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Coordinated Control of Networked Mechanical Systems with Unstable Dynamics

Sujit Nair and Naomi Ehrich Leonard[1]
Mechanical and Aerospace Engineering
Princeton University
Princeton, NJ 08544
USA
nair@princeton.edu, naomi@princeton.edu

Luc Moreau[2]
Mechanical Engineering Department
Eindhoven University of Technology
PO Box 513, WH -1.127
5600 MB Eindhoven, The Netherlands
L.P.M.Moreau@tue.nl

***Abstract*— In this paper we present a coordinating control law for a network of mechanical systems with unstable dynamics. The control law is derived using the Method of Controlled Lagrangians together with potential shaping designed to couple the mechanical systems. The coupled system is Lagrangian with symmetry, and energy methods are used to prove stability and coordinated behavior. The class of mechanical systems we consider includes the planar inverted pendulum on a cart as well as the spherical inverted pendulum on a 2D cart. For these examples, the control law stabilizes each inverted pendulum and coordinates the relative motion of the carts.**

1. Introduction

Coordinated motion and cooperative control have become important topics of late because of growing interest in the possibility of faster data processing and more efficient decision-making by a network of autonomous systems. For example, mobile sensor networks are expected to provide better data about a distributed environment if the sensors can be made to cooperate towards optimal coverage and efficient coordination.

Much of the recent work explores coordination and cooperative control with very simple dynamical systems, e.g., single or double integrator models (e.g., [6], [9], [10]) or nonholonomic models (e.g., [3]). These authors deliberately choose to focus on the coordination issues independent of stabilization issues.

On the other hand, for networks of autonomous systems such as unmanned helicopters or underwater vehicles, stability issues are important, and it may not always be possible (or desirable) to decouple the stabilization problem from the coordination problem. In [5], an extension to a previous work ([4]) on UAV motion planning is presented for identical multiple-vehicle stabilization and coordination. The single vehicle motion planning was based on the interconnection of a finite number of suitably defined motion primitives. The problem was set in such a way that multiple-vehicle motion coordination primitives are obtained from the single-vehicle primitives. The technique is applied to motion planning for a group of small model helicopters.

In this paper, we investigate the problem of coordination of mechanical systems with unstable dynamics and make use of the Method of Controlled Lagrangians. The Method of Controlled Lagrangians and the equivalent IDA-PBC method use energy shaping for stabilization of underactuated mechanical systems (see [1], [11] and references therein). The Method of Controlled Lagrangians provides a control law for underactuated mechanical systems such that the closed-loop dynamics derive from a Lagrangian. The approach is to choose the control law to shape the controlled kinetic and potential energy for stability.

The class of mechanical systems we consider in this paper includes the planar or spherical inverted pendulum on a cart. The goal of the development in this paper is to stabilize unstable dynamics and coordinate the actuated configuration variables across the network. So, for a network of pendulum/cart systems, the problem is to stabilize each pendulum in the upright position while synchronizing the motion of the carts.

The organization of the paper is as follows. In §2, we give a brief background on mechanical systems which satisfy the simplified matching conditions and discuss how unstable dynamics are stabilized with feedback control that preserves Lagrangian structure. In §3, we study a network of n systems, each of which satisfies the simplified matching conditions. We show that potentials that couple the individual systems can be prescribed so that the complete coupled system still satisfies the simplified matching conditions. Using this result, we choose coupling potentials in §4, and we prove stability and coordination of

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the network. Asymptotic stabilization is investigated in §5. We illustrate the theory with the example of two planar, inverted pendulum/cart systems in §6. In §7 we introduce an alternative approach to coordinating these systems which relies on a kinematic model.

2. Simplified Matching Conditions

In [1] the method of controlled Lagrangian is used to derive a control law that asymptotically stabilizes a class of underactuated mechanical systems with otherwise unstable dynamics. This class of systems satisfies a set of "simplified matching conditions", and we will denote such systems as SMC systems. SMC systems lack gyroscopic forces; the planar inverted pendulum on a cart is one such system.

Consider a mechanical system with an $(m+r)$-dimensional configuration space. Let x^α denote the coordinates for the unactuated directions with index α going from 1 to m. θ^a denotes the coordinates for the actuated directions with index a going from 1 to r. Let the Lagrangian for this system be given by

$$\begin{aligned} & L(x^\alpha, \theta^a, \dot{x}^\beta, \dot{\theta}^b) \\ & = \frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta + g_{\alpha a}\dot{x}^\alpha \dot{\theta}^a + \frac{1}{2} g_{ab}\dot{\theta}^a\dot{\theta}^b - V(x^\alpha, \theta^a) \end{aligned}$$

where summation over indices is implied, g is the kinetic energy metric and V is the potential energy. It is assumed that the actuated directions are symmetry directions for the kinetic energy, that is, we assume $g_{\alpha\beta}$, $g_{\alpha a}$, g_{ab} are all independent of θ^a.

For such a system, the simplified matching conditions are

- $g_{ab} = \text{ constant}$
- $\dfrac{\partial g_{\alpha a}}{\partial x^\delta} = \dfrac{\partial g_{\delta a}}{\partial x^\alpha}$
- $\frac{\partial^2 V}{\partial x^\alpha \partial \theta^a} g^{ad} g_{\beta d} = \frac{\partial^2 V}{\partial x^\beta \partial \theta^a} g^{ad} g_{\alpha d}$.

Satisfaction of these simplified matching conditions allows for a structured feedback shaping of kinetic *and* potential energy. In particular, a control law is given in [1] such that the closed-loop system is a Lagrangian system. The controlled Lagrangian, parametrized by constant parameters κ and ρ and by a potential term V_ϵ, is given by

$$\begin{aligned} & L_c(x^\alpha, \theta^a, \dot{x}^\beta, \dot{\theta}^b) = \\ & \frac{1}{2}\left(g_{\alpha\beta} + \rho(\kappa+1)(\kappa + \frac{\rho-1}{\rho}) g_{\alpha a} g^{ab} g_{b\beta}\right) \dot{x}^\alpha \dot{x}^\beta \\ & + \rho(\kappa+1) g_{\alpha a}\dot{x}^\alpha\dot{\theta}^a + \frac{1}{2}\rho g_{ab}\dot{\theta}^a\dot{\theta}^b - V(x^\alpha, \theta^b) - V_\epsilon(x^\alpha, \theta^b) \end{aligned}$$

where V_ϵ must satisfy

$$-\left(\frac{\partial V}{\partial \theta^a} + \frac{\partial V_\epsilon}{\partial \theta^a}\right)(\kappa + \frac{\rho-1}{\rho}) g^{ad} g_{\alpha d} + \frac{\partial V_\epsilon}{\partial x^\alpha} = 0.$$

The results in [1] further give conditions on ρ, κ and V_ϵ that ensure stability of the equilibrium in the full state space. Without loss of generality, we assume that the equilibrium of interest is the origin. We further assume that it is a *maximum* of the original potential energy V (the case when the origin is a minimum can be handled similarly). The inverted pendulum systems fall into this category. In this case, $\kappa > 0$ and $\rho < 0$ and the potential V_ϵ can be chosen such that the energy function for the controlled Lagrangian E_c has a maximum at the origin of the full state space. Asymptotic stability is obtained by adding a dissipative term to the control law which drives the controlled system to the maximum value of the energy E_c.

In [1], it is also shown how to select new, useful coordinates $(x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a)$. In particular, for any SMC system, there exists a function $h(x^\alpha)$ defined on an open subset of the configuration space of the unactuated variables such that

$$\frac{\partial h^a}{\partial x^\alpha} = \left(\kappa + 1 - \frac{1}{\rho}\right) g^{ac} g_{\alpha c}, \quad h^a(0) = 0.$$

The new coordinates are defined as

$$(x^\alpha, y^a) = (x^\alpha, \theta^a + h^a(x^\alpha)).$$

Note that if the origin is an equilibrium in the original coordinates it is also an equilibrium in the new coordinates. In these coordinates, the closed-loop Lagrangian takes the form

$$\begin{aligned} L_c = & \frac{1}{2}\left(g_{\alpha\beta} - (\kappa + \frac{\rho-1}{\rho}) g_{\alpha a} g^{ab} g_{b\beta}\right) \dot{x}^\alpha \dot{x}^\beta \\ & + g_{\alpha a}\dot{x}^\alpha \dot{y}^a + \frac{1}{2}\rho g_{ab}\dot{y}^a\dot{y}^b - V(x^\alpha, y^a - h^a(x^\alpha)) - V_\epsilon(y^a) \\ & = \frac{1}{2}\tilde{g}^i_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + \tilde{g}^i_{\alpha a}\dot{x}^\alpha\dot{y}^a + \frac{1}{2}\tilde{g}^i_{ab}\dot{y}^a_i\dot{y}^b_i \\ & - V(x^\alpha, y^a - h^a(x^\alpha)) - V_\epsilon(y^a). \end{aligned}$$

Further, after adding dissipation $u_a^{\rm diss}$, the Euler-Lagrange equations in the new coordinates become

$$\begin{aligned} EL_c(x^\alpha) &= 0 \\ EL_c(y^a) &= u_a^{\rm diss} \end{aligned}$$

where

$$EL(q) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}.$$

3. Matching for Network of SMC Systems

In this section we examine a network of n systems each of which satisfies the simplified matching conditions and determine what control design freedom remains under the constraint that the complete network dynamics are Lagrangian and satisfy the simplified matching conditions.

Consider n SMC systems and let the ith system have dynamics described by Lagrangian L_i where

$$\begin{aligned} & L_i(x_i^\alpha, \theta_i^a, \dot{x}_i^\beta, \dot{\theta}_i^b) \\ = \; & \frac{1}{2} g^i_{\alpha\beta}\dot{x}_i^\alpha\dot{x}_i^\beta + g^i_{\alpha a}\dot{x}_i^\alpha\dot{\theta}_i^a + \frac{1}{2} g^i_{ab}\dot{\theta}_i^a\dot{\theta}_i^b - V_i(x_i^\alpha, \theta_i^a) \end{aligned}$$

The Lagrangian for the total (uncontrolled, uncoupled) system is $L = \sum_{i=1}^n L_i = \frac{1}{2}\dot{\boldsymbol{x}}^T M \dot{\boldsymbol{x}} - \sum_{i=1}^n V_i(x_i^\alpha, \theta_i^a)$, where $\boldsymbol{x} = (x_1^\alpha, \ldots, x_n^\beta, \theta_1^a, \ldots, \theta_n^b)^T$, and

$$M = \left(\begin{array}{cccc|cccc} g^1_{\alpha\beta} & & 0 & & g^1_{\alpha a} & & 0 & \\ & \ddots & & & & \ddots & & \\ 0 & & g^n_{\alpha\beta} & & 0 & & g^n_{\alpha a} & \\ \hline g^1_{a\alpha} & & 0 & & g^1_{ab} & & 0 & \\ & \ddots & & & & \ddots & & \\ 0 & & g^n_{a\alpha} & & 0 & & g^n_{ab} & \end{array}\right).$$

Since each system satisfies the simplified matching conditions, $g^i_{ab} =$ constant for each $i = 1, \ldots, n$. It can be easily verified that the simplified matching conditions are satisfied for the total system L, since they are satisfied for each individual system.

For the total system, the symmetry coordinates are $(\theta_1^a, \ldots, \theta_n^b)$. As in [1], we can find a control law and a change of coordinates $(x_1^\alpha, \ldots, x_n^\beta, \theta_1^a, \ldots, \theta_n^b) \mapsto (x_1^\alpha, \ldots, x_n^\beta, y_1^a, \ldots, y_n^b)$ such that the closed-loop system is equivalent to another Lagrangian system with mass matrix and potential, respectively, given by

$$M_c = \left(\begin{array}{cccc|cccc} \tilde{g}^1_{\alpha\beta} & & 0 & & \tilde{g}^1_{\alpha a} & & 0 & \\ & \ddots & & & & \ddots & & \\ 0 & & \tilde{g}^n_{\alpha\beta} & & 0 & & \tilde{g}^n_{\alpha a} & \\ \hline \tilde{g}^1_{a\alpha} & & 0 & & \tilde{g}^1_{ab} & & 0 & \\ & \ddots & & & & \ddots & & \\ 0 & & \tilde{g}^n_{a\alpha} & & 0 & & \tilde{g}^n_{ab} & \end{array}\right), \tag{3.1}$$

$$V'_\epsilon = \sum_{i=1}^n \Big(V_i(x_i^\alpha, y_i^a - h_i^a(x_i^\alpha)) + V_{\epsilon i}(x_i^\alpha, y_i^a) \Big).$$

The control gains κ_i and ρ_i and control potentials $V_{\epsilon i}$ can be chosen such that the mass matrix M_c is negative definite and the potential has a maximum when the configuration of each system, i.e., (x_i^α, θ_i^a), is at the origin. This means the control law brings each system independently to the origin without coordination.

To determine what additional freedom exists in the choice of the control, notably in the choice of control potentials $V_{\epsilon i}$, such that the network system satisfies the simplified matching conditions, we specialize to a network of SMC systems which each satisfy the following condition.

AS. *Assume that the potential energy for each system in the original coordinates satisfies* $V_i(x_i^\alpha, \theta_i^a) = V_{1i}(x_i^\alpha) + V_{2i}(\theta_i^a)$.

The inverted pendulum examples all satisfy this assumption in the general case that the cart moves on an inclined plane. In the case that the cart moves in the horizontal plane, $V_2 = 0$.

As shown in [1], given the assumption **AS**, each $V_{\epsilon i}$ in the new coordinates can be chosen to take the form

$$V_{\epsilon i}(x_i^\alpha, y_i^a) = -V_{2i}(y_i^a - h_i^a(x_i)) + \bar{V}_{\epsilon i}(y_i^a)$$

where $\bar{V}_{\epsilon i}$ is an arbitrary function and $h_i^a(x_i)$ satisfies

$$\frac{\partial h_i^a}{\partial x_i^\alpha} = \left(\kappa_i + 1 - \frac{1}{\rho_i}\right) g_i^{ac} g^i_{\alpha c}, \quad h_i^a(0) = 0.$$

Proposition 3.1: Under assumption **AS**, the potential $V'_\epsilon = V + V_\epsilon$ satisfies the simplified matching condition with

$$\begin{aligned} V &= \sum_{i=1}^n \left(V_{1i}(x_i) + V_{2i}(y_i^a - h_i^a(x_i))\right) \\ V_\epsilon &= -\left(\sum_{i=1}^n V_{2i}(y_i^a - h_i^a(x_i))\right) + \tilde{V}_{\epsilon i}(y_1^a, \ldots, y_n^a) \end{aligned} \tag{3.2}$$

and each $\tilde{V}_{\epsilon i}$ an arbitrary function.

Proof. By [1], the simplified matching condition for the potential shaping of the total system is equivalent to

$$\frac{\partial V_\epsilon}{\partial x_i^\alpha} = \frac{\partial V}{\partial y_i^a} \frac{\partial h_i^a(x_i)}{\partial x_i^\alpha}$$

By a direct computation, one can check that both sides of the above equation equal $\frac{\partial V_{2i}}{\partial v_i^a}\frac{\partial v_i^a}{\partial x_i^\alpha}$ where $v_i^a = y_i^a - h_i^a(x_i)$ $\blacksquare$

Prop 3.1 implies that *we can couple the n vehicles in the network using the freedom in our choice of $\tilde{V}_{\epsilon i} = \tilde{V}_{\epsilon i}(y_1^a, \ldots, y_n^a)$, and the network dynamics will still satisfy the simplified matching conditions*. This result is completely independent of the degree of coupling, i.e., it extends from a network of uncoupled systems to a network of completely connected systems.

4. Stable Coordination of SMC Network

In this section we make use of Prop 3.1 to design coupling potentials $\tilde{V}_{\epsilon i}$ for stable coordination of the network of SMC systems. Here stable coordination refers to stabilization of the unstable dynamics associated with the unactuated variables x_i^α and coordination of the actuated variables θ_i^a.

The goal of coordination is to synchronize the variables θ_i^a with the variables θ_j^a for all $i, j = 1, \ldots, n$. Here, we will assume that the configuration space for the actuated variables is $\mathbb{R}^r$. We define synchronization of these variables as stabilization of $\theta_i^a - \theta_j^a = 0$ for all $i \neq j$. Since we can allow for coordinate transformations, synchronization can be interpreted to mean that we stabilize the equilibrium associated with constant relative actuated variables. For example, in the case of a network of n inverted pendulum/cart systems, we seek to stabilize each pendulum in the network in the upright position and each cart to a position that may change with time but is constant relative to all the other carts. Note that the center of mass of the network is unconstrained. For example, in the case of the n pendulum/cart network, we can consider coordinated maneuvers in which the carts move together with constant linear momentum.

To synchronize the actuated variables we use the results of Prop. 3.1 and design coupling potentials for stabilization of $y_i^a - y_j^a = 0$, for all $i \neq j$. Note that $y_i^a - y_j^a = 0$ for all $i \neq j$ if $\theta_i^a - \theta_j^a = 0$ for all $i \neq j$ and if $x_i^\alpha = 0$ for all i. This is the desired result. However, $y_i^a - y_j^a = 0$ for all $i \neq j$ under more general conditions, e.g., if $\theta_i^a - \theta_j^a = 0$ for all $i \neq j$ and $h_i(x_i^\alpha) = h_j(x_j^\alpha) \neq 0$, $i \neq j$. This more general case makes possible interesting synchronized dynamics, when we add dissipation for asymptotic stability, as will be discussed in §5.

We choose $\tilde{V}_\epsilon$ such that the closed-loop potential V'_ϵ, defined in Prop. 3.1, has a maximum when $x_i^\alpha = 0$ and $y_i^a - y_j^a = 0$ for all $i \neq j$. This is possible since from (3.2), the closed loop potential is $V'_\epsilon = \sum_{i=1}^n (V_{1i}(x_i)) + \tilde{V}_{\epsilon i}(y_1^a, \ldots, y_n^a)$ and the V_{1i} are assumed to already be maximized at $x_i^\alpha = 0$. So, for example, we could choose $\tilde{V}_\epsilon$ to be quadratic in terms of the form $(y_i^a - y_j^a)$. In this case, consider a graph with one node corresponding to each individual system in the network. There is an edge between nodes k and l if the term $(y_k^a - y_l^a)$ appears in the quadratic function $\tilde{V}_\epsilon$. Then, V'_ϵ has a maximum when $x_i^\alpha = 0$ and $y_i^a - y_j^a = 0$ for all $i \neq j$, if the graph is connected.

By coupling the systems in this way, we have introduced a network system symmetry. The closed-loop Lagrangian is invariant to translations of y_i^a, $i = 1, \ldots, n$, each by the same amount. Consider a new set of coordinates given by

$$
\begin{aligned}
&(x_1, \ldots, x_n, z_1, \ldots, z_n) = \\
&(x_1, \ldots, x_n, y_1 - y_2, \ldots, y_1 - y_n, y_1 + y_2 + \ldots + y_n).
\end{aligned}
$$

In this coordinate system, the controlled Lagrangian for the total system (with abuse of notation for V'_ϵ) is

$$
\begin{aligned}
\tilde{L}_c &= \frac{1}{2}\dot{\boldsymbol{x}}_c^T \tilde{M}_c \dot{\boldsymbol{x}}_c - V'_\epsilon(x_1, \ldots, x_n, z_1, z_2, \ldots, z_{n-1}) \\
&= \frac{1}{2}\dot{\boldsymbol{x}}_r^T \tilde{M}_{11} \dot{\boldsymbol{x}}_r + \dot{\boldsymbol{x}}_r^T \tilde{M}_{12} \dot{z}_n + \frac{1}{2}\dot{z}_n \tilde{M}_{22} \dot{z}_n \\
&\quad - V'_\epsilon(x_1, \ldots, x_n, z_1, z_2, \ldots, z_{n-1})
\end{aligned}
\tag{4.1}
$$

where $\boldsymbol{x}_c = (x_1^\alpha, \ldots, x_n^\beta, z_1^a, \ldots, z_n^b)^T$ and $\boldsymbol{x}_r = (x_1^\alpha, \ldots, x_n^\beta, z_1^a, \ldots, z_{n-1}^b)$ and

$$
\tilde{M}_c = \left(\begin{array}{c|c} \tilde{M}_{11} & \tilde{M}_{12} \\ \hline \tilde{M}_{12}^T & \tilde{M}_{22} \end{array} \right) \tag{4.2}
$$

Note that in these coordinates z_n^a is the symmetry variable. We are interested in the relative equilibria given by

$$
\begin{aligned}
&(x_1^\alpha, \ldots, x_n^\gamma, z_1^a, \ldots, z_{n-1}^c, \dot{x}_1^\alpha, \ldots, \dot{x}_n^\gamma, \dot{z}_1^a, \ldots, \dot{z}_{n-1}^c, \dot{z}_n^d) \\
&= (0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, \zeta^d)
\end{aligned}
\tag{4.3}
$$

where ζ^d corresponds to (n times) the constant velocity of the center of mass of the network.

Let $V_\mu = V'_\epsilon(x_1^\alpha, \ldots, x_n^\beta, z_1^a, \ldots, z_{n-1}^a b) + \tilde{g}^{cd}\mu_c\mu_d$ be the amended potential [7]. Here μ_a is the momentum conjugate to z_n^a at the relative equilibrium when $\dot{z}_n^a = \zeta^a$. Using the Routh criteria, one can see that the relative equilibrium is stable if the second variation of

$$
E_\mu := \frac{1}{2}\dot{\boldsymbol{x}}_r^T (\tilde{M}_{11} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{M}_{12}^T)\dot{\boldsymbol{x}}_r + V_\mu \tag{4.4}
$$

evaluated at origin is definite. Since g_{ab}^i is a constant, the second term in the amended potential V_μ does not contribute to the second variation. So it follows that the relative equilibrium with momentum μ_a is stable if $(\tilde{M}_{11} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{M}_{12}^T)$ evaluated at the origin is negative definite, since the potential is already maximum at equilibrium. We now prove that this matrix is negative definite using the following results from linear algebra.

Result 4.1: Consider the negative definite symmetric matrix

$$
T = \left(\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{12}^T & T_{22} \end{array} \right)
$$

Here, this is any partition of the matrix T. Then T_{11} and T_{22} are also negative definite.

Proof. This follows by evaluating the definite matrix T on the vectors $(x, 0)$ and $(0, y)$ respectively. ∎

Result 4.2: $T_{11} - T_{12}T_{22}^{-1}T_{12}^T$ is also negative definite.

Proof. Let $(T'_{22})^2 = -T_{22}$. Then,

$$
\begin{aligned}
(x, y)^T T(x, y) &= x^T T_{11} x + 2y^T T_{12}^T x + y^T T_{22} y \\
&= x^T (T_{11} - T_{12}T_{22}^{-1}T_{12}^T) x \\
&\quad - (T'_{22} y - T'^{-1}_{22} T_{12}^T x)^T (T'_{22} y - T'^{-1}_{22} T_{12}^T x).
\end{aligned}
$$

For any x, one can choose $y = -T_{22}^{-1}T_{12}^T x$ so that the second term is made zero. Hence, it follows that $T_{11} - T_{12}T_{22}^{-1}T_{12}^T < 0$ since the left hand side is less than zero for all nonzero vectors (x, y). ∎

Theorem 4.3: Consider a network of n SMC systems that each satisfy Assumption **AS**. Suppose for each system that the origin is an equilibrium and that the original potential energy is maximum at the origin. Consider the kinetic energy shaping defined in §3 and potential energy coupling defined above with connected graph so that the closed-loop dynamics derive from the Lagrangian $\tilde{L}_c$ given by (4.1) and the potential energy V'_ϵ is maximized at the relative equilibrium (4.3). Then, the relative equilibrium (4.3) is stable for any ζ^d.

Proof. By Results 4.1 and 4.2, $(\tilde{M}_{11} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{M}_{12}^T)$ evaluated at the origin is negative definite. Thus, the second variation of E_μ evaluated at the origin is definite. Hence, the relative equilibrium (4.3) is stable for the total network system independent of momentum value μ_a.

5. Asymptotic stability

In this section we investigate asymptotic stabilization of the coordinated network to a constant velocity (or momentum), i.e., asymptotic stabilization of the relative equilibrium (4.3). The first step is to choose a controlled dissipation by means of a Lyapunov function analysis. E_μ as defined in (4.4) has a maximum at the origin in its variables $(\boldsymbol{x}_r, \dot{\boldsymbol{x}}_r)$. One can construct a Lyapunov function which will be maximum at the origin in the variables $(\boldsymbol{x}_r, \dot{\boldsymbol{x}}_r, \dot{z}_n^d - \zeta^d)$ as follows. Let

$$\tilde{E}_\mu := E_\mu + \lambda c^{ab}(J_a - \mu_a)(J_b - \mu_b)$$

where λ is a positive constant, J_a is the momentum conjugate to z_n^a and c^{ab} is a constant negative definite matrix. $\tilde{E}_\mu$ serves as the required Lyapunov function.

Let the Euler-Lagrange equations in the original coordinates for the uncontrolled systems be

$$EL_i(x_i^\alpha) = 0 \;\; ; \;\; EL_i(\theta_i^a) = u_{a,i}^{\text{cons}} + u_{a,i}^{\text{diss}}.$$

Here, $u_{a,i}^{\text{cons}}$ is the conservative part of the control law for the i^{th} vehicle that we already have defined and $u_{a,i}^{\text{diss}}$ is the dissipative part to be designed. Then, in the new coordinates, we have

$$E\tilde{L}_c(x_i^\alpha) = 0 \;\; ; \quad E\tilde{L}_c(z_i^a) = \frac{\tilde{u}_{a,i}^{\text{diss}}}{n} \quad i = 1, \ldots, n$$

where

$$\tilde{u}_{a,i}^{\text{diss}} = \sum_{j=i+1}^{n} u_{a,j}^{\text{diss}} - (n-1)u_{a,i+1}^{\text{diss}} \quad i = 1, 2, \ldots, n-1$$

$$\tilde{u}_{a,n}^{\text{diss}} = \sum_{j=1}^{n} u_{a,j}^{\text{diss}}.$$

It can be calculated that

$$n\frac{d}{dt}\tilde{E}_\mu = u_{a,1}^{\text{diss}}\left(\sum_{j=1}^{n-1}\dot{z}_j^a + 2\lambda c^{ab}(J_b - \mu_b)\right)$$
$$+ \sum_{j=2}^{n} u_{a,j}^{\text{diss}}\left(-(n-1)\dot{z}_{j-1}^a + \sum_{k=1,k\neq j-1}^{n-1}\dot{z}_k^a + 2\lambda c^{ab}(J_b - \mu_b)\right)$$

Choose

$$u_{a,1}^{\text{diss}} = d_{ab}\left(\sum_{j=1}^{n-1}\dot{z}_j^b + 2\lambda c^{be}(J_e - \mu_e)\right)$$

$$u_{a,j}^{\text{diss}} = d_{ab}\left(-(n-1)\dot{z}_{j-1}^b + \sum_{k=1,k\neq j-1}^{n-1}\dot{z}_k^b + 2\lambda c^{be}(J_e - \mu_e)\right)$$

where d_{ab} is a positive definite, possibly x_i^α,z_i^a dependent, control gain matrix. With this dissipation, $\frac{d}{dt}\tilde{E}_\mu \geq 0$.

Let $M = \{(x, z, \dot{x}, \dot{z}) | \frac{d}{dt}\tilde{E}_\mu = 0\}$. On this set, it is easy to show that $\dot{z}_i^a = 0$ for $i = 1, \ldots, n-1$ and $J_a = \mu_a$.

By the LaSalle Invariance Principle, solutions approach the set M. The trivial solution on a constant momentum surface is contained in the set M. However, trivial solutions are not the only solutions in this set. A deeper investigation of asymptotic convergence will be pursued in future work.

6. Coordination of Two Inverted Pendulum/Cart Systems

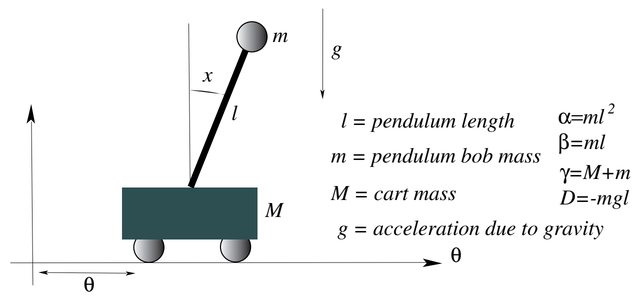

Fig. 6.1. The planar pendulum on a cart.

As an illustration, we now consider the coordination of two identical inverted pendulum/cart systems. Let the Lagrangian for each system shown in Figure 6.1 be

$$L_i = \frac{1}{2}\alpha\dot{x}_i^2 + \beta\cos x_i\dot{x}_i\dot{\theta}_i + \frac{1}{2}\gamma\dot{\theta}_i^2 + D\cos x_i \;; \;\; i = 1, 2.$$

One can see that θ_i, which is the position of the ith cart on the line is a symmetry variable. It can be easily verified that each pendulum/cart system satisfies the simplified matching conditions [1], [2]. The closed-loop Lagrangian for the total system in the coordinates $\boldsymbol{x} = (x_1, x_2, z_1, z_2) = (x_1, x_2, y_1 - y_2, y_1 + y_2)$ where $y_i = \theta_i + p\sin x_i$ and $p = \kappa + 1 - \frac{1}{\rho}$ is

$$\tilde{L}_c = \frac{1}{2}\dot{\boldsymbol{x}}^T\tilde{M}_c\dot{\boldsymbol{x}} - V'_\epsilon(x_1, x_2, z_1) \tag{6.1}$$

where $\tilde{M}_c$ is as in (4.2) and M_c is as in (3.1),

$$\tilde{g}_{\alpha\beta}^i = \alpha - (\kappa + 1 - \frac{1}{\rho})\frac{\beta^2}{\gamma}\cos^2(x_i), \;\; \tilde{g}_{\alpha a}^i = \beta\cos(x_i)$$

$$\tilde{g}_{ab}^i = \rho\gamma, \;\; V'_\epsilon = -D\cos(x_1) - D\cos(x_2) + \frac{1}{2}\epsilon D\frac{\gamma^2}{\beta^2}z_1^2.$$

with $\epsilon > 0$.

The final control law is

$$u_i = \frac{\kappa\beta\left(\sin x_i\left(\alpha\dot{x}_i^2 + \cos x_i D\right) - B_i\left(\frac{\partial V'_\epsilon}{\partial\theta_i} - u_i^{\text{diss}}\right)\right)}{\alpha - \frac{\beta^2}{\gamma}(1+\kappa)\cos^2 x_i} \tag{6.2}$$

where $B_i = \frac{1}{\rho}\left(\alpha - \frac{\beta^2\cos^2(x_i)}{\gamma}\right)$ and

$$u_i^{\text{diss}} = c\left((-1)^{i+1}\frac{\dot{z}_1}{2} + \frac{\lambda}{\rho\gamma}(J - \mu)\right).$$

This control law stabilizes the relative equilibrium corresponding to the two pendula in the upright position and the two carts moving together at a constant velocity. Simulations suggest that there are stable periodic solutions nearby the stable equilibrium. In particular, the simulation in Figure 6.2 shows asymptotic stabilization to a periodic solution in which the cart positions synchronize and the pendulum angles oscillate in phase as the whole network moves with a steady average forward velocity. In this MATLAB simulation, parameters for the pendulum/cart systems as well as values of κ, ρ and ϵ are the same as those described in [1]. Remaining control parameters are $\lambda = 2$, $\mu = -1$ kg-m/s, $c = 0.05$ kg/s.

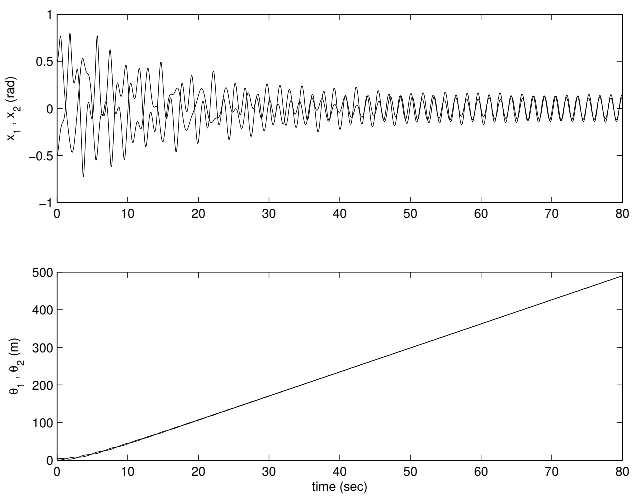

Fig. 6.2. Simulation of coordination and stabilization of two pendulum/cart systems.

7. Kinematic Coordination Approach

An alternative line of research that we are currently investigating concerns the coordination of mechanical systems via a “kinematic coordination” approach. In this approach, we first endow each individual mechanical system with a set-point controller $u_i^{\mathrm{sp}}(x_i, \theta_i, \overline{\theta}_i, \dot{x}_i, \dot{\theta}_i)$, which steers $\theta_i \to \overline{\theta}_i$ and $x_i \to 0$ when the set-point $\overline{\theta}_i$ is constant. In addition we also require that, when the set-point $\overline{\theta}_i$ is time-varying but approaches θ_i exponentially, then the mechanical system comes to a rest configuration $\theta_i \to$ constant and $x_i \to 0$. Such a set-point controller could possibly be designed via the Method of Controlled Lagrangians, at least for SMC systems.

Under these assumptions, we may then propose a kinematic coordination module that guarantees coordination of the mechanical systems on a slow time-scale. This would replace the coupling potentials described in §4. The kinematic coordination module consists of first-order dynamics for the set-points which take the following form:

$$\dot{\overline{\theta}}_i = \delta u_i^{\mathrm{comm}}(\overline{\theta}_1, \ldots, \overline{\theta}_n) + \dot{\theta}_i + A_i(\overline{\theta}_i - \theta_i).$$

where A_i are Hurwitz matrices of appropriate dimension. The first term on the right-hand side represents inter-system communication whereas the second term is decoupled from the other mechanical systems. The parameter δ is chosen to be small and enables the analysis of the closed-loop system via time-scale separation techniques.

First, ignoring the slow time-scale (that is, setting $\delta = 0$) we obtain that $\overline{\theta}_i - \theta_i$ converges exponentially to zero and thus that each individual system approaches a rest configuration $\theta_i \to$ constant and $x_i \to 0$. The reduced system on the slow time-scale then becomes a dynamical system for the set-points $\overline{\theta}_i$ which depends on the communication terms $\delta u_i^{\mathrm{comm}}(\overline{\theta}_1, \ldots, \overline{\theta}_n)$. The coordination problem now corresponds to the design of $u_i^{\mathrm{comm}}(\overline{\theta}_1, \ldots, \overline{\theta}_n)$ such that for the reduced system all set-points $\overline{\theta}_i$ converge to a common value. This is essentially a fully actuated kinematic control problem. With this approach, we expect to be able to incorporate both bi-directional and uni-directional, time-dependent communication between the agents [8].

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9. References

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