, which can be rewritten as $\eta_{0T} = \eta_{0T}^{1/2} \int_0^1 B_n(t) B_n^T(t) , dt$, where $B_n(t) = B_n(t) - \theta_1 \cdot \theta_2 t$.

Turning to the term $T^{-1} \sum W^T_{t-1} \Delta W^T_t$, use $\Delta W^T_t = S_k[\Delta \omega_t^{T} (\theta_2 - \theta_2)]$ to write,

$$
T^{-1} \sum W^T_{t-1} \Delta W^T_t = S_k T^{-1} \sum \omega_{t-1}^{T} (\theta_1 - \theta_1) \cdot (\theta_2 - \theta_2) (t-1)) \eta_{0T} (\theta_1 - \theta_2) S_k'.
$$

(A.16)

Expanding (A.16) using (A.12), defining $\eta_{4T} = T^{-1/2} \xi_t$ and $\eta_{5T} = T^{-1/2} (t/T) \nu_t$, and using Chebyshev's inequality, one obtains:
\[ T^{-1} \sum_{t=1}^{T} \Delta \hat{Y}_t^T = \mathcal{C}(1) \left[ \Psi_{nT} - \Theta_{1T} \Theta_{1T}' - \Theta_{2T} \Theta_{2T}' - \Theta_{0T} \Theta_{2T}' + \Theta_{1T} \Theta_{2T}' + 2 \Theta_{2T} \Theta_{2T}' \right] \mathcal{C}(1)' + M + o_p(1) \]

(A.17)

\[
- \mathcal{C}(1) \Psi_{nT} \mathcal{C}(1)' = M + o_p(1)
\]

which in combination with (A.15) gives the desired result. By direct calculation and Theorem 2.4 of Chan and Wei (1988),

\[ \Psi_{nT}^r \rightarrow \Phi_{nT}^r = \int_0^1 \Phi_{nT}^r(t) dB_n(t)' \]

(i) The proof of (i) is similar to the proof of (ii) but simpler. By assumption, \( \beta_2 = 0 \) so that \( Y_t = \beta_1 + \omega_t \). Letting \( \beta_1 = T^{-1} \sum_{t=1}^{T} Y_t \), \( Y_t = \beta_1 + \omega_t - (\beta_1 - \beta_1) \),

one obtains

\[ T^{-1} (\beta_1 - \beta_1) = T^{-3/2} \sum_{t=1}^{T} \omega_t = \beta_3 \Theta_{0T} + o_p(1) \rightarrow \beta_3 \Theta_0 . \]

Thus the limits of the two matrices comprising \( \Phi_{nT}^r \) can be computed as in the proof of (ii):

\[ T^{-2} \sum_{t=1}^{T} \omega_t \omega_t' = S_k T^{-2} \sum_{t=1}^{T} \left[ \omega_t \left( \beta_1 - \beta_1 \right) \right] \omega_t ' \left( \beta_1 - \beta_1 \right) S_k ' \]

\[ = \mathcal{C}(1) \left[ \Gamma_{nT} - \Theta_0 \Theta_0' \right] \mathcal{C}(1)' + o_p(1) \]

\[ = \mathcal{C}(1) \Gamma_{nT} \mathcal{C}(1)' + o_p(1) \]

\[ \rightarrow \mathcal{C}(1) \Gamma_{nT} \mathcal{C}(1)' \quad (A.18) \]

\[ T^{-1} \sum_{t=1}^{T} \Delta \omega_t \omega_t' = S_k T^{-1} \sum_{t=1}^{T} \left[ \omega_t - 1 \left( \beta_1 - \beta_1 \right) \right] \Delta \omega_t S_k ' \]

\[ = \mathcal{C}(1) \left[ \Psi_{nT} - \Theta_0 \Theta_0' \right] \mathcal{C}(1)' + M + o_p(1) \]

\[ = \mathcal{C}(1) \Psi_{nT} \mathcal{C}(1)' + M + o_p(1) \]

\[ \rightarrow \mathcal{C}(1) \Psi_{nT} \mathcal{C}(1)' + M \quad (A.19) \]
where $r_{nT}^\mu$ and $\Psi_{nT}^\mu$ are given in the statement of the theorem and where

\[ r_{nT}^\mu = \int_0^1 B_n^\mu(t)B_n^\mu(t)\,dt \quad \text{and} \quad \Psi_{nT}^\mu = \int_0^1 B_n^\mu(t)dB_n^\mu(t), \]

where $B_n^\mu(t) = B_n(t) - \Theta_0$. The desired result obtains from (A.18), (A.19), and the definition of $\Phi^\mu_c$.

\[ \square \]

**Proof of Theorem 5.1.**

(b.i) We prove (b.i) first, initially considering the case that $D$ and $\Pi(L)$ are known. To examine the OLS coefficient matrix based on $\hat{W}_t^\mu$, let

\[ T(\hat{\Phi}^\mu_w) = T(\hat{W}_t^\mu)^{-1}, \]

where $\hat{W}_t^\mu = T^{-2} \sum_{t=1}^T \Delta \hat{\gamma}^\mu_t$, and

\[ \hat{\gamma}^\mu_t = \Pi(L) \hat{W}_t^\mu. \]

Use (3.1) and the definition $\xi_t^\mu = \sum_{s=1}^t \eta_s$ to write $\Pi(L) \hat{W}_t^\mu = \Pi(1) \hat{W}_0 - \Pi(L) \xi_t^\mu + \xi_t^\mu$. Also let $\hat{W}_0$ and $\hat{\gamma}$ denote the coefficients from a regression of $\hat{W}_t^\mu$ onto $(1,t)$. Noting that $\Pi(L) \hat{W}_0 = \Pi(1) \hat{W}_0 - \Pi(1) \xi_t^\mu + \xi_t^\mu$, one obtains:

\[ \xi_t^\mu = \xi_t^\mu - \Pi(1) \hat{W}_0 - \Pi(1) \xi_t^\mu + \xi_t^\mu - \Pi(1) \xi_t^\mu + \xi_t^\mu. \]

Analysis like that leading to (A.12) shows that

\[ \begin{bmatrix} T^{-1}(\hat{W}_0 - \hat{W}_0) \\ T(\hat{\gamma}-\gamma) \end{bmatrix} = \begin{bmatrix} \Pi(1)^{-1} \Xi_{1T} \\ \Pi(1)^{-1} \Xi_{2T} \end{bmatrix} + o_p(1) \tag{A.21} \]

where $\Xi_{iT} = T^{-3/2} \sum_{a=1}^T \xi_t^\mu$, $i=1,2$. Using (A.21) to provide rates of convergence for $\hat{W}_0 - \hat{W}_0$ and $\hat{\gamma} - \gamma$ and applying Chebyshev's inequality, one obtains

\[ T\hat{\Phi}_T^\mu = T^{-1} \sum (\xi_t^\mu - \Xi_{1T} - \Xi_{2T}(t-1))[\eta_t^\mu - \Xi_{2T}] + o_p(1) \]

\[ = \Psi_{kT}^\mu + o_p(1) \to \Psi_k^\mu \]
\[ \psi_{IT}^r = T^{-2} \sum \xi_{t-1}^2 \xi_{1T}^r \xi_{2T}^r(t-1) + \xi_{t-1}^2 \xi_{1T}^r \xi_{2T}^r(t-1) + o_p(1) \]
\[ = \Gamma_{IT}^r + o_p(1) \Rightarrow \Gamma_k^r . \]

Thus \( T(\psi_{IT}^r - I_k) \Rightarrow \psi_k^r (\Gamma_k^r)^{-1} \).

The extension of this result to the case that \( D \) and \( \Pi(L) \) are consistently estimated (up to the normalization matrix \( R \)) parallels the proof of Theorem 3.1 (i) and is omitted.

(a.i) The proof of (a.i) is similar to the proof of (b.i), with modifications like those used to extend the proof of Lemma 5.1 (ii) to Lemma 5.1 (i).

(a.ii), (a.iii), (b.ii), (b.iii). The proofs of (a.ii), (a.iii), (b.ii), (b.iii) parallel the proofs of Theorem (3.1) (ii) and (iii) and are omitted.

**Proof of Theorem 5.2.**

The proof of Theorem 5.2 parallels the proofs of Theorem 4.1 and is omitted.
References


150D A Reexamination of Friedman’s Consumption Puzzle (February 1986), by James H. Stock.

151D Fine Tuning (April 1986) by Francis M. Bator.

152D The State of Macroeconomics (May 1986), by Francis M. Bator.

153D Functional Finance (June 1986), by Francis M. Bator.

154D The Organizational Implications of New Technologies: Remote Work Centers at AT&T Communications (September 1986), by John Paul MacDuffie and Michael Maccoby.


158D Imperfect Competition, Scale Economies, and Trade Policy in Developing Countries (March 1987), by Dani Rodrik.


161D The Dilemma of Government Responsiveness (October 1987), by Dani Rodrik and Richard Zeckhauser.

162D Promises, Promises: Credible Policy Reform Via Signaling (October 1987), by Dani Rodrik.

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167D Clearly Heard on the Street: The Effect of Takeover Rumors on Stock Prices (April 1988), By John Pound and Richard Zeckhauser.