Lecture Notes 5.B[1]

Linear Transformations and the Geometry of Motion

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1 Framework Concept

An important objective in providing a framework for describing motion from shape change is to make it possible to compute in a straightforward manner where a system will end up after a given non-reciprocal shape change sequence. For studying problems in biology, a central goal is to analyze how the animal moves as a function of a given shape change. In a controller design problem, we'd like to understand how to systematically assign shape change sequences (i.e., to prescribe our control input) to effect a desired system maneuver.

Consider again the spacecraft reorientation problem discussed at the end of Lecture Notes 5.A. Recall that the problem was to prescribe commands to drive two internal wheels in such a way as to reorient the spacecraft as desired. Suppose, for example, that we have available one wheel that spins about the spacecraft's roll axis and a second wheel that spins about the spacecraft's pitch axis. By controlling the first wheel's rotation we can make the spacecraft rotate about its roll axis by any amount α_1. Similarly, by controlling the second wheel's rotation we can make the spacecraft rotate about its pitch axis by any amount α_2. Now suppose we would like to reorient the spacecraft such that it experiences a net rotation about its yaw axis, precisely the axis for which we have no wheel available.

It turns out (as we will see more systematically later) that a non-reciprocal shape change sequence of the form described at the end of Lecture Notes 5.A., with shape variables α_1 and α_2 will produce a net rotation about the spacecraft's yaw axis. This sequence could be something of the form (1) use the first wheel to rotate the spacecraft about its roll axis by $\alpha_1 = \beta$, (2) use the second wheel to rotate the spacecraft about its pitch axis by $\alpha_2 = \gamma$, (3) use the first wheel to rotate the spacecraft about its roll axis by $\alpha_1 = -\beta$ and (4) use the second wheel to rotate the spacecraft about its pitch axis by $\alpha_2 = -\gamma$. The net result for β and γ relatively small (i.e., less than 20 degrees each) is that the spacecraft will have experienced a net yaw approximately equal to $\alpha_3 = \beta\gamma$. (Note that if β and γ are given in degrees, the formula should be modified to read $\alpha_3 = (\beta\gamma)(\pi/180)$ degrees). One can test this experimentally by performing this sequence of rotations by hand on a book and then observing the final orientation as compared with the initial orientation!

The question then remains how to define a mathematical language with elements to represent roll, pitch and yaw and operations to represent the application of these elements to a spacecraft (or a book) such that the "product" of a sequence of the mathematical operations gives the mathematical element that represents the corresponding resulting physical motion. Specifically, we would like to define a roll element R as a function of α_1, i.e., $R(\alpha_1)$, a pitch element $P(\alpha_2)$ and a yaw element $Y(\alpha_3)$ together with a "multiplication" of these elements such that

$$P(-\gamma) \times R(-\beta) \times P(\gamma) \times R(\beta) = Y(\beta\gamma).$$

If the mathematical language is general enough, we will then be able to check out other shape change sequences and see what net result they produce. For example, what do you think will happen if we have a wheel about the roll and yaw axes but none about the pitch axis, and we apply a shape change sequence as above but with shape variables α_1 and α_3? A really general framework will allow us to consider more complicated problems such as the problem of reorienting *and* repositioning a hovering underwater vehicle that has a limited number of propellers.

2 Matrices

The mathematical language that we introduce here for the purpose outlined above is called *matrix algebra*. A matrix is a collection (array) of numbers much as you would see in a spreadsheet table. An individual matrix (i.e., a particular choice of the numbers in the array) will be used to represent an individual motion (e.g., roll). Multiplication of matrices will be defined so that the product of two matrices (each representing an individual motion) correctly represents the motion that results from the corresponding sequence of the two individual maneuvers performed in the appropriate order.

Matrices have associated to them a geometrical interpretation which not only makes them easier to appreciate but also more clearly illustrates their connection to shape change and motion. This geometrical interpretation is developed and emphasized in the lab.

In the lab we focus on planar motion, i.e., motion of an object on a flat plane. In this setting any given matrix represents a *linear transformation* on the plane. A transformation operates on points on the plane and changes them into new points. More concretely, a transformation assigns to any given point on the plane described by the coordinates (x, y), a new point with coordinates (x', y'). For example, a linear transformation is given by the assignment:

$$\begin{aligned} x' &= x - y \\ y' &= 2y. \end{aligned} \tag{1}$$

This means that the point $(1, 1)$ is transformed into the point $(0, 2)$, the point $(7, -3)$ is transformed into the point $(10, -6)$, the point $(0.1, 0.5)$ is transformed into the point $(-0.4, 1)$, etc.

What makes the transformation linear is that all the points in a given straight line are transformed into points that still lie on a straight line. This is useful if we want to apply the transformation to a picture (e.g., of a stick figure or a robotic vehicle) made up of points in straight lines and we don't want the transformed picture to be warped (i.e., we want straight lines to remain straight in the transformed picture).

A more general linear transformation is given by the assignment:

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy, \end{aligned} \tag{2}$$

where a, b, c, d are any real numbers. This general assignment can be written equivalently as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}. \tag{3}$$

Here, the 2 by 2 array of numbers

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is called a 2×2 matrix. The operation between the matrix and the coordinate pair (x, y) is called multiplication of the coordinate pair by the matrix. The rules of the multiplication are defined by the right hand side of (3), i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}. \tag{4}$$

We can, therefore, see that the matrix associated to the linear transformation given by (1) is

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

since according to the rules of multiplication (4)

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ 2y \end{pmatrix}.$$

A somewhat more general linear transformation assignment is given by

$$\begin{aligned} x' &= ax + by + e \\ y' &= cx + dy + f, \end{aligned} \tag{5}$$

where e and f are any real numbers. Essentially, the addition of e and f introduces a fixed translation of every point in the x and y direction after the transformation associated with a, b, c, d is done. In the case that $a = d = 1$ and $b = c = 0$, the transformation reads

$$\begin{aligned} x' &= x + e \\ y' &= y + f. \end{aligned}$$

This transformation simply translates every point (x, y) by the same amount e in the x direction and f in the y direction.

We can again write the linear transformation (5) equivalently using a matrix. In this case we use a 3×3 matrix, i.e., a 3 by 3 array of numbers as follows:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} := \begin{pmatrix} ax + by + e \\ cx + dy + f \\ 1 \end{pmatrix}. \tag{6}$$

Note that we have taken a coordinate pair (x, y) and turned it into a coordinate triplet of the form $(x, y, 1)$. The rules for multiplication of the triplet by the 3×3 matrix are given by the right hand side of (6), i.e.,

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} := \begin{pmatrix} ax + by + e \\ cx + dy + f \\ 1 \end{pmatrix}. \tag{7}$$

Thus, the matrix associated with the transformation defined by

$$\begin{aligned} x' &= 7x - 2.3y + 4.1 \\ y' &= 3.3x + 0.1y - 16 \end{aligned}$$

is given by

$$\begin{pmatrix} 7 & -2.3 & 4.1 \\ 3.3 & 0.1 & -16 \\ 0 & 0 & 1 \end{pmatrix}$$

since according to the multiplication rules of (7)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 7 & -2.3 & 4.1 \\ 3.3 & 0.1 & -16 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 7x - 2.3y + 4.1 \\ 3.3x + 0.1y - 16 \\ 1 \end{pmatrix}.$$

3 Geometry

While it is clear that e and f in the general linear transformation lead to translations of points, it is not immediately clear how different choices of a, b, c and d in the linear transformation affect the position of points. The role of a, b, c and d is explored in the lab by looking at the geometry of various standard choices. In the lab you can type in any choice of a, b, c and d and see how a given picture is transformed.

Rotation of points (x, y) about a fixed point (for example, the origin (0,0) of the plane) is a particularly useful category of linear transformations in the study of motion control. For rotations in the plane, a particular choice of a, b, c and d (i.e., a particular choice of a 2×2 matrix) is associated with a rotation of any point (x, y) about $(0, 0)$ by a particular angle

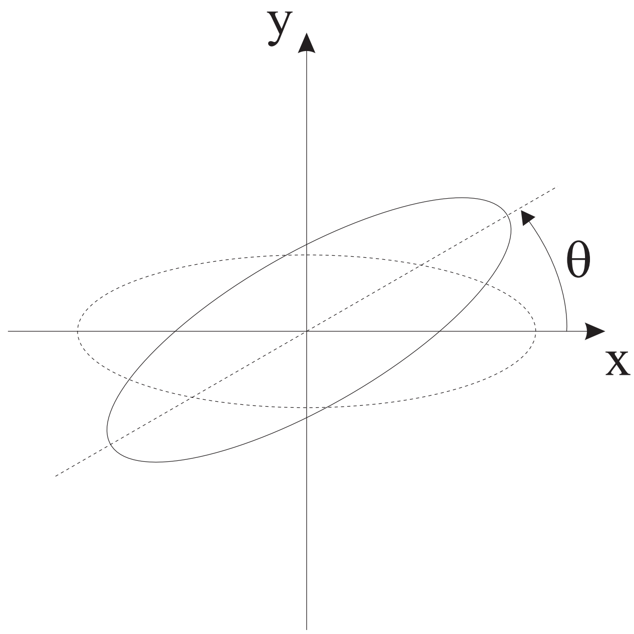

Figure 1: Rotation in the plane by θ degrees.

θ as in Figure 1. For example, if we are interested in the orientation of a planar spacecraft we could use a 2×2 linear transformation matrix to describe at what angle θ the spacecraft has rotated relative to the original x and y axes of the plane.

The 2×2 matrix that provides rotation by θ degrees is given by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

For example, if $\theta = 90$ degrees, then $\cos\theta = \cos(90) = 0$ and $\sin\theta = \sin(90) = 1$. Therefore, the transformation matrix that rotates every point 90 degrees about $(0, 0)$ is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

So, the nose of the spacecraft which originally lies on the x-axis at the point $(1, 0)$ will be transformed to the new location defined by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Similarly, the tail of the spacecraft which originally lies on the x-axis at the point $(-1, 0)$ will be transformed to the new location defined by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

I.e., the nose and the tail of the spacecraft will be rotated so that they lie on the y-axis as in Figure 2. The rest of the points on the spacecraft will be rotated accordingly such that the size and shape of the spacecraft will be unchanged.

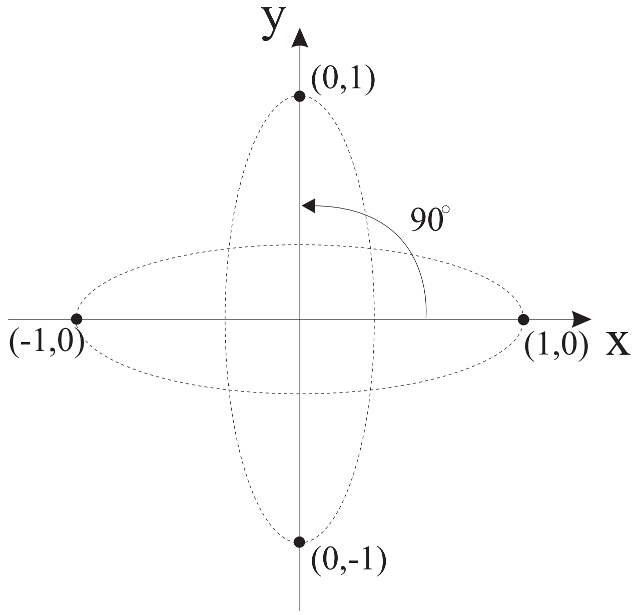

Figure 2: Rotation in the plane by 90 degrees.

It is interesting to note that we need not restrict ourselves to planar rotations. In the case of a three-dimensional spacecraft, we can identify a rotation with a 3×3 matrix. In this case we are interested in how the 3×3 matrix transforms a triplet (x, y, z) which gives the coordinates of a point in three-dimensional space. The rules of multiplication in this case are a generalization of the rules provided above. The new coordinates (x', y', z') are given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a & b & h \\ c & d & i \\ j & k & l \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} ax + by + hz \\ cx + dy + iz \\ jx + ky + lz \end{pmatrix}.$$

So, for example, the 3×3 matrix that corresponds to rotation of the spacecraft in the x-y plane by an angle α_3 (see Figure 3) is

$$\begin{pmatrix} \cos\alpha_3 & -\sin\alpha_3 & 0 \\ \sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Any point (x, y, z) on the spacecraft given is transformed into the point

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha_3 & -\sin\alpha_3 & 0 \\ \sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\alpha_3 - y\sin\alpha_3 \\ x\sin\alpha_3 + y\cos\alpha_3 \\ z \end{pmatrix}.$$

Note that this rotation does not affect the height of the spacecraft i.e., the value of z is unchanged by the transformation. If $\alpha_3 = 90$ degrees, the nose of the spacecraft originally at point $(1, 0, 0)$ is transformed into the point

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} (1)\cos(90) - (0)\sin(90) \\ (1)\sin(90) + (0)\cos(90) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

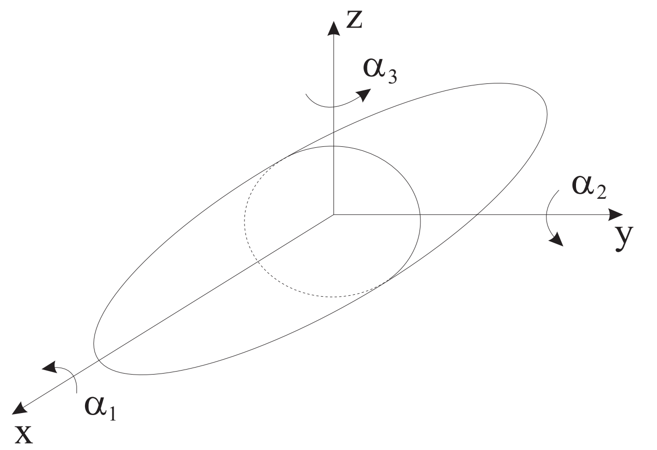

Figure 3: Three rotations of a spacecraft in 3D.

and the tail of the spacecraft originally at the point $(-1, 0, 0)$ is transformed into the point

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} (-1)\cos(90) - (0)\sin(90) \\ (-1)\sin(90) + (0)\cos(90) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

The matrix that corresponds to rotation of the spacecraft in the x-z plane by an angle α_2 is

$$\begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & 0 & \cos\alpha_2 \end{pmatrix}.$$

Note that this transformation when applied to a point (x, y, z) leaves the value of y unchanged (as it should since the spacecraft is turning *about* the y-axis). Similarly, the matrix that corresponds to rotation of the spacecraft in the y-z plane by an angle α_1 is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & -\sin\alpha_1 \\ 0 & \sin\alpha_1 & \cos\alpha_1 \end{pmatrix}.$$

This transformation when applied to a point (x, y, z) leaves the value of x unchanged (as it should since the spacecraft is turning *about* the x-axis).

There are lots of other interesting linear transformations. Linear transformations in the plane including reflection, contraction/expansion, and shear are detailed in the lab.