# A Framework for Inductive Logic[1]

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The first essay on our web site provides an informal introduction to the acceptance-based inductive logic elaborated in the sequel. We get down to business in the present essay. An inductive paradigm is defined formally and some of its basic properties are investigated. The discussion presupposes only elementary mathematical logic (as in [8, 9]).

A roadmap might be useful. Section 1 establishes conventions and notation necessary for the rest of the discussion. Section 2 formalizes the inductive game that underlies our theory. The first two sections are thus indispensable for further reading. They suffice for understanding much of the discussion in the third essay, which bears on belief revision. Hence, you may wish to skip the remaining sections of the present essay. Section 3 makes some observations about the model of inquiry advanced in Section 2. The main technical work is carried out in Section 4, which offers a characterization of the winnable inductive games. As noted below, these results serve as a tool for subsequent analyses but need not be assimilated by readers who plan to skirt the proofs of theorems. Many of our results bear on inductive games with special properties that simplify their form. Section 5 discusses two kinds of special games. Efficient inquiry is the topic of Section 6. Scientists that can be simulated by a computable process occupy Section 7. Proofs are relegated to Section 8.

Be advised that the game terminology of the first essay will give way to more sober expressions in the formal definitions of the present essay. For example, what were “games” will be called “problems,” and they will be “solved” rather than “won.” Games will still be invoked, however, for informal explanations.

## 1. Logical setting

We start, inevitably, with definitions and conventions. We have endeavored to find the right compromise between generality and naturalness. Some generalizations of our basic setup are discussed in the fourth essay. Others will be indicated along the way in this essay (not all have been explored).

Section 2 defines a "paradigm" of inductive inference, by which is meant a formal model of scientific inquiry. The paradigm is built from concepts and terminology adapted from mathematical logic. The latter material is presented in the present section.

## 1.1. A first-order framework

### 1.1.1. Language

Central to all our concepts is the choice of a first-order language. The language serves to delimit the kind of reality the scientist might confront and the kind of data he might receive about it. We use $\mathcal{L}$ to denote this language, and make the following assumptions about it.

(a) $\mathcal{L}$ is countable.

(b) $\mathcal{L}$ includes the identity symbol $=$,

(c) $\mathcal{L}$ includes a countably infinite set of (individual) variables (distinct from all the other symbols).

The set of variables will be denoted by *Var*. The variables themselves are $v_0, v_1, v_2 \cdots$, sometimes abbreviated to $x, y, z$ when convenient.

To finish specifying $\mathcal{L}$, it remains to choose its nonlogical vocabulary, denoted by **Sym**. But how do we choose the members of **Sym**? Our approach is to leave **Sym** as a free parameter of inductive logic. Thus, you may choose whatever members of **Sym** seem appropriate, and then interpret our theory as relating to that choice. Some of our results impose special hypotheses on **Sym** (typically, that it include predicates of various arities). Theorems stated without such hypotheses are true for any choice of (countable) **Sym**.

We summarize as follows.

(1) NOTATION:

(a) Our first-order language is called $\mathcal{L}$. It is countable and decidable, and includes identity.

(b) The variables of $\mathcal{L}$ are $v_0, v_1, v_2 \cdots$, collectively denoted by *Var*.

(c) The nonlogical vocabulary of $\mathcal{L}$ is brought together in **Sym**. It consists of predicates and function symbols of various arities, along with constants. Any of these sets of symbols may be empty, finite, or countably infinite. We also assume that **Sym** is computably decidable.

By the latter clause we mean that it must be possible to determine via an algorithm whether a given symbol belongs to the class of unary predicates, etc. With this notation in hand, we can think of **Sym** and *Var* as fixed throughout the discussion.

Some familiar notation concerning our language $\mathcal{L}$ will be used in what follows.

(2) NOTATION:

(a) $\mathcal{L}_{form}$ denotes the set of formulas of $\mathcal{L}$.

(b) $\mathcal{L}_{sen}$ denotes the set of sentences of $\mathcal{L}$ (no free variables).

(c) $\mathcal{L}_{atomic}$ denotes the set of atomic formulas of $\mathcal{L}_{form}$.

(d) $\mathcal{L}_{basic}$ denotes the set of basic formulas of $\mathcal{L}_{form}$.[1]

(e) The set of variables occurring free in $\varphi \in \mathcal{L}_{form}$ is denoted $Var(\varphi)$.

[1]Reminder: a formula is *basic* iff it is an atomic formula or the negation of an atomic formula.

(f) An $\exists$ formula is a formula equivalent to a formula in prenex form whose quantifier prefix is limited to existentials. The same terminology applies to $\exists\forall$ formulas and sentences, etc.

(g) Given $\varphi \in \mathcal{L}_{form}$ and variables $x, y$, the expression $\varphi[y/x]$ denotes the result of substituting $y$ for every free occurrence of $x$ in $\varphi$.

### 1.1.2. Potential data

There is another set to be fixed at the outset of our discussion. Recall our picture of scientific inquiry. Nature provides the scientist with an enumeration of facts about the reality chosen to be "actual." (See the first essay, Section 3.) But which facts get enumerated? In the first essay we took them to be basic formulas. This is a natural choice since it supports the picture of data being the affirmation or denial of "atomic facts." Such a picture may be excessively positivistic, however, and our results typically support other choices. Often it suffices that the data merely include the basic formulas, or even just the atomic formulas. Other times it is only necessary that the data be closed under negation. We therefore proceed as for **Sym**, namely, by fixing a set of formulas to be used as data. The reader can choose whatever set she likes for this purpose provided that it meets a few conditions. We proceed officially as follows.

(3) Notation: We fix **Obs** $\subseteq \mathcal{L}_{form}$ to serve as potential observations available to scientists. It is assumed that **Obs** is nonempty, and that $\varphi[y/x] \in$ **Obs** for every $\varphi \in$ **Obs** and $x, y \in Var$.

The last clause means that **Obs** is not biased about variables. If $Rv_0v_1$ is a potential observation then so is $Rv_{18}v_5$. Which (if either) of the formulas ends up on Nature's list depends on what the variables name (see below). In general terms, Nature lists every

$\varphi \in$ **Obs** that is made true by the reality chosen at the outset of the game. If **Obs** consists of the existential formulas, for example, then Nature reveals to the scientist an enumeration of the true existential formulas.[2]

Similarly to **Sym**, some of our results depend on assumptions about **Obs** that go beyond (3). Stronger assumptions will appear as needed in the formulation of theorems. One condition occasionally needed is that **Obs** be "closed under negation." Let us be clear about what this is supposed to mean.

(4) DEFINITION: Let **Obs** $\subseteq \mathcal{L}_{form}$ be given. We say that **Obs** is *closed under negation* just in case for every $\varphi \in$ **Obs**, there is $\psi \in$ **Obs** that is logically equivalent to $\neg\varphi$.

The point is that **Obs** can be closed under negation without being "closed under $\neg$." In the latter case, every $\varphi \in$ **Obs** would be accompanied by $\neg\varphi$, $\neg\neg\varphi$, etc. To be closed under negation it suffices that there be just one copy of each polarity of $\varphi$.

Certain choices of **Obs** can be used to model the distinction between "theoretical" and "observational" vocabulary. For example, it is allowed that certain predicates of **Sym** (e.g., "is an electron") appear in no formula of **Obs**. Such missing predicates might be considered theoretical in the sense that the "raw" data provide no direct information about their application. Such a distinction, of course, yields only a crude approximation to scientific inquiry. In real inquiry, the observational/theoretical distinction is not fixed and sharp (if intelligible at all), but shifts with the character and stage of investigation. It may nonetheless be interesting to observe that many of our theorems make no assumptions about **Obs** [beyond (3)]. They are therefore true for any proposed distinction between observational and theoretical vocabulary.

[2] Inductive paradigms with data varying in complexity were first explored in [13].

Similar remarks apply to the issue of missing data. For example, to model a situation in which facts of the form $f(x) = g(x)$ are not shown to scientists, it suffices to exclude all such formulas from **Obs**. This is not the same as excluding formulas with any occurrence of $f$ and $g$. The latter exclusion is the same as declaring $f$ and $g$ to be theoretical vocabulary whereas all that is intended in the present case is that certain facts involving these symbols be unavailable. Again, this kind of case is embraced by theorems formulated without assumptions on **Obs**.

We admit to fondness for the choice **Obs** $= \mathcal{L}_{basic}$. Our inductive logic was originally developed under this hypothesis, then generalized subsequently. Many of the examples and results below assume that **Obs** $= \mathcal{L}_{basic}$.

### 1.1.3. Structures

We turn now to the semantic side of our first-order framework. With one exception, our concepts are standard. Thus, structure $\mathcal{S}$ is a model of $\Gamma \subseteq \mathcal{L}_{form}$ just in case there is an assignment $h : Var \to |\,\mathcal{S}\,|$ with $\mathcal{S} \models \Gamma[h]$; in this case $\mathcal{S}$ is said to "satisfy" $\Gamma$.[3] As usual, a set $\Gamma$ of formulas implies a set $\Delta$ just in case for every structure $\mathcal{S}$ and every assignment $h : Var \to |\,\mathcal{S}\,|$, if $\mathcal{S} \models \Gamma[h]$ then $\mathcal{S} \models \Delta[h]$. We often rely on the following notation.

(5) NOTATION:

(a) The class of models of $\Gamma \subseteq \mathcal{L}_{form}$ is denoted $MOD(\Gamma)$. In particular $MOD(\emptyset)$ is the class of all structures.

(b) For $\varphi \in \mathcal{L}_{form}$, we write $MOD(\varphi)$ in place of $MOD(\{\varphi\})$.

(c) A class of structures of the form $MOD(\Gamma)$ for some $\Gamma \subseteq \mathcal{L}_{form}$ is called *elementary*.

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[3] The symbol $|\,\mathcal{S}\,|$ denotes the domain of the structure $\mathcal{S}$.

An example of an elementary class may be helpful. First, recall that a *total order* is reflexive, connected, transitive, and antisymmetric (like $\leq$ on the integers). To assert within $\mathcal{L}$ that a binary relation symbol $R$ stands for a total order, it suffices to write the formulas:

(6) $\forall x Rxx$ (reflexivity)
$\forall x \forall y (x \neq y \rightarrow (Rxy \vee Ryx))$ (connectedness)
$\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$ (transitivity)
$\forall x \forall y ((Rxy \wedge Ryx) \rightarrow x = y)$ (antisymmetry)

Let $T$ be the set of these four formulas. Then $MOD(T)$ is the class of structures that interpret $R$ as a total order, and this class is elementary.

We often rely on two familiar facts about first-order logic, namely, the compactness and Löwenheim-Skolem theorems. As a reminder, we state them here.

(7) Theorem: (Compactness) Let a set $\Delta$ of formulas be given.

(a) $\Delta$ is satisfiable iff every finite subset of $\Delta$ is satisfiable.

(b) $\Delta$ implies a given formula $\varphi$ iff some finite subset of $\Delta$ implies $\varphi$.

(8) Theorem: (Löwenheim-Skolem theorem, downward) A set $\Delta$ of formulas is satisfiable iff there is a structure with countable domain that satisfies $\Delta$.[4]

There is one liberty we take with standard terminology. It simplifies much of our discussion to limit attention to countable structures that interpret **Sym**, that is, to structures

[4] We depend here on Convention (1), requiring $\mathcal{L}$ to be countable.

with finite or denumerable domains. In the fourth essay we generalize our paradigm to structures of arbitrary cardinality, but until then the countability assumption is pivotal. So we record the following convention.

(9) CONVENTION: By "structure" will always be meant a *countable* structure that interprets the symbol set **Sym**. Similarly, when we write $MOD(\Gamma)$ for some $\Gamma \subseteq \mathcal{L}_{form}$, we refer to the class of *countable* structures that satisfy $\Gamma$. Such a class is still called "elementary."

Let $T$ be the set of formulas in (6). Then according to our new convention, $MOD(T)$ is the class of total orders *over a countable domain* with respect to the relation symbol $R$, and this class is elementary. In other words, Convention (9) means: just forget about uncountable domains (until the fourth essay).

Does Convention (9) perturb the definition of implication between formulas (since there are fewer potential counterexamples to an implication)? No. Theorem (8) ensures that implication is the same relation between sets of formulas whether are not we limit attention to countable domains.

In the way of background logic, this is all we need to get started.

## 2. The paradigm

The present section formalizes the inductive game described in Section 3 of the first essay. To proceed, we step through five components of the game, defining each within the first-order setting established above. The components are as follows.

(a) *Worlds*, that is, the potential realities from which Nature chooses one to be "actual."

(b) *Problem*, that is, a partition of potential realities that defines the idea of a "correct conjecture."

(c) *Environment*, that is, a stream of clues made available about reality.

(d) *Scientist*, that is, a method for converting clues into conjectures.

(e) *Success*, that is, a criterion for determining who wins the game.[5]

## 2.1. Worlds

This one is easy. The potential realities of our paradigm are all the (countable!) structures that interpret the symbol set **Sym**. Any such structure might turn out to be the scientist's world, so the scientist's question is always roughly: "What's true in my structure?"

## 2.2. Problems

It is standard fare in philosophy to construe a proposition as a collection of possible worlds [6]. The proposition corresponding to the assertion that lions are carnivores, for example, is identified with the collection of worlds in which lions *are* carnivores. To exploit this snappy terminology (without taking a position on the associated philosophy), recall that worlds in the present setting are just structures. We therefore take a proposition to be a collection of structures.

When does it make sense to call a proposition "true?" Well, a proposition is true if it is entailed by the "facts." The facts are embodied by the structure that Nature chose to be actual. Moreover, entailment between propositions comes down to inclusion: one

[5]The idea of analyzing inductive paradigms into these components is due to [17].

proposition entails another if the first is a subclass of the second.[6] Putting the pieces together, proposition $P$ is true iff $\{\mathcal{S}\} \subseteq P$, where $\mathcal{S}$ is Nature's choice of structure (the "real world"). That is, proposition $P$ is true iff $\mathcal{S} \in P$ for this $\mathcal{S}$.

By a "problem" is meant a choice among mutually exclusive propositions. Intuitively, the problem is to figure out which proposition is true. Let us record these concepts officially.

(10) DEFINITION: A nonempty class of structures is a *proposition*. A *problem* is a nonempty collection of disjoint propositions.

An example will make the definition clearer.[7]

(11) EXAMPLE: Suppose that **Sym** consists of a binary predicate $R$. Let $T$ be the theory of total orders (with respect to $R$) with either a least point or a greatest point. but not both.[8] Let $\theta = \exists x \forall y Rxy$ and $\mathbf{P} = \{MOD(T \cup \{\theta\}), MOD(T \cup \{\neg\theta\})\}$. Then $\mathbf{P}$ is a problem consisting of the propositions $MOD(T \cup \{\theta\})$ and $MOD(T \cup \{\neg\theta\})$.

The two propositions in $\mathbf{P}$ are the class of total orders with a least but no greatest point and the class of total orders with a greatest but no least point. In this example $\mathbf{P}$ is composed of propositions that are elementary classes, i.e., specified by sets of sentences. This was the kind of case envisioned in the informal discussion of the first essay (Section 3). In the

[6]This definition of entailment generalizes the usual idea of logical implication between formulas. In the more general setting we don't need formulas to name classes of structures, which are then compared via inclusion. Instead of relying on formulas, the classes themselves serve as the relata of implication.

[7]We love this example. It already figured in the first essay as the game defined by Partition (9). It appears again as we progress.

[8]More explicitly, $T$ can be understood as the set of formulas in (6) along with the formula $(\exists x \forall y Rxy \vee \exists x \forall y Ryx) \wedge \neg(\exists x \forall y Rxy \wedge \exists x \forall y Ryx)$.

present, more general setting, problems need not be elementary classes since they are just arbitrary collections of structures. For example, let $P_{<\omega}$ be the collection of structures whose domains are finite. It is well known that $P_{<\omega}$ is not elementary.[9] Yet $P_{<\omega}$ is a perfectly legal proposition and may appear in a problem (e.g., the problem consisting of $P_{<\omega}$ and the collection of structures with countably infinite domain).

You see that a problem is a partition of some class of structures. The partitioned class is the union of the propositions that make it up. The cells of the partition are the propositions themselves. In this respect, our set-up is similar to that of the Bayesians. A problem is a partition over an event-space, where the events are structures and each cell of the partition is a proposition. A problem thus embodies the question: which of the propositions is true in my world? The partition embodied by a problem can be finite or infinite. It can even have just one member, although this makes the problem trivial. Notice, by the way, how Example (11) makes explicit the assumption we needed about **Sym**, namely, that it consists of just the binary relation symbol $R$.

Finally, it is worth pointing out that Definition (10) rules the the empty set out of the realm of propositions. The empty set might have been named "the contradictory proposition" but such a status leads to complications elsewhere. So we leave it as a non-proposition.

## 2.3. Environments

Let us now consider the information made available to a scientist about a given structure (namely, the structure chosen as the "actual" world). The information is organized as a stream of formulas called an "environment."[10] To explain environments we must first explain how Nature refers to the elements in her chosen structure.

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[9] That is, there is no $\Gamma \subseteq \mathcal{L}_{form}$ with $MOD(\Gamma) = P_{<\omega}$. This fact follows from Theorem (7); see [8, IV.3].

[10] We might have used the term "data-environment" except that it is cumbersome. Another possibility would be "data stream." Forget these possibilities, however; just remember "environment."

### 2.3.1. Variables as temporary names

Similarly to the informal games introduced in the first essay, each element of the structure's domain is given a temporary name, and the variables of $\mathcal{L}$ are used for this purpose. Tagging every member of the domain with one or more variables amounts to an "assignment" in the sense of model theory. Since every member must get tagged, the assignment is onto the domain. Officially:

(12) DEFINITION: Given structure $\mathcal{S}$, a *full assignment* to $\mathcal{S}$ is any mapping of *Var* onto $|\mathcal{S}|$.

Thus, a full assignment $h$ to $\mathcal{S}$ provides temporary names for all the elements of $|\mathcal{S}|$, namely, $s \in |\mathcal{S}|$ is assigned nonempty $h^{-1}(s) \subseteq \mathit{Var}$.

Suppose that Nature chooses a full assignment $h$ to name the domain elements of her chosen structure $\mathcal{S}$. She is then required to provide information about the interpretation $(\mathcal{S}, h)$. For this purpose, she lists the members of **Obs** that are true in $(\mathcal{S}, h)$. Such is the essential idea behind the definition of an "environment."

### 2.3.2. Pauses and data content

There is a slight complication, however. It will be technically useful to allow the flow of data to be interrupted by pauses from time to time. Perhaps the pauses reflect the scientist's sleep or budgetary cycle, but whatever their provenance, pauses are moments at which no new data are presented. They will be represented by the special symbol $\sharp$, assumed to be different from every other symbol appearing in $\mathcal{L}$. Officially:

(13) CONVENTION: We fix a symbol $\sharp$, assumed to be disjoint from every symbol (logical and nonlogical) appearing in $\mathcal{L}$.

Since pauses have no content, we'll often need to ignore them when considering data. The following definition provides what's needed.

(14) DEFINITION: Let $s$ be a sequence (finite or infinite) over $\mathcal{L}_{form} \cup \{\sharp\}$. We let $content(s)$ denote the range of the sequence without $\sharp$. That is, $content(s)$ is the set of all formulas that occur in $s$.

For example, if $s = \langle Pv_2,\ \sharp,\ Rv_1v_3\rangle$ then $content(s) = \{Pv_2,\ Rv_1v_3\}$.

### 2.3.3. Definition of environments

At last we can define the stream of data made available by Nature. Such streams are called "environments."

(15) DEFINITION: Let structure $\mathcal{S}$ and full assignment $h$ to $\mathcal{S}$ be given.

(a) An *environment* for $\mathcal{S}$ and $h$ is an infinite sequence $e$ such that

$$content(e) = \{\beta \in \mathbf{Obs} \mid \mathcal{S} \models \beta[h]\}.$$

(b) An *environment* for $\mathcal{S}$ is an environment for $\mathcal{S}$ and $h$, for some full assignment $h$ to $\mathcal{S}$.

(c) An *environment* is an environment for some structure.

(d) An *environment* for proposition $P$ is an environment for some $\mathcal{S} \in P$.

(e) An *environment* for problem $\mathbf{P}$ is an environment for some $P \in \mathbf{P}$.

Here are some examples. Recall that $N$ denotes the set $\{0, 1, 2, 3 \cdots\}$ of natural numbers.

(16) EXAMPLE: Suppose that binary predicate $R$ is the only member of **Sym**, and that structure $\mathcal{S}$ with $|\,\mathcal{S}\,| = N$ interprets $R$ as $<$.

(a) Suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$, the set of basic formulas. Suppose also that full assignment $h$ to $\mathcal{S}$ is $\{(v_i, i) \,|\, i \in N\}$. Then one environment for $\mathcal{S}$ and $h$ begins this way:

$$v_3 \neq v_4,\ \neg Rv_0v_0,\ Rv_1v_9,\ v_9 = v_9,\ Rv_7v_8,\ v_0 \neq v_3,\ v_5 = v_5,\ \neg Rv_{23}v_8,\ \cdots$$

If full assignment $g$ to $\mathcal{S}$ is $\{(v_{2i}, i), (v_{2i+1}, i) \,|\, i \in N\}$ then one environment for $\mathcal{S}$ and $g$ begins this way:

$$v_2 = v_3,\ \neg Rv_4v_5,\ Rv_1v_9,\ v_9 = v_9,\ Rv_7v_{19},\ v_0 \neq v_3,\ \neg Rv_{33}v_2,\ \neg Rv_{23}v_8,\ \cdots$$

If $P$ is the proposition containing every strict total order, then this same environment is for $P$.[11] If **P** is a problem that includes $P$ as a component proposition, then the environment is also for **P**.

(b) Now suppose that $\mathbf{Obs} = \mathcal{L}_{atomic}$, and that $h$ is as above. Then one environment for $\mathcal{S}$ and $h$ begins this way:

$$Rv_1v_9,\ v_9 = v_9,\ Rv_7v_8,\ v_5 = v_5,\ Rv_2v_3,\ Rv_7v_8 \cdots$$

Let us note confusion that can be nipped in the bud. The assumption that $\mathbf{Obs} = \mathcal{L}_{basic}$ doesn't mean that a given environment includes every basic formula. Only the basic formulas true in the underlying interpretation (structure plus assignment) show up in the environment.

### 2.3.4. Remarks about environments

As a final aid to intuition, suppose that **Sym** contains the unary predicate $H$, the binary predicate $R$, and the individual constant $c$. Suppose also that $\mathbf{Obs} = \mathcal{L}_{basic}$. Then, abstract-

[11] A *strict total order* is irreflexive, connected, transitive, and asymmetric (like $<$ on the integers).

ing away from the arbitrary choice of temporary names for domain elements (embodied in the underlying full assignment), an environment provides information like this:[12]

> "The first object encountered falls under $H$. The second object encountered does not fall under $H$, and is related to the first object by $R$. The third object encountered is identical to the first object. The fourth object encountered is not identical to the second object, and is not related to the third object by $R$. The thing denoted by $c$ is related to the second object. ..."

Here we picture temporary names as arising from ordinal position in the data stream. Alternatively, we can imagine the scientist affixing numbered *PostIt* notes to each object falling into his hands (each note has a unique number). Sometimes he discovers that the same object has come back to him with multiple notes, leading to the affirmation of an identity ("the object with PostIt #47 is the same object as the one with PostIt #82"). In the other cases, he affirms diversity ("the object with PostIt #19 is different from the object with PostIt #22"). The point is that there is nothing essential about our use of variables as temporary names. Any scheme that allows the scientist to keep track of objects would do as well. In the same connection, let us observe that our theory would not change if we allowed Nature to use only a subset of the variables as names, or to use a new set of constants for this purpose. The use of full assignments in Definition (15) simply cuts down clutter later on. We could also require Nature to use $v_0$ for the first name to appear in the data, $v_1$ for the second name, etc. Our theory would not be affected thereby.

Before returning to technical matters, let us address another conceptual issue. In our inductive games, environments are controlled by Nature. The scientist plays a passive role inasmuch as he receives but does not generate data. An alternative picture would empower the scientist at each stage to choose a formula. Nature would then have to declare it either

[12] The present remarks are similar to those on page 19 of the first essay.

true or false. One way to formalize the latter picture is explored in [16, §3.4.3], but the matter is not pursued here. To keep things simple, we focus on the passive model of data collection represented by the environments of Definition (15). Do not feel badly for the scientist. He will have enough to do!

### 2.3.5. The information available in environments

An environment for a structure offers considerable information about that structure. Indeed, if the environment includes every true atomic formula (relative to some full assignment), then it essentially determines the underlying structure. This is the content of the following lemma. Its proof is easy (and also an immediate consequence of [12, Prop 3.2(i)]).

(17) LEMMA: Let structures $\mathcal{S}$ and $\mathcal{T}$ be given.

(a) If $\mathcal{S}$ and $\mathcal{T}$ are isomorphic then the set of environments for $\mathcal{S}$ is identical to the set of environments for $\mathcal{T}$.

(b) Suppose that $\mathcal{L}_{atomic} \subseteq \mathbf{Obs}$. If some environment is for both $\mathcal{S}$ and $\mathcal{T}$ then $\mathcal{S}$ and $\mathcal{T}$ are isomorphic.

Lemma (17)b is also true if it is assumed that $(\mathcal{L}_{basic} - \mathcal{L}_{atomic}) \subseteq \mathbf{Obs}$.

The lemma does not say that a scientist can infer the underlying structure (up to isomorphism) by progressively examining an environment for it (assuming $\mathcal{L}_{atomic} \subseteq \mathbf{Obs}$). This is because no proper initial segment of an environment need provide enough information to pin down the structure. For example, no finite piece of the environment indicated in Example (16)a reveals whether the underlying structure has a least point or a greatest point. (See also the discussion on page 20 of the first essay.)

### 2.3.6. Data

Environments present their data neatly, offering one formula (or the pause symbol) per position. Nothing hinges on such tidiness. We could just as well allow arbitrary finite sets of formulas in the positions currently reserved for just single formulas (in this case, pauses would correspond to the empty set). Sticking with our current conception, we will often need to denote the datum available in an environment at a given position, and also the sequence of data available up to that point. This is achieved as follows.

(18) DEFINITION: Let environment $e$ and $k \in N$ be given.

(a) $e(k)$ denotes the member of $e$ that falls in its $k$th position.

(b) $e[k]$ is the initial finite segment of $e$ of length $k$.

Notice that $e(k)$ comes right after $e[k]$ in $e$.

(19) EXAMPLE: If $e$ is the environment of Example (16)a, then $e(2) = Rv_1v_9$, $e[2] = (v_3 \neq v_4, \neg Rv_0v_0)$, $e(0) = (v_3 \neq v_4)$, and $e[0] = \emptyset$.

We can think of $e[n]$ as the scientist's data at stage $n$ of his examination of environment $e$. His conjectures depend on nothing more than the available data, so it is convenient to have a notation for the collection of all such $e[n]$. The following definition serves this purpose.

(20) DEFINITION:

(a) We let *SEQ* denote the collection of proper initial segments of any environment.

(b) Given $\sigma \in SEQ$, we denote by $\bigwedge \sigma$ the conjunction (in order of appearance in $\sigma$) of $content(\sigma)$. To cover degenerate cases, we define $\bigwedge \sigma$ to be $\forall v_0 (v_0 = v_0)$ in case $\sigma$ contains no formulas (either because $length(\sigma) = 0$ or because only $\sharp$ appears in $\sigma$).

(c) We denote by $Var(\sigma)$ the set of free variables appearing in $\sigma$.

(d) Let problem $\mathbf{P}$, proposition $P$ and $\sigma \in SEQ$ be given. We say that $\sigma$ is *for* $P$ just in case $\sigma$ is an initial segment of an environment for $P$. We say that $\sigma$ is *for* $\mathbf{P}$ just in case $\sigma$ is for some $P \in \mathbf{P}$.

Thus, $\sigma$ is for $P$ if and only if $\bigwedge \sigma$ is satisfiable in some member of $P$. Likewise, $\sigma$ is for $\mathbf{P}$ if and only if there is $\mathcal{S} \in \bigcup \mathbf{P}$ that satisfies $\bigwedge \sigma$. If everything is clear to you, the following observation should be evident.

(21) Observation: Let $\mathbf{Obs}$ be given. Then $SEQ$ is the countable set of all finite sequences $\sigma$ over $\mathbf{Obs} \cup \{\sharp\}$ such that $content(\sigma)$ is consistent.

## 2.4. Scientists

We adopt the *extensional* approach to scientists, discussed in Section 2.3 of the first essay. Scientists are thus conceived as mappings from evidence to conjectures. How the mapping is implemented does not concern us at the present stage (later on it will). Possible evidence is embodied by $SEQ$. Possible conjectures are embodied by the collection of all propositions. Scientists map one to the other. This is how we would like our definition to go, but there is an annoyance. The empty set is not a proposition; see Definition (10). Yet it will be convenient for us to give scientists the option of conjecturing the empty set. The official definition of "scientist" is thus the following.

(22) Definition: A scientist is a (possibly partial) mapping of *SEQ* into classes of structures.

The possibility of conjecturing the empty set will be ignored when there is no risk of confusion. So we can say informally that a scientist maps *SEQ* into the collection of propositions. Faced with evidence $\sigma$, $\Psi$ asserts the proposition expressed by $\Psi(\sigma)$. Should $\Psi$ be undefined on $\sigma$, then $\Psi(\sigma)$ does not denote anything. (Scientists are allowed to be partial functions.)

## 2.5. Success

To solve a problem $\mathbf{P}$, we require the scientist's conjectures to stabilize on the one true proposition of $\mathbf{P}$, namely, the proposition holding the structure Nature chose at the outset of the game. To express such belief it is sufficient to announce a consistent proposition that implies the true one from $\mathbf{P}$; there is no penalty for saying something stronger than the target proposition, provided it is consistent. This added flexibility will be exploited in our discussion of computable inquiry, and of belief revision. Before further comment, let us present the official definition of success.

(23) Definition: Let scientist $\Psi$ be given.

(a) Let environment $e$ for proposition $P$ be given. We say that $\Psi$ *solves* $P$ *in* $e$ just in case for cofinitely many $k$, $\emptyset \neq \Psi(e[k]) \subseteq P$. We say that $\Psi$ *solves* $P$ just in case $\Psi$ solves $P$ in every environment for $P$.

(b) Let problem $\mathbf{P}$ be given. We say that $\Psi$ *solves* $\mathbf{P}$ just in case $\Psi$ solves every member of $\mathbf{P}$. In this case we say that $\mathbf{P}$ is *solvable*, and otherwise *unsolvable*.

Hence, solving $\mathbf{P}$ requires solving every $P \in \mathbf{P}$ in every environment for $P$. Equivalently: $\Psi$ solves $\mathbf{P}$ just in case for every $P \in \mathbf{P}$, every $\mathcal{S} \in P$, and every environment $e$ for $\mathcal{S}$, there are cofinitely many $k$ such that $\emptyset \neq \Psi(e[k]) \subseteq P$. Requiring success on all environments instead of just some eliminates the possibility of communicating the correct answer to a scientist via a coding scheme involving the order in which formulas are presented. (We discuss such collusion at greater length in the fourth essay.) Let's consider an example.

(24) EXAMPLE: Suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$.

(a) Suppose that unary predicate $H$ is the only member of **Sym**. Given $n \in N$, let $P_n$ be the class of all structures $\mathcal{S}$ such that $card(H^{\mathcal{S}}) = n$.[13] Let $\mathbf{P} = \{P_n \,|\, n \in N\}$. Then $\mathbf{P}$ is solvable. To see this, given $\sigma \in SEQ$, let $t(\sigma)$ denote the smallest $n \in N$ such that $\sigma$ implies there are at least $n$ things falling in $H$. Define scientist $\Psi$ such that for all $\sigma \in SEQ$, $\Psi(\sigma) = P_{t(\sigma)}$. Then it is easy to see that for all $n \in N$ and all environments $e$ for $P_n$, $\Psi(e[k]) = P_n$ for cofinitely many $k$. Thus, $\Psi$ solves $\mathbf{P}$.

(b) Suppose that binary predicate $R$ is the only member of **Sym**. Set:

$$P_f = \{\langle N, \preceq\rangle \,|\, \preceq \text{ is isomorphic to less-than-or-equals on } \omega\}\,,$$

$$P_b = \{\langle N, \preceq\rangle \,|\, \preceq \text{ is isomorphic to less-than-or-equals on } \omega^*\}\,.$$

(Think of the subscripts as "forward" and "backward," respectively.) Then it is easy show that $\{P_f, P_b\}$ is solvable.[14] Observe that each of $P_f, P_b$ contains uncountably many structures. For example, $P_f$ contains the ordering $1 \preceq 0 \preceq 3 \preceq 2 \preceq 5 \preceq 4 \cdots$.

(c) Again suppose that binary predicate $R$ is the only member of **Sym**. Let $\mathbf{P}$ be as specified in Example (11). Then $\mathbf{P}$ is solvable. (If the solvability of $\mathbf{P}$ isn't clear to you, review Section page 21 of the first essay.) This fact implies

[13] $H^{\mathcal{S}}$ is the set interpreting $H$ in the structure $\mathcal{S}$. The cardinality of that set is $card(H^{\mathcal{S}})$.
[14] $\omega^*$ is the set of natural numbers ordered backwards, that is, as $\cdots 4, 3, 2, 1, 0$.

that $\{P_f, P_b\}$ is solvable since $P_f \subset MOD(T \cup \{\theta\})$ and $P_b \subset MOD(T \cup \{\neg\theta\})$, where $T$ and $\theta$ are given in (11).

The solvability of these problems depends on the assumption that **Obs** $= \mathcal{L}_{basic}$. Other assumptions lead to unsolvability. For example, suppose that **Obs** contains nothing but existential and universal sentences (no free variables). Then $\{P_f, P_b\}$ in (24)b is unsolvable. (You should be able to figure out why.)

One aspect of Definition (23) needs further discussion. Let $P$ be the proposition consisting of every total order (over a binary predicate $R$). Let $e$ be an environment for $\langle N, \leq \rangle$, and suppose that scientist $\Psi$ issues the class of *dense* total orders on cofinitely many initial segments of $e$.[15] Our success criterion credits $\Psi$ with solving $P$ in $e$, yet $\Psi$ is systematically mistaken in one respect about the structure underlying $e$, namely, it is not a dense order. Is the criterion too liberal? We think not. If $\Psi$'s behavior is deemed unsatisfactory in this example, the excessive liberality is inherent in $P$, not in our success criterion. When, for particular purposes, density is an important property of orders, the problem should be defined in such a way that dense and non-dense orders belong to different cells. In contrast, if the class of all total orders is one of the propositions of a problem, this can only signify that density is irrelevant to the inquiry in question. The criterion introduced in Definition (23) allows the problem itself to determine what counts as an accurate conjecture.

## 3. Three observations

That's all there is to it! Our basic inductive paradigm is henceforth in place. (We realize that the journey through the preceding definitions was arduous, but now you're done.) We

[15] A total order is dense iff for every pair of objects in the relation there is a third object between them. The rational numbers ordered by $\leq$ are dense whereas the integers are not.

are therefore free to consider what kinds of problems are solvable and by what kinds of scientists. This task will occupy much of the remainder of the essay. First, we make three observations about our paradigm.

## 3.1. Countability

Let $\Psi$ be a given scientist. Then $\{\Psi(\sigma) \mid \sigma \in SEQ\}$ is a countable set because $SEQ$ is countable and $\Psi$ is a function. Suppose that problem $\mathbf{P}$ contains uncountably many propositions (pairwise disjoint, of course). Then because of the excess cardinality there is $P \in \mathbf{P}$ such that for no $\sigma \in SEQ$:

$$\emptyset \neq \Psi(\sigma) \subseteq P.$$

It follows immediately that $\Psi$ does not solve $\mathbf{P}$ because it can't even conjecture one of its propositions (no matter what evidence it receives). We thus have:

(25) Lemma: Every solvable problem is countable.

Of course, by a problem being "countable" is meant that it contains countably many propositions. Each of the propositions may have arbitrary cardinality (or even be a "proper class," as will often be the case in our examples).

## 3.2. Normal problems

If two structures are isomorphic then every environment for one is also for the other. In such circumstances it is impossible for the scientist to detect which of the two structures was chosen by Nature. It is therefore impossible to solve a problem that places the two structures in different propositions. The point is worth stating precisely.

(26) Definition: For any proposition $P$, let $\mathcal{I}(P)$ denote the closure of $P$ under isomorphism.[16] Problem $\mathbf{P}$ is *normal* just in case for all $P_1, P_2 \in \mathbf{P}$, $\mathcal{I}(P_1) \cap \mathcal{I}(P_2) = \emptyset$.

Thus, in a normal problem there are no isomorphic structures appearing in different propositions. Directly from Lemma (17)a:

(27) Lemma: Only normal problems are solvable.

Normal problems will be the focus of the our discussion. It remains that case, however that a problem is *any* collection of disjoint propositions.

## 3.3. Separation

Let $P_f, P_b$ be as specified in Example (24)b. Suppose $\mathbf{Obs} = \mathcal{L}_{basic}$. Let $e$ be an environment for $P_f \cup P_b$. Then no formula $\bigwedge e[k]$, with $k \in N$, "separates" $P_f$ and $P_b$, in the sense of being satisfiable in just one of them. Indeed, every initial segment of $e$ is satisfiable in every member of $P_f \cup P_b$. The solvability of $\{P_f, P_b\}$ thus illustrates that within our paradigm, separability is not required for successful inquiry.[17] The paradigm differs in this respect from the framework established in [10], in which separation plays a major role. Roughly speaking, one motivation for the developments to follow is to understand inquiry without requiring the scientist's data to rule out any theoretically possible world.

[16] In other words, $\mathcal{S} \in \mathcal{I}(P)$ implies that every structure isomorphic to $\mathcal{S}$ is also in $\mathcal{I}(P)$.
[17] We made a similar point in the first essay on page 20.

# 4. Characterization

Which problems are solvable? To respond nontrivially to this question, we seek a necessary and sufficient condition that does not simply repeat Definition (23) by invoking the behavior of scientists. The present section offers such a condition. It will be used immediately to demonstrate the unsolvability of some simple problems. A wider range of examples is considered in Section 4.4.

To be truthful with you, our condition has a slightly technical character. It is important mainly as a tool for demonstrating more memorable results, e.g., about belief revision (see the third essay). Consequently, we have organized the material so that you can jump over the present section, and pick up the discussion in Section 5.

## 4.1. Locking pairs

As a first step towards characterizing the class of solvable problems, we need the idea of data that "lock" a scientist onto the correct conjecture.

(28) DEFINITION: Let scientist $\Psi$, proposition $P$, $\mathcal{S} \in P$, $\sigma \in SEQ$, and finite assignment $a : Var \to |\mathcal{S}|$ be given. We say that $(\sigma, a)$ is a *locking pair* for $\Psi$, $\mathcal{S}$, and $P$ just in case the following conditions hold.

(a) $domain(a) \supseteq Var(\sigma)$.

(b) $\mathcal{S} \models \bigwedge \sigma[a]$.

(c) For every $\tau \in SEQ$, if $\mathcal{S} \models \exists \bar{x} \bigwedge(\sigma * \tau)[a]$, where $\bar{x}$ contains the variables in $Var(\tau) - domain(a)$, then $\emptyset \neq \Psi(\sigma * \tau) \subseteq P$.

You might think that locking pairs would be reserved for scientists suffering from an *idée fixe*. To the contrary, every successful scientist is prey to them.

(29) LEMMA: Let scientist $\Psi$, proposition $P$, and $\mathcal{S} \in P$ be given. Suppose that $\Psi$ solves $P$ in every environment for $\mathcal{S}$. Then there is a locking pair for $\Psi$, $\mathcal{S}$, and $P$.

The proof is given in Section 8.1, below.[18]

## 4.2. Tip-offs

With Lemma (29) in hand, we are ready to characterize solvability. The solvability of a problem requires that each of its propositions be associated with a faithful signal.[19] The signals are called "tip-offs," and defined in the present subsection. After tip-offs are defined, we show that their existence is both sufficient and necessary for solvability. A preliminary definition is needed.

(30) DEFINITION: We say that $\varphi \in \mathcal{L}_{form}$ is *refutable* just in case for all structures $\mathcal{S}$ and full assignments $h$ to $\mathcal{S}$, the following holds. Suppose that $\mathcal{S} \not\models \varphi[h]$. Then there exists finite $D \subseteq \mathbf{Obs}$ such that $\mathcal{S} \models D[h]$ and $D \models \neg\varphi$.

Thus, a refutable formula is contradicted by any environment for a structure in which the formula is false. To get a grip on the idea, let us see what refutability means when environments are composed of basic formulas or atomic formulas. (These are perhaps the most natural choices of **Obs**.) The following lemma is demonstrated in Section 8.2.

[18]It is inspired by an argument appearing in [2] for a similar lemma in a numerical setting.
[19]The idea of such signals comes from [1], which investigates a numerical paradigm.

(31) LEMMA:

(a) Suppose that **Obs** $= \mathcal{L}_{atomic}$. Then an invalid formula is refutable iff it is logically equivalent to a formula of form $\forall x_1 \ldots \forall x_n \neg\psi$, where $\psi$ is a positive formula.[20]

(b) Suppose that **Obs** $= \mathcal{L}_{basic}$. Then a formula is refutable iff it is logically equivalent to a $\forall$ formula.

Now let us define the key combinatorial concept, embodying the "signal" that must accompany each proposition of a solvable problem.

(32) DEFINITION: Let problem **P** and $P \in \mathbf{P}$ be given. A *tip-off* for $P$ in **P** is a countable collection $\boldsymbol{t}$ of subsets of $\mathcal{L}_{form}$ such that:

(a) Each member of $\boldsymbol{t}$ has the form $Y \cup Z$, where (i) $Y$ is a finite subset of **Obs**, (ii) every member of $Z$ is refutable, and (iii) the set of free variables occurring in $Y \cup Z$ is finite.

(b) For every $\mathcal{S} \in P$ and full assignment $h$ to $\mathcal{S}$, there is $\pi \in \boldsymbol{t}$ with $\mathcal{S} \models \pi[h]$.

(c) For all $\mathcal{U} \in P' \in \mathbf{P}$ with $P' \neq P$, all full assignments $g$ to $\mathcal{U}$, and all $\pi \in \boldsymbol{t}$, $\mathcal{U} \not\models \pi[g]$.

If every member of **P** has a tip-off in **P**, then we say that **P** *has tip-offs.*

When **Obs** is closed under negation (as when **Obs** $= \mathcal{L}_{basic}$), tip-offs have a simpler form.[21] This is shown by the following lemma and example. They follow directly from

[20] Aformula is "positive" if it is equivalent to a formula with no conditionals or biconditionals and in which all atomic subformulas appear under the scope of an even number of negations.

[21] Recall from Definition (4) that **Obs** is closed under negation just in case for every $\varphi \in \mathbf{Obs}$, there is $\psi \in \mathbf{Obs}$ that is logically equivalent to $\neg\varphi$.

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Definition (32).

(33) Lemma: Suppose that **Obs** is closed under negation. Let problem $\mathbf{P}$ and $P \in \mathbf{P}$ be given. Then if there is a tip-off for $P$ in $\mathbf{P}$, there is one with $Y = \emptyset$ in Definition (32)a.

(34) Example: Suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$, and that $P$ in $\mathbf{P}$ has a tip-off. Then by Lemma (33), there is a tip-off $\boldsymbol{t}$ for $P$ in $\mathbf{P}$ of the following form.

(a) Each member of $\boldsymbol{t}$ is a set of universal formulas with only finitely many free variables occuring in the set.

(b) For every $\mathcal{S} \in P$ and full assignment $h$ to $\mathcal{S}$, there is $\pi \in \boldsymbol{t}$ with $\mathcal{S} \models \pi[h]$.

(c) For all $\mathcal{U} \in P' \in \mathbf{P}$ with $P' \neq P$, all full assignments $g$ to $\mathcal{U}$, and all $\pi \in \boldsymbol{t}$, $\mathcal{U} \not\models \pi[g]$.

The next example shows what tip-offs look like in case $\mathbf{Obs} = \mathcal{L}_{atomic}$.

(35) Example: Suppose that $\mathbf{Obs} = \mathcal{L}_{atomic}$. Let problem $\mathbf{P}$ and $P \in \mathbf{P}$ be given. Then a tip-off for $P$ in $\mathbf{P}$ is a countable collection $\boldsymbol{t}$ of subsets of $\mathcal{L}_{form}$ such that:

(a) Each member of $\boldsymbol{t}$ has the form $Y \cup Z$, where (i) $Y$ is a finite subset of $\mathcal{L}_{atomic}$, (ii) every member of $Z$ is equivalent to a formula of form $\forall x_1 \ldots \forall x_n \neg\varphi$, where $\varphi$ is positive quantifier-free formula, and (iii) the set of free variables occurring in $Y \cup Z$ is finite.

(b) For every $\mathcal{S} \in P$ and full assignment $h$ to $\mathcal{S}$, there is $\pi \in \boldsymbol{t}$ with $\mathcal{S} \models \pi[h]$.

(c) For all $\mathcal{U} \in P' \in \mathbf{P}$ with $P' \neq P$, all full assignments $g$ to $\mathcal{U}$, and all $\pi \in \boldsymbol{t}$, $\mathcal{U} \not\models \pi[g]$.

Here is an example of a problem with tip-offs, and one without. Their proofs are given in Section 8.3.

(36) Example: Suppose that $\mathcal{L}_{basic} \subseteq \mathbf{Obs}$. Then the problem defined in Example (11) has tip-offs.

(37) Example: Suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$, and that binary predicate $R$ is the only symbol of **Sym**. Let $\mathcal{S} = \langle N, \preceq \rangle$ be isomorphic to $\omega$, let $\mathcal{T} = \langle Z, \preceq^* \rangle$, with $Z$ the set of integers, be isomorphic to $\omega^*\omega$, and let disjoint propositions $P_1, P_2$ be such that $\mathcal{S} \in P_1$ and $\mathcal{T} \in P_2$. Then $\{P_1, P_2\}$ does not have tip-offs.

## 4.3. Tip-offs and solvability

Countable problems are solvable if and only if they have tip-offs. We state the matter in two propositions, first concerning sufficiency then necessity.

(38) Proposition: If problem $\mathbf{P}$ is countable and has tip-offs, then $\mathbf{P}$ is solvable.

The proof is given in Section 8.4. As an application, from Example (36) we obtain the following.

(39) Proposition: Suppose that $\mathcal{L}_{basic} \subseteq \mathbf{Obs}$. Then the problem defined in Example (11) is solvable.

Now necessity:

(40) Proposition: Every solvable problem has tip-offs.

See Section 8.5 for the proof. As an application, we obtain the following from Example (37).

(41) Proposition: Suppose that binary predicate $R$ is the only symbol of **Sym**, and that **Obs** $= \mathcal{L}_{basic}$, Let $\mathcal{S} = \langle N, \preceq \rangle$ be isomorphic to $\omega$, and let $\mathcal{T} = \langle Z, \preceq^* \rangle$, with $Z$ the set of integers, be isomorphic to $\omega^*\omega$. Suppose that propositions $P_1, P_2$ are such that $\mathcal{S} \in P_1$ and $\mathcal{T} \in P_2$. Then $\{P_1, P_2\}$ is not solvable.

Along with Lemma (25), Propositions (38) and (40) yield the following characterization of solvability.

(42) Theorem: A problem is solvable if and only if it is countable and has tip-offs.

## 4.4. Examples

Let's put Theorem (42) to work by considering a range of examples. Actually, we prefer to put *you* to work by treating the examples as exercises. If you can't stand this kind of thing, just skip to Section 5.

For the first two exercises, suppose that **Sym** consists of a binary predicate $R$ and that **Obs** $= \mathcal{L}_{atomic}$.

(43) Exercise: Let $T$ be the theory of strict total orders (with respect to $R$) with either a least point or a greatest point, but not both. Let $\theta = \exists x \forall y (x = y \vee Rxy)$ and $\mathbf{P} = \{MOD(T \cup \{\theta\}), MOD(T \cup \{\neg\theta\})\}$. Show that **P** is solvable.

(44) Exercise: Let $T$ be the theory of total orders (with respect to $R$) with either a least point or a greatest point, but not both. Let $\theta = \exists x \forall y Rxy$ and $\mathbf{P} = \{MOD(T \cup \{\theta\}), MOD(T \cup \{\neg\theta\})\}$.

(a) Show that $\mathbf{P}$ is not solvable

(b) Suppose that environment $e$ is for structure $\mathcal{S}$ and full assignment $h$. We say that $e$ is *bijective* just in case $h$ is a bijection between $Var$ and $|\,\mathcal{S}\,|$. Specify a scientist $\Psi$ such that for all $P \in \mathbf{P}$, $\Psi$ solves $P$ in every bijective environment for $P$.

For the remaining exercises, suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$.

(45) Exercise: Suppose that **Sym** is the vocabulary of Boolean algebra. Let $T \subseteq \mathcal{L}_{sen}$ be such that $MOD(T)$ is the class of Boolean algebras, and let $\theta \in \mathcal{L}_{sen}$ be true of a Boolean algebra iff it is atomic. Show that $\{MOD(T \cup \{\theta\}), MOD(T \cup \{\neg\theta\})\}$ is not solvable. (For Boolean algebra, see [14, Ch. VIII].)

(46) Exercise: Suppose that **Sym** is limited to a unary constant $\overline{0}$, a unary function symbol $s$, and two binary function symbols $\oplus, \otimes$. Let $Q$ be the conjunction of the seven axioms of "Robinson's arithmetic" (see [3, Ch. 14]). Show that the problem $\{MOD(Q), MOD(\neg Q)\}$ is not solvable.

(47) Exercise: Suppose that $\mathbf{Sym} = \emptyset$. Let $P_1 = \{N\}$, $P_2 = \{\emptyset \neq D \subseteq N \mid D \text{ finite}\}$. Show that $\{P_1, P_2\}$ is not solvable.

(48) Exercise: Suppose that **Sym** is limited to the binary predicate $R$. Let $T$ be the theory of strict total orders (with respect to $R$). Denote by $\theta$ the sentence $\forall xz[Rxz \rightarrow \exists y(Rxy \wedge Ryz)]$ ("$R$ is dense"). Show that $\mathbf{P} = \{MOD(T \cup \{\theta\}), MOD(T \cup \{\neg\theta\})\}$ is not solvable.

(49) Exercise: Suppose that **Sym** is limited to a binary predicate $R$. Say that structure $\mathcal{S}$ is *standard* if $|\,\mathcal{S}\,| = N$. We specify a countable collection $\{\mathcal{S}_j \mid j \in N\}$ of standard

structures by specifying the extension $R^{\mathcal{S}_j}$ of $R$ for all $j \in N$. $R^{\mathcal{S}_0}$ is the relation $\{(i, i+1) \mid i \in N\}$. For $j > 0$, $R^{\mathcal{S}_j}$ is the finite relation $\{(i, i+1) \mid i < j\}$. Let $P_1 = \{\mathcal{S}_0\}$ and $P_2 = \{\mathcal{S}_j \,|\, j > 0\}$. Show that $\{P_1, P_2\}$ is not solvable.

(50) EXERCISE: Given a well-ordering $\prec$ over collection $\mathbf{K}$ of structures, we define an associated scientist $\Psi_\prec$ as follows. For all $\sigma \in SEQ$, $\Psi_\prec(\sigma)$ is the $\prec$-least member $\mathcal{S}$ of $\mathbf{K}$ such that $\bigwedge \sigma$ is satisfiable in $\mathcal{S}$. $\Psi_\prec(\sigma)$ is undefined if there is no such. Problem $\mathbf{P}$ is *solvable by enumeration* just in case there is a well-ordering $\prec$ of $\bigcup \mathbf{P}$ such that $\Psi_\prec$ solves $\mathbf{P}$. Exhibit structures $\mathcal{S}$ and $\mathcal{T}$ such that $\{\{\mathcal{S}\}, \{\mathcal{T}\}\}$ is solvable, but not by enumeration.

(51) EXERCISE: Let scientist $\Psi$ and problem $\mathbf{P}$ be given. We say that $\Psi$ is $\mathbf{P}$-*conservative* just in case for all $P \in \mathbf{P}$, $\sigma$ for $\mathbf{P}$ and $\beta \in \mathcal{L}_{basic}$, if $\emptyset \neq \Psi(\sigma) \subseteq P$ and every structure in $\Psi(\sigma)$ satisfies $\{\bigwedge \sigma, \beta\}$, then $\emptyset \neq \Psi(\sigma{*}\beta) \subseteq P$. Exhibit structures $\mathcal{S}$ and $\mathcal{T}$ such that $\mathbf{P} = \{\{\mathcal{S}\}, \{\mathcal{T}\}\}$ is solvable, but no $\mathbf{P}$-conservative scientist solves $\{\{\mathcal{S}\}, \{\mathcal{T}\}\}$.

(52) EXERCISE: Call problem $\mathbf{P}$ "separated" (respectively "elementary separated") just in case for all $P_1 \neq P_2 \in \mathbf{P}$, $\mathcal{S} \in P_1$, and $\mathcal{T} \in P_2$, there is no isomorphic (respectively elementary) embedding from $\mathcal{S}$ into $\mathcal{T}$. [For background on embeddings, see [11, Secs. 1.2, 2.5].]

(a) Exhibit a separated, unsolvable problem.

(b) Show that every solvable problem is elementary separated.

The present exercise thus strengthens Lemma (27).

(53) EXERCISE: Call problem $\mathbf{P}$ *elementary* just in case for all $\mathcal{S}, \mathcal{U} \in \bigcup \mathbf{P}$, $\mathcal{S} \equiv \mathcal{U}$. Exhibit an elementary, solvable problem consisting of more than one proposition.

(54) EXERCISE: Scientist $\Psi$ *weakly* solves problem $\mathbf{P}$ just in case for every environment $e$ for $P \in \mathbf{P}$, the following conditions are met:

(a) There are infinitely many $k$ such that $\emptyset \neq \Psi(e[k]) \subseteq P$.

(b) For every $P' \in \mathbf{P}$ with $P' \neq P$, there are only finitely many $k$ such that $\emptyset \neq \Psi(e[k]) \subseteq P'$.

In this case $\mathbf{P}$ is said to be *weakly solvable*.

(a) Exhibit a problem that is weakly solvable but not solvable.

(b) Show that no unsolvable problem consisting of finitely many propositions is weakly solvable.

# 5. Special Problems

The preceding section was devoted to characterizing solvability in terms of a combinatorial concept called "tip-offs." In the present section we exploit the characterization in the service of analyzing the solvability of a restricted class of problems.

## 5.1. Problems of form $(T, \{\theta_0 \ldots \theta_n\})$

The next definition introduces notation for problems of an appealing form. They become increasingly important as our theory develops.

(55) DEFINITION: Let problem $\mathbf{P}$, $T \subseteq \mathcal{L}_{sen}$, and $\theta_0 \ldots \theta_n \in \mathcal{L}_{sen}$ be given. We say that $\mathbf{P}$ has the form $(T, \{\theta_0 \ldots \theta_n\})$ just in case:

(a) for every model $\mathcal{S}$ of $T$ there is exactly one $i \in \{0 \ldots n\}$ such that $\mathcal{S} \models \theta_i$;
(b) $\mathbf{P} = \{MOD(T \cup \{\theta_i\}) \mid 0 \leq i \leq n\}$.

By Definition (10) of "problem," $\mathbf{P}$ can have the form $(T, \{\theta_0 \ldots \theta_n\})$ only if $MOD(T \cup \{\theta_i\})$ is nonempty for all $i \leq n$. Hence $T \cup \{\theta_i\}$ must be consistent for all $i \leq n$, which implies that $T$ is consistent. Intuitively, given problem $(T, \{\theta_0 \ldots \theta_n\})$, we may think of $T$ as an accepted background theory that leaves open which among the $\theta_i$'s is true. The $\theta_i$'s thus partition the models of $T$. The informal overview provided by the first essay concentrated on problems of form $(T, \{\theta_0 \ldots \theta_n\})$.

A yet more special case arises when the partition has just two members. The problem then has the form $(T, \{\theta, \neg\theta\})$. In contrast, the more general form $(T, \{\theta_0, \theta_1, \ldots\})$ is of no use since an easy compactness argument shows that there is no infinite partition of $MOD(T)$ of the form $\{MOD(T \cup \{\theta_i\}) \mid i \in N\}$.

To make sure that our notation is clear, let us consider an example.

(56) Example: Suppose that **Sym** is limited to a binary function symbol $\circ$, and let $T$ be the theory of groups. Let $\theta$ be the sentence $\forall xy(x \circ y = y \circ x)$. Then $(T, \{\theta, \neg\theta\})$ is the problem whose propositions are:

- the class of abelian groups, namely, $MOD(T \cup \{\theta\})$;
- the class of non-abelian groups, namely, $MOD(T \cup \{\neg\theta\})$.

If **Obs** includes $\mathcal{L}_{basic}$, then $(T, \{\theta, \neg\theta\})$ is trivially solvable.

For a more substantial illustration, Propositions (39) and (41) yield the following.

(57) Proposition: Suppose that **Sym** is limited to the binary predicate $R$ and that $\mathbf{Obs} = \mathcal{L}_{basic}$. Let $\theta = \exists x \forall y Rxy$, let $T_0 \subseteq \mathcal{L}_{sen}$ be the theory of total orders over $R$,

and let $T_1 \subseteq \mathcal{L}_{sen}$ be the theory of total orders with either a greatest point or a least point, but not both. Then the problem $(T_1, \{\theta, \neg\theta\})$ is solvable but $(T_0, \{\theta, \neg\theta\})$ is not.

We can exploit Proposition (57) to reformulate a point about inquiry that we made in Section 2.3.5 above. Let a problem of form $(T, \{\theta_0 \ldots \theta_n\})$ be given, and suppose that $e$ is one of its environments. If there is $k \in N$ such that $T \cup \{\bigwedge e[k]\}$ implies the right choice among $\theta_0 \ldots \theta_n$, then the problem can be solved in $e$ by simply issuing $MOD(T \cup \{\bigwedge e[k]\})$ at each stage $k$ of inquiry. If the same is true of all the environments, then it may be said that $(T, \{\theta_0 \ldots \theta_n\})$ is solved by "waiting for deduction to work." For, it suffices to wait for the background theory and current data to imply the right answer. Our inductive paradigm would have a trivial character if the solvability of $(T, \{\theta_0 \ldots \theta_n\})$ implies solvability via such a strategy. But in fact, awaiting deduction is not sufficient. This can be seen from the fact that $(T_1, \{\theta, \neg\theta\})$ of Proposition (57) is solvable, with **Obs** $= \mathcal{L}_{basic}$, even though for all $\sigma \in SEQ$, $\bigwedge \sigma$ implies neither $\theta$ nor $\neg\theta$ in the models of $T_1$. So, some kind of genuinely inductive strategy is needed to solve $(T_1, \{\theta, \neg\theta\})$.

## 5.2. Solvability of problems of form $(T, \{\theta_0 \ldots \theta_n\})$

Theorem (42) offers an infinitary condition on solvability since the tip-offs it evokes are infinite sets of formulas. The restricted form of problems $(T, \{\theta_0 \ldots \theta_n\})$, however, leads us to hope for deeper characterization of their solvability. In fact, for such problems we can prove various finitary conditions depending on the choice of **Obs**. For example, we will demonstrate the following.

(58) Proposition: Suppose that **Obs** $= \mathcal{L}_{basic}$. Then a problem of form $(T, \{\theta_0 \ldots \theta_n\})$ is solvable if and only if for every $i \leq n$, $\theta_i$ is equivalent in $T$ to an $\exists\forall$ sentence.

The proposition follows from a more general theorem.

(59) THEOREM: A problem of form $(T, \{\theta_0 \ldots \theta_n\})$ is solvable if and only if for every $i \leq n$, $\theta_i$ is equivalent in $T$ to the existential closure of any of the following kinds of formulas.

(a) a formula built from **Obs** using only conjunctions and disjunctions,

(b) a refutable formula, or

(c) a formula of form $\chi \wedge \varphi$, where $\chi$ is built from **Obs** using only conjunctions and disjunctions and $\varphi$ is refutable.

For the proof, see Section 8.6. We have the following corollaries.[22]

(60) COROLLARY: A problem of form $(T, \{\theta_0 \ldots \theta_n\})$ is solvable if and only if for every $i \leq n$, $\theta_i$ is equivalent in $T$ to the existential closure of a formula of form $\chi$ or $\forall y \neg \varphi$ or $\chi \wedge \forall y \neg \varphi$, where $\chi$ and $\varphi$ are built from **Obs** using conjunctions and disjunctions only.

(61) COROLLARY: Suppose that **Obs** is closed under negation. A problem of form $(T, \{\theta_0 \ldots \theta_n\})$ is solvable if and only if for every $i \leq n$, $\theta_i$ is equivalent in $T$ to the existential closure of a refutable formula.

Proposition (58), given above, follows from Theorem (59) in conjunction with Corollary (31)b. From (59) and (31)a, we get:

[22] The first follows directly from the theorem and Lemma (92). The latter is proved in Section 8.2. The second comes from from Theorem (59), Lemma (91)a, and Corollary (93). The latter two results are proved in Section 8.2.

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(62) Corollary: Suppose that $\mathbf{Obs} = \mathcal{L}_{atomic}$. Then a problem of form $(T, \{\theta_0 \dots \theta_n\})$ is solvable if and only if for every $i \leq n$, $\theta_i$ is equivalent in $T$ to the existential closure of a positive quantifier-free formula or of a formula of form $\chi \wedge \forall y \neg \varphi$, where $\chi$ and $\varphi$ are positive quantifier-free formulas.

Here is another useful way to formulate Proposition (58).

(63) Corollary: Suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$. Then a problem of form $(T, \{\theta_0 \dots \theta_n\})$ is solvable if and only if for all $i \leq n$, $\theta_i$ is equivalent in $T$ to a boolean combination of $\exists$ sentences.

To get (63) from (58), we rely on the following fact, which is an easy adaptation of [5, Theorem 3.1.16].

(64) Fact: Let $T \subseteq \mathcal{L}_{sen}$ and $\theta \in \mathcal{L}_{sen}$ be given. Suppose that $\theta$ is equivalent in $T$ both to an $\exists\forall$ sentence and to a $\forall\exists$ sentence. Then $\theta$ is equivalent in $T$ to a boolean combination of $\exists$ sentences.

## 5.3. Problems of form $(T, \{P_0, P_1, \dots\})$

We now define another special class of problems.

(65) Definition: Let $T \subseteq \mathcal{L}_{sen}$ be given. We say that problem $\mathbf{P}$ has the form $(T, \{P_0, P_1, \dots\})$ just in case $\mathbf{P} = \{P_0, P_1, \dots\}$ and $\bigcup \mathbf{P} = MOD(T)$.

A problem can have the form $(T, \{P_0, P_1, \dots\})$ only if $T$ is a consistent theory (otherwise the propositions $P_i$ are empty). For consistent $T$, a problem of form $(T, \{P_0, P_1, \dots\})$ is a partition of the models of $T$. Plainly, any problem of form $(T, \{\theta_0 \dots \theta_n\})$ also has the more general form $(T, \{P_0, P_1, \dots\})$. For examples, see Section 5.5.

### 5.4. Solvability of problems of form $(T, \{P_0, P_1, \dots\})$

The solvable problems of form $(T, \{P_0, P_1, \dots\})$ share an interesting logical feature. Let us first state the matter under the assumption that **Obs** $= \mathcal{L}_{basic}$. Then we will formulate a more general result.

(66) PROPOSITION: Suppose that **Obs** $= \mathcal{L}_{basic}$. Let there be given a solvable problem **P** of form $(T, \{P_0, P_1, \dots\})$. Then for all $P \in \mathbf{P}$, there is an enumeration $\{X_i \mid i \in N\}$ of sets of $\exists\forall$ sentences such that $P = \bigcup_{i \in N} MOD(T \cup X_i)$.

The matter might be put this way: first-order sentences are sufficient to distinguish among any collection **P** of propositions that partition an elementary class, provided only that **P** is solvable. It is striking that no further constraint need be placed upon the propositions.

Proposition (66) follows directly from the next fact, along with Corollary (31).

(67) PROPOSITION: Let solvable problem **P** of form $(T, \{P_0, P_1, \dots\})$ be given. Then for all $P \in \mathbf{P}$, there is an enumeration $\{X_i \mid i \in N\}$ of sets of existential closures of formulas of form $\bigwedge D \wedge \varphi$, where $D$ is a (possibly empty) finite subset of **Obs** and $\varphi$ a refutable formula, such that $P = \bigcup_{i \in N} MOD(T \cup X_i)$.

The proposition is proved in Section 8.7.

Two other easy corollaries of Proposition (67) include the following.[23]

[23] Corollary (68) follows from Proposition (67) and Lemma (92). The latter is proved in Section 8.1. Corollary (69) follows from Proposition (67) and Lemma (31).

(68) COROLLARY: Let solvable problem **P** of form $(T, \{P_0, P_1, \ldots\})$ be given. Then for all $P \in \mathbf{P}$, there is an enumeration $\{X_i \mid i \in N\}$ of sets of existential closures of formulas of form $\chi$ or $\forall y \neg\varphi$ or $\chi \wedge \forall y \neg\varphi$, where $\chi$ and $\varphi$ are built from **Obs** using conjunctions and disjunctions only, such that $P = \bigcup_{i \in N} MOD(T \cup X_i)$.

(69) COROLLARY: Suppose that $\mathbf{Obs} = \mathcal{L}_{atomic}$. Let solvable problem **P** of form $(T, \{P_0, P_1, \ldots\})$ be given. Then for all $P \in \mathbf{P}$, there is an enumeration $\{X_i \mid i \in N\}$ of sets of existential closures of positive quantifier-free formulas or of formulas of form $\chi \wedge \forall y \neg\varphi$, where $\chi$ and $\varphi$ are positive quantifier-free formulas, such that $P = \bigcup_{i \in N} MOD(T \cup X_i)$.

For technical interest, let it be noted that the converse of Proposition (67) is false. To see this, suppose that $\mathbf{Sym} = \emptyset$, $\mathbf{Obs} = \mathcal{L}_{basic}$, and for $i > 0$ let $\chi_i$ be an $\exists\forall$ sentence true in just the structures of cardinality $i$. Let $X = \{\exists x_0 \ldots x_n \bigwedge_{0 \leq i < j \leq n} x_i \neq x_j \mid n > 0\}$. Then $\mathbf{P} = \{MOD(X), MOD(\chi_1), MOD(\chi_2) \ldots\}$ partitions the class of all structures, hence has the form $(T, \{P_0, P_1, \ldots\})$. But an easy consequence of Exercise (47) shows **P** to be unsolvable.

## 5.5. Examples

Here are some examples and facts related to the material in this section. We leave them as exercises.

(70) EXERCISE: Suppose that **Sym** consists of a unary function symbol $s$ and a constant $\bar{0}$. The term that results from $n$ applications of $s$ to $\bar{0}$ is denoted $\bar{n}$. Let $P_0$ be the class of models of $\{\bar{n} \neq \bar{0} \mid n > 0\}$, and for $i > 0$, let $P_i$ be the class of models of $\{\bar{n} \neq \bar{0} \mid 0 < n < i\} \cup \{\bar{i} = \bar{0}\}$. It is easy to verify that if $\mathcal{L}_{atomic} \subseteq \mathbf{Obs}$ then $(\emptyset, \{P_0, P_1, \ldots\})$ is solvable.

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(71) EXERCISE: Suppose that **Sym** contains just two binary function symbols. For $i \in N$ either 0 or prime, let proposition $P_i$ be the collection of all fields of characteristic $i$ in this vocabulary. Show that if **Obs** $= \mathcal{L}_{basic}$, $\{P_i \mid i \text{ is either 0 or prime}\}$ is solvable.

(72) EXERCISE: Suppose that **Sym** is limited to a binary function symbol. Let **G** be all groups, **T** all torsion groups, and **F** all torsion-free groups. Show that if **Obs** $= \mathcal{L}_{basic}$ then $\{\mathbf{F}, \mathbf{G} - \mathbf{F}\}$ is solvable but that $\{\mathbf{T}, \mathbf{G} - \mathbf{T}\}$ is not.

(73) EXERCISE: Suppose **Obs** $= \mathcal{L}_{basic}$. Show that any problem of form $(T, \{\theta_0 \dots \theta_n\})$ is solvable if $T$ is model-complete. (For model-completeness, see [11, §8.3].)

(74) EXAMPLE: It is interesting to see how Theorem (59) — in particular, its corollary (58) — can be generalized to the case of problems of arbitrary form. We just consider the case **Obs** $= \mathcal{L}_{basic}$. Show that a problem **P** is solvable iff it consists of countably many propositions, each of which is equivalent in $\bigcup \mathbf{P}$ to an $\mathcal{L}_{\omega_1\omega}$-sentence of form:

$$\bigvee_{i<\omega} \exists \bar{x}_i \bigwedge_{j<\omega} \forall \bar{y}_j \varphi_{i,j},$$

where $\chi_i$ and $\varphi_{i,j}$ are quantifier-free members of $\mathcal{L}_{form}$, for all $i, j < \omega$.[24]

# 6. Efficiency

The solvability criterion advanced in Section 2.5 imposes no requirement of speedy convergence, so is open to the objection that tardy success is often no better than failure. We

[24] For $\mathcal{L}_{\omega_1\omega}$, see [8, §IX.2].

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are therefore led to reinforce our conception of success by including a standard of efficiency. Specifically, efficient investigation will be defined as inquiry whose use of data cannot be uniformly reduced. Efficiency in this sense will be relevant to our study of belief revision in the third essay.

The needed definitions are introduced in the next subsection. Then we prove that the class of solvable problems coincides with the class of problems that can be solved efficiently.

## 6.1. Success points, dominance, and efficiency

Given proposition $P$ and environment $e$ for $P$, scientist $\Psi$ solves $P$ in $e$ just in case $\Psi$'s successive conjectures ultimately become non-void subsets of $P$ [see Definition (23)]. The particular propositions that $\Psi$ announces, however, may vary indefinitely as $\Psi$ progresses through $e$. So, success does not imply convergence to a single proposition. There is nonetheless a sense in which success requires convergence to the right proposition. The matter can be formulated as follows.

(75) Definition: Let proposition $P$, environment $e$ for $P$, and scientist $\Psi$ be given. We say that $\Psi$ is *$P$-correct* on $e$ at $k \in N$ just in case $\emptyset \neq \Psi(e[k]) \subseteq P$. If $\Psi$ solves $P$ in $e$, we write $SP(\Psi, e, P)$ to denote the least $k_0 \in N$ such that $\Psi$ is $P$-correct on $e$ at $k$ for all $k \geq k_0$.

Such a $k_0$ may be called the "success point" of $\Psi$ on $e$, relative to $P$. The earlier the success point, the fewer data required for the scientist to figure out Nature's choice of proposition (even if the scientist remains uncertain whether convergence to the correct conjecture has in fact been achieved).

It would be nice if we could associate a single success point with a given scientist. But this is not possible because of the multiplicity of environments for a given structure.

An environment that begins with many pauses or repetitions, for example, will typically yield a greater success point than an environment which presents denser information. For this reason, we proceed indirectly, defining efficient induction in terms of a strong form of *in*efficiency.

(76) DEFINITION: Suppose that scientist $\Psi$ solves problem $\mathbf{P}$. We say that $\Psi$ is *dominated* on $\mathbf{P}$ just in case there is scientist $\Psi'$ such that:

(a) $\Psi'$ solves $\mathbf{P}$;
(b) for all $P \in \mathbf{P}$ and environments $e$ for $P$, $SP(\Psi', e, P) \leq SP(\Psi, e, P)$;
(c) for some $P \in \mathbf{P}$ and environment $e$ for $P$, $SP(\Psi', e, P) < SP(\Psi, e, P)$.

We say that $\Psi$ solves $\mathbf{P}$ *efficiently* just in case no scientist dominates $\Psi$ on $\mathbf{P}$. In this case, $\mathbf{P}$ is solvable *efficiently*.

An efficient scientist, in other words, cannot be improved *uniformly*. No other scientist sometimes consumes less data and never consumes more.

## 6.2. Solvability implies efficient solvability

Requiring scientists to be efficient as well as successful does not reduce the class of solvable problems. This is the content of the following proposition

(77) PROPOSITION: Every solvable problem is solvable efficiently.

Let us head off a misunderstanding. The proposition does not assert that every scientist who solves a problem does so efficiently. It says only that if some scientist solves a given problem then some scientist (not necessarily the same one) solves it efficiently.

To prove Proposition (77), it will be convenient to demonstrate a stronger result. This requires a stronger concept of efficiency that is interesting in its own right.

(78) DEFINITION: Suppose that scientist $\Psi$ solves problem $\mathbf{P}$. We say that $\Psi$ solves $\mathbf{P}$ *strongly efficiently* just in case for every scientist $\Psi'$ that solves $\mathbf{P}$, if there is an environment $e_0$ for $P_0 \in \mathbf{P}$ with $k_0 = SP(\Psi', e_0, P_0) < SP(\Psi, e_0, P_0)$ then there is $P_1 \in \mathbf{P}$, $\mathcal{S} \in P_1$, and full assignment $h$ to $\mathcal{S}$ such that:

(a) $\mathcal{S} \models \bigwedge e_0[k_0][h]$ and

(b) $SP(\Psi, e, P_1) \leq k_0 < SP(\Psi', e, P_1)$ for every environment $e$ for $\mathcal{S}$ and $h$ that extends $e_0[k_0]$.

In this case, $\mathbf{P}$ is solvable *strongly efficiently*.

Intuitively, $\Psi$ is strongly efficient if it succeeds faster than any rival on many environments — where a "rival" is a scientist that succeeds faster than $\Psi$ on some environment. Plainly, strong efficiency implies efficiency. Proposition (77) thus follows from:

(79) THEOREM: Every solvable problem is solvable strongly efficiently.

The theorem is demonstrated in Section 8.8.

# 7. Computability

It is an engaging (but far from proven) hypothesis that human scientific behavior is Turing simulable. If the hypothesis is true, then our paradigm would benefit from greater realism by isolating the computable scientists for separate study. Our first goal is thus to formalize the idea that a scientist [in the sense of Definition (22)] is Turing simulable. The question will then arise: Are there solvable problems that cannot be solved by computable scientists?

## 7.1. Finitization of inputs and outputs

In order to conceive of a scientist as Turing simulable, we must find a way to represent his inputs and outputs as finite objects. It might be thought that inputs pose no problem since *SEQ* contains just finite sequences of formulas, hence is a countable set. But there is a difficulty. If **Obs** is sufficiently rich (e.g., including all formulas with two quantifiers) then *SEQ* will not be effectively decidable. This is because *SEQ* contains just the *consistent* finite sequences over **Obs** [see Definition (20)]. To give our computable processes a more suitable domain we therefore introduce an extension of *SEQ*.

(80) Definition: The set of all finite sequences over $\mathcal{L}_{form} \cup \{\sharp\}$ is denoted $SEQ^*$. As in Definition (14), for $\sigma \in SEQ^*$ we let $content(\sigma)$ denote the set of formulas appearing in $\sigma$ ($\sharp$ is excluded).

The inputs to the computable processes used to simulate scientists will thus be drawn from $SEQ^*$.

Now consider the scientist's outputs, namely propositions. Propositions, it will be recalled, are vast objects that comprehend an arbitrary class of structures. [see Definition (10)]. To represent them finitely we limit attention to elementary classes of structures, that is, to classes composed of every structure that satisfies some given set of formulas. However, if the set of formulas is not recursively enumerable (*r.e.*), then such a proposition remains ineffable (at least, in one important sense) by computable scientists. So we must further restrict propositions to classes of structures that are definable by an *r.e.* set $S$ of formulas.

The *r.e.* index of $S$ can then be used to express the proposition defined by $S$. Finitizing outputs in this way makes it possible for some scientists to be simulated by computable functions from $SEQ^*$ to the set of indices. These will be called the "computable scientists."

To keep things tidy, it is necessary to distinguish the indices used for *r.e.* sets of formulas from those used for *r.e.* sets of numbers. So we fix for the remainder of our discussion an acceptable indexing $\{\mathrm{W}_i^{form} \mid i \in N\}$ of the *r.e.* subsets of $\mathcal{L}_{form}$, along with an acceptable indexing $\{\mathrm{W}_i^{num} \mid i \in N\}$ of the *r.e.* subsets of $N$.[25] Observe that for $i \in N$ with $\mathrm{W}_i^{form}$ consistent, $MOD(\mathrm{W}_i^{form})$ is a proposition; namely, it is the class of structures that satisfy all the formulas enumerated by the $i$th Turing Machine. [Also note that $MOD(\mathrm{W}_i^{form})$ is the universal proposition (all structures) if $\mathrm{W}_i^{form} = \emptyset$.] Hence, given mapping $\psi : SEQ^* \to N$ and $\sigma \in domain(\psi)$, $MOD(\mathrm{W}_{\psi(\sigma)}^{form})$ is also a proposition (when nonempty), namely, $MOD(\mathrm{W}_i^{form})$ for $i = \psi(\sigma)$. In this way, $\psi$ uses $i$ to express its view that a certain proposition is true.

## 7.2. Computable scientists

Pursuant to the preceding discussion, the class of computable scientists is defined as follows.

(81) DEFINITION: Scientist $\Psi$ is *computable* just in case there is computable $\psi : SEQ^* \to N$ such that for all $\sigma \in SEQ$, $\psi(\sigma)$ is defined if $\Psi(\sigma)$ is defined, and when both are defined $\Psi(\sigma) = MOD(\mathrm{W}_{\psi(\sigma)}^{form})$. In this case, we say that $\psi$ *underlies* $\Psi$. A problem that is solved by a computable scientist is *computably solvable.*

[25] For "acceptable indexing," see [4, 15].

So, a computable scientist $\Psi$ is associated with a computable function $\psi : SEQ^* \to N$ that simulates $\Psi$'s behavior. If $\Psi$ issues proposition $P$ on input $\sigma \in SEQ$, then $\psi$ issues an index $i$ on $\sigma$ with $P = MOD(\mathrm{W}_i^{form})$. It makes no difference how $\psi$ behaves on $SEQ^* - SEQ$. (The latter set holds the inconsistent sequences of formulas.)

Computability imposes a double constraint on scientists. On the one hand, conjectures must be expressed via (indices for) recursively axiomatizable theories.[26] On the other hand, the conversion of data into theories must be Turing simulable. The two conditions are necessary and jointly sufficient for computer implementation of a scientist, at least in principle.

It is evident that not every scientist is computable according to Definition (81). Indeed, there are only countably many computable scientists (one for each Turing Machine) but uncountably many scientists in general.

## 7.3. Competence of computable scientists

Now we may consider the inductive competence of computable scientists. Let us start with the most basic question, namely: Is there a solvable problem that is solved by no computable scientist? Yes, of course. Let $T$ be a complete, nonrecursive theory. Then the degenerate problem $\{MOD(T)\}$ is trivially solvable but not computably. This is because there is no consistent, *r.e.* extension of $T$, hence no means whereby a computable scientist can express a nonempty subset of $MOD(T)$. It is a more trenchant fact that computable scientists have limited competence even when there is no computability obstacle to their announcing an adequate theory. To prove the point, we rely on the following extension of our notation (5).

[26] Let us recall that a set $S$ of formulas is *r.e.* if and only if $S$ is recursively axiomatizable, i.e., if and only if $S$ is a equivalent to a recursive subset of $\mathcal{L}_{form}$ [7]. Therefore, $S$ has a recursive set of axioms if and only if $S$ has an *r.e.* set of axioms.

(82) DEFINITION: A proposition is *strongly elementary* just in case it has the form $MOD(\theta)$ for some $\theta \in \mathcal{L}_{sen}$.

Thus, strongly elementary propositions can be expressed by just a single sentence. A fortiori, for every proposition $P$ that is strongly elementary, there is an index $i$ with $P = MOD(\mathrm{W}_i^{form})$. Indeed, it suffices to take $i$ such that $\mathrm{W}_i^{form} = \{\theta\}$, where $\theta$ defines $P$. The upshot is that problems consisting of strongly elementary propositions pose no difficulty specifically linked to expressing the propositions. This leaves open the possibility that a *collection* of strongly elementary propositions is difficult for a computable scientist to manipulate, and indeed we shall see that such collections can be solvable but not computably. The following result is proved in Section 8.9.

(83) THEOREM: Suppose that **Sym** is limited to a binary predicate, two constants, and a unary function symbol. Let $\mathbf{Obs} = \mathcal{L}_{basic}$. Then for every countable collection $\Sigma$ of scientists there is a problem $\mathbf{P}$ with the following properties.

(a) Every member of $\mathbf{P}$ is strongly elementary.

(b) $\mathbf{P}$ is solvable.

(c) No member of $\Sigma$ solves $\mathbf{P}$.

As noted above, the computable scientists form a countable set, since their number is bounded by the number of Turing Machines. So we obtain the immediate corollary:

(84) COROLLARY: Let **Sym** and **Obs** be as in Theorem (83). Then there is a solvable problem whose members are strongly elementary but which is not solvable computably.

## 7.4. Efficiency and computability

We have so far conceived efficiency in terms of the number of data examined prior to convergence (see Section 6). In the context of computable scientists, however, a variety of additional efficiency concepts may be defined. The new concepts are based on the time that a given scientist spends examining individual $\sigma \in SEQ$. For example, given a function $f : SEQ \to N$, it may be said that scientist $\Psi$ is "$f$-fast" just in case the running time of $\Psi$ on $\sigma \in SEQ$ is bounded by $f(\sigma)$. The running-time conception of efficiency can be studied in conjunction with the data-use conception to provide an overall picture of resource consumption during inquiry. For our part, however, we shall continue to focus just on data-use, leaving running-time to one side. The reason is the shape of the ensuing theory of revision-based inquiry, to be developed in the next essay. Only data-use will be relevant.

Recall that Proposition (77) shows every solvable problem to be solvable efficiently. This reassuring fact does not survive the transition to computable solvability. Instead, the following proposition shows that there are computable, solvable problems that cannot be solved efficiently by computable scientist.

(85) PROPOSITION: Suppose that **Sym** is limited to the vocabulary of arithmetic (including $\bar{0}$ and a unary function symbol $s$) plus the additional constant $a$. Suppose also that **Obs** includes all identities (that is, formulas of form $t_1 = t_2$, for terms $t_1, t_2$). Then there is a problem **P** with the following properties.

(a) Every member of **P** is strongly elementary.

(b) **P** is solvable computably.

(c) Every computable scientist that solves **P** is dominated on **P**.

The proof is given in Section 8.10.

## 7.5. *R.e.* problems of form $(T, \{\theta_0 \ldots \theta_n\})$

Problems of form $(T, \{\theta_0 \ldots \theta_n\})$ are central to our theory so it is interesting to consider their computable solvability. Of course, if $T$ is not recursively axiomatizable then $(T, \{\theta_0 \ldots \theta_n\})$ may not be computably solvable for want of the ability to name its propositions. To set aside this kind of inexpressible case, we rely on the following definition.

(86) DEFINITION: A problem of form $(T, \{\theta_0 \ldots \theta_n\})$ is called *r.e.* just in case $T$ is a recursively enumerable set of sentences.

All the propositions of an *r.e.* problem of form $(T, \{\theta_0 \ldots \theta_n\})$ are recursively axiomatizable, so computable scientists can express any of them. Do there remain further obstacles to solving such problems, obstacles that are specific to computable scientists? Surprisingly, this question receives a negative answer. In other words, the computable solvability of an *r.e.* problem of form $(T, \{\theta_0 \ldots \theta_n\})$ follows from its solvability *tout court*. The matter may be summarized as follows.

(87) THEOREM: Suppose that **Obs** is recursive. Then every solvable, *r.e.* problem of form $(T, \{\theta_0 \ldots \theta_n\})$ is computably solvable.

For the proof, see Section 8.11. A little adjustment to the proof allows the hypothesis of Theorem (87) to be weakened to: **Obs** is *r.e.*

Here is a nice exercise to test your grasp of the concepts in the present subsection.

(88) EXERCISE: Assume that **Obs** is a recursive set. Suppose that scientist $\Psi$ solves problem **P**. We say that $\Psi$ *respects* **P** just in case for all $\sigma \in SEQ$ for **P**, $\Psi(\sigma)$ is a nonempty subset of some $P \in$ **P**. Show that there is a computably solvable problem **P** such that no computable scientist that solves **P** respects **P**.

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# 8. Proofs

## 8.1. Proof of Lemma (29)

(29) LEMMA: Let scientist $\Psi$, proposition $P$, and $\mathcal{S} \in P$ be given. Suppose that scientist $\Psi$ solves $P$ in every environment for $\mathcal{S}$. Then there is a locking pair for $\Psi$, $\mathcal{S}$, and $P$.

*Proof:* Suppose there is no locking pair for $\Psi$, $\mathcal{S}$, and $P$. Then:

(89) For every $\sigma \in SEQ$ and finite assignment $a : Var \to |\mathcal{S}|$, if $domain(a) \supseteq Var(\sigma)$ and $\mathcal{S} \models \bigwedge \sigma[a]$, then there is $\tau \in SEQ$ and finite extension $a' : Var \to |\mathcal{S}|$ of $a$ such that:

- (a) $Var(\tau) \subseteq domain(a')$,
- (b) $\mathcal{S} \models \bigwedge \tau[a']$, and
- (c) either $\Psi(\sigma * \tau)$ is not defined or $\Psi(\sigma * \tau) = \emptyset$, or $\Psi(\sigma * \tau) \not\subseteq P$.

We shall use (89) to construct an environment $e$ for $\mathcal{S}$ such that $\Psi$ does not solve $P$ in $e$. This suffices to finish the proof.

Recall our enumeration $\{v_i \mid i \in N\}$ of $Var$. Let $\{s_i \mid i \in N\}$ enumerate $|\mathcal{S}|$. Let $\{\alpha_i \mid i \in N\}$ enumerate **Obs**. Set $a^{-1} = \emptyset$ and $\sigma^{-1} = \emptyset$. The construction of $e$ proceeds by defining $\sigma^k \in SEQ$ and finite assignment $a^k : Var \to |\mathcal{S}|$ for each $k \in N$ such that:

(90) (a) $a^k$ extends $a^{k-1}$;

(b) $v_k \in domain(a^k)$ and $s_k \in content(a^k)$;

(c) $\sigma^k \supset \sigma^{k-1}$;

(d) $Var(\sigma^k) \subseteq domain(a^k)$ and $\mathcal{S} \models \bigwedge \sigma^k[a^k]$;

(e) $\alpha_k \in content(\sigma^k)$ iff $\mathcal{S} \models \alpha_k[a^k]$.

(f) either $\Psi(\sigma^k)$ is not defined or $\Psi(\sigma^k) = \emptyset$, or $\Psi(\sigma^k) \not\subseteq P$.

By (90)a,b, $\bigcup_{k \in N} a^k$ is a full assignment to $\mathcal{S}$. By (90)c-e, $\bigcup_{k \in N} \sigma^k$ is an environment $e$ for $\mathcal{S}$ and $\bigcup_{k \in N} a^k$. By (90)f, $\Psi$ does not solve $P$ in $e$.

Let $k \in N$ be given, and suppose that $a^{-1} \ldots a^{k-1}, \sigma^{-1} \ldots, \sigma^{k-1}$ have been defined. We define $a^k$ and $\sigma^k$. Let $n \geq k$ be least such that every variable appearing in $\alpha_k$ has index less than or equal to $n$. Let $a$ be any finite extension of $a^{k-1}$ such that $\{v_i \mid i \leq n\} \subseteq domain(a)$ and $s_k \in content(a) \subseteq |\,\mathcal{S}\,|$. Define $\sigma \in SEQ$ to be $\sigma^{k-1} * \alpha_k$ if $\mathcal{S} \models \alpha_k[a]$; otherwise, define $\sigma$ to be $\sigma^{k-1} * \sharp$. By (89), let $\tau \in SEQ$ and finite extension $a' : Var \to |\,\mathcal{S}\,|$ of $a$ be such that $Var(\tau) \subseteq domain(a')$, $\mathcal{S} \models \bigwedge \tau[a']$, and either $\Psi(\sigma * \tau)$ is not defined or $\Psi(\sigma * \tau) = \emptyset$, or $\Psi(\sigma * \tau) \not\subseteq P$. Define $a^k = a'$ and $\sigma^k = \sigma * \tau$. It is easy to verify that $a^k$ and $\sigma^k$ satisfy (90). ■

## 8.2. Proof of Lemma (31)

(31) Lemma:

(a) Suppose that **Obs** $= \mathcal{L}_{atomic}$. Then an invalid formula is refutable iff it is logically equivalent to a formula of form $\forall x_1 \ldots \forall x_n \neg\psi$, where $\psi$ is a positive formula.

(b) Suppose that **Obs** $= \mathcal{L}_{basic}$. Then a formula is refutable iff it is logically equivalent to a $\forall$ formula.

We obtain Lemma (31) from the following lemmas, which will also be useful later on. The first is evident.

(91) Lemma:

(a) A conjunction of refutable formulas is refutable.

(b) A disjunction of refutable formulas is refutable.

(92) Lemma: An invalid formula is refutable iff it is logically equivalent to a formula of form $\forall x_1 \ldots \forall x_n \neg\psi$, where $\psi$ is a formula built from **Obs** using conjunctions and disjunctions only and $x_1 \ldots x_n$ is a (possibly empty) sequence of variables.

*Proof of Lemma (92):* Let invalid $\varphi \in \mathcal{L}_{form}$ be given. Suppose that $\varphi$ is refutable. Let $X$ be the set of pairs $(\mathcal{S}, h)$, where $\mathcal{S}$ is a structure, $h$ is a full assignment to $\mathcal{S}$, and $\mathcal{S} \not\models \varphi[h]$. For all $(\mathcal{S}, h) \in X$ we can choose by hypothesis a finite subset $D_{\mathcal{S},h}$ of **Obs** such that $\mathcal{S} \models D_{\mathcal{S},h}[h]$ and $D_{\mathcal{S},h} \models \neg\varphi$. Given $(\mathcal{S}, h) \in X$ set $\psi_{\mathcal{S},h} = \forall x_1 \ldots \forall x_n \neg \bigwedge D_{\mathcal{S},h}$, where $x_1 \ldots x_n$ are all the variables that occur free in $D_{\mathcal{S},h}$ but not free in $\varphi$. It is easy to verify that $\{\psi_{\mathcal{S},h} \mid (\mathcal{S}, h) \in X\} \cup \{\neg\varphi\}$ is not satisfiable. Since only finitely many variables occur free in that set and $X$ is nonempty, it follows by compactness that there exists a nonempty, finite subset $Y$ of $X$ such that $\{\psi_{\mathcal{S},h} \mid (\mathcal{S}, h) \in Y\} \cup \{\neg\varphi\}$ is inconsistent. We show that $\varphi$ is logically equivalent to $\bigwedge_{\mathcal{S},h \in Y} \psi_{\mathcal{S},h}$. Trivially, $\bigwedge_{\mathcal{S},h \in Y} \psi_{\mathcal{S},h} \models \varphi$. Let $(\mathcal{S}, h) \in Y$ be given. Since $\varphi \models \neg \bigwedge D_{\mathcal{S},h}$ by the definition of $D_{\mathcal{S},h}$, it follows from the definition of $\psi_{\mathcal{S},h}$ that $\varphi \models \psi_{\mathcal{S},h}$. Clearly $\bigwedge_{\mathcal{S},h \in Y} \psi_{\mathcal{S},h}$ is logically equivalent to a formula of form $\forall x_1 \ldots \forall x_n \neg\psi$, where $\psi$ is a disjunction of conjunctions of members of **Obs** and $x_1 \ldots x_n$ is a (possibly empty) sequence of variables.

Conversely, suppose that $\varphi$ is logically equivalent to $\forall x_1 \ldots \forall x_n \neg\psi$, where $\psi$ is a formula built from **Obs** using conjunctions and disjunctions only and $x_1 \ldots x_n$ is a (possibly empty) sequence of variables. Without loss of generality we can assume that $\psi = \psi_1 \vee \ldots \vee \psi_p$, where $\psi_1 \ldots \psi_p$ are conjunctions of members of **Obs**. Let structure $\mathcal{S}$ and full assignment $h$ to $\mathcal{S}$ be such that $\mathcal{S} \not\models \varphi[h]$. Hence there exists $\chi \in \{\psi_1 \ldots \psi_p\}$ such that

$\mathcal{S} \not\models \forall x_1 \ldots \forall x_n \neg\chi[h]$. So $\mathcal{S} \models \exists x_1 \ldots \exists x_n \chi[h]$, and we can choose variables $y_1 \ldots y_n$ such that $\mathcal{S} \models \chi[y_1/x_1 \ldots y_n/x_n][h]$. By Convention (3), $\chi[y_1/x_1 \ldots y_n/x_n]$ is a conjunction of members of **Obs**. Moreover, since $\varphi \models \forall x_1 \ldots \forall x_n \neg\psi$, we infer that $\varphi \models \forall x_1 \ldots \forall x_n \neg\chi$, hence $\varphi \models \neg\chi[y_1/x_1 \ldots y_n/x_n]$, hence $\chi[y_1/x_1 \ldots y_n/x_n] \models \neg\varphi$, and we conclude that $\varphi$ is refutable. ■

The following corollary is immediate. So is Lemma (31), which we wanted to prove.

(93) COROLLARY: Suppose that **Obs** is closed under negation. An invalid formula is refutable iff it is logically equivalent to a formula of form $\forall x_1 \ldots \forall x_n \psi$, where $\psi$ is a boolean combination of members of **Obs** and $x_1 \ldots x_n$ is a (possibly empty) sequence of variables.

## 8.3. Proof of Examples (36) and (37)

(36) EXAMPLE: Suppose that $\mathcal{L}_{basic} \subseteq$ **Obs**. Then the problem defined in Example (11) has tip-offs.

*Proof:* Let **P** be the problem defined in Example (11). As a tip-off for $MOD(T \cup \{\theta\})$ in **P** we may take $\{\{\forall x R v_i x\} \mid i \in N\}$. A tip-off for $MOD(T \cup \{\neg\theta\})$ in **P** is $\{\{\forall x R x v_i\} \mid i \in N\}$. ■

(37) EXAMPLE: Suppose that **Obs** $= \mathcal{L}_{basic}$, and that binary predicate $R$ is the only symbol of **Sym**. Let $\mathcal{S} = \langle N, \preceq \rangle$ be isomorphic to $\omega$, let $\mathcal{T} = \langle Z, \preceq^* \rangle$, with $Z$ the set of integers, be isomorphic to $\omega^*\omega$, and let disjoint propositions $P_1, P_2$ be such that $\mathcal{S} \in P_1$ and $\mathcal{T} \in P_2$. Then $\{P_1, P_2\}$ does not have tip-offs.

*Proof:* We show that there is no tip-off for $P_2$ in $\{P_1, P_2\}$. Let $n \in N$, variables $x_0 \ldots x_n$, and set $\pi$ consisting of $\forall$ formulas all of whose free variables are taken from $\{x_0 \ldots x_n\}$

be given. Suppose that $\pi$ is satisfiable in $\mathcal{T}$. By Example (34), and since $\mathcal{S} \in P_1$ and $\mathcal{T} \in P_2$, we can conclude that there is no tip-off for $P_2$ in $\{P_1, P_2\}$ if we show that $\pi$ is satisfiable in $\mathcal{S}$. Choose $a_o \dots a_n \in Z$ such that for all $\varphi \in \pi$, $\mathcal{T} \models \varphi[a_0/x_0 \dots a_n/x_n]$. Choose $k \geq 0$ such that $\{a_i + k \,|\, i \leq n\} \subseteq N$. It is easy to verify that for all $\varphi \in \pi$, $\mathcal{T} \models \varphi[a_0 + k/x_0 \dots a_n + k/x_n]$. Since $\forall$ formulas are preserved in substructures (see [11, Cor. 2.4.2]), it follows that for all $\varphi \in \pi$, $\mathcal{S} \models \varphi[a_0 + k/x_0 \dots a_n + k/x_n]$. Hence $\pi$ is satisfiable in $\mathcal{S}$. ■

## 8.4. Proof of Proposition (38)

(38) PROPOSITION: If problem **P** is countable and has tip-offs, then **P** is solvable.

*Proof:* Let $\{P_j \,|\, j < \kappa\}$ be a repetition-free enumeration of the countably many propositions in **P**, where $\kappa = card(\mathbf{P})$. Let $\{\boldsymbol{t}_j \,|\, j < \kappa\}$ enumerate tip-offs corresponding to the $P_j$. Since **P** is nonempty and countable, and since each tip-off is countable, we may enumerate $\bigcup_{j<\kappa} \boldsymbol{t}_j$ as $\{\pi_i \,|\, i \in N\}$. For $i \in N$, we let $\pi_i = Y_i \cup Z_i$, where $Y_i$ and $Z_i$ are as specified in Definition (32)a. By the same definition we have:

(94) (a) For every $\mathcal{S} \in \bigcup \mathbf{P}$ and full assignment $h$ to $\mathcal{S}$, there is $i \in N$ such that $\mathcal{S} \models \pi_i[h]$.

(b) For every $i$ such that $\pi_i$ is satisfiable in some $\mathcal{S} \in \bigcup \mathbf{P}$, there is exactly one $j$ such that $\pi_i \in \boldsymbol{t}_j$.

We specify scientist $\Psi$ that solves **P**. Let $\sigma \in SEQ$ be given. If for all $i$, $\{\bigwedge \sigma\} \cup \pi_i$ is not satisfiable in any $\mathcal{S} \in \bigcup \mathbf{P}$ or $\bigwedge \sigma$ does not imply $Y_i$, then $\Psi(\sigma)$ is undefined. Otherwise, let $i$ be least such that $\{\bigwedge \sigma\} \cup \pi_i$ is satisfiable in some $\mathcal{S} \in \bigcup \mathbf{P}$ and $\bigwedge \sigma$ implies $Y_i$. Then $\Psi(\sigma) = P_j$, where by (94)b, $j$ is unique with $\pi_i \in \boldsymbol{t}_j$. To see that $\Psi$ solves **P**, let $j < \kappa$,

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$\mathcal{S} \in P_j$, full assignment $h$ to $\mathcal{S}$, and environment $e$ for $\mathcal{S}$ and $h$ be given. By (94)a, let $i_0$ be least with $\mathcal{S} \models \pi_{i_0}[h]$. Then by Definition (32), $j$ is unique with $\pi_{i_0} \in \boldsymbol{t}_j$. We need to show that $\Psi(e[k]) = P_j$ for cofinitely many $k$. By the definition of $\Psi$, it suffices for this purpose to show:

(95) (a) there are cofinitely many initial segments $\sigma$ of $e$ such that $\bigwedge \sigma$ implies $Y_{i_0}$;

(b) for all $i < i_0$ there are cofinitely many initial segments $\sigma$ of $e$ such that $\{\bigwedge \sigma\} \cup \pi_i$ is inconsistent or $\bigwedge \sigma$ does not imply $Y_{i_0}$.

(95)a follows immediately from the facts that $Y_{i_0} \subseteq \mathbf{Obs}$, $e$ is an environment for $\mathcal{S}$ and $h$, and $\mathcal{S} \models \pi_{i_0}[h]$. To prove (95)b, let $i < i_0$ be given. If $\mathcal{S} \not\models Y_i[h]$, then all initial segments $\sigma$ of $e$ are such that $\bigwedge \sigma \not\models Y_i$. So suppose that $\mathcal{S} \models Y_i[h]$. Then $\mathcal{S} \not\models (\pi_i - Y_i)[h]$. So there is $\psi \in \pi_i - Y_i$ such that $\mathcal{S} \not\models \psi[h]$. Since $\psi \in \pi_i - Y_i$, $\psi$ is refutable by Definition (32)a. So by Definition (30) there is finite $D \subseteq \mathbf{Obs}$ such that $\mathcal{S} \models D[h]$ and $D \models \neg\psi$. Hence, since $e$ is an environment, there is $k \in N$ such that $\bigwedge e[k] \models D[h]$. So $\{\bigwedge e[k], \psi\}$ is inconsistent. So $\{\bigwedge e[k'], \psi\}$ is inconsistent for all $k' \geq k$, implying (95)b. ■

## 8.5. Proof of Proposition (40)

(40) Proposition: Every solvable problem has tip-offs.

*Proof:* Let problem $\mathbf{P}$ and scientist $\Psi$ be such that $\Psi$ solves $\mathbf{P}$. Let $P \in \mathbf{P}$ be given. We exhibit a tip-off for $P$ in $\mathbf{P}$. Given $\sigma \in \mathit{SEQ}$, define $X_\sigma$ to be the collection of pairs $(\mathcal{S}, a)$ such that $\mathcal{S} \in P$, $a$ is a finite assignment to $\mathcal{S}$, and $(\sigma, a)$ is a locking pair for $\Psi$, $\mathcal{S}$ and $P$. Define $\Pi_\sigma$ to be the set consisting of *content*$(\sigma)$ together with all refutable formulas $\psi$ such that:

(a) $\mathit{Var}(\psi) \subseteq \mathit{Var}(\sigma)$;

(b) for all $(\mathcal{S}, a) \in X_\sigma$, $\mathcal{S} \models \psi[a]$.

Given $\sigma \in SEQ$ and $f : Var(\sigma) \to Var$, set $\Pi_{\sigma,f} = \{\xi[f(x)/x, x \in Var(\sigma)] \mid \xi \in \Pi_\sigma\}$. We claim that $\boldsymbol{t} = \{\Pi_{\sigma,f} \mid \sigma \in SEQ, f : Var(\sigma) \to Var\}$ is a tip-off for $P$ in $\mathbf{P}$. By Convention (3), for all $\psi \in content(\sigma)$, $\psi[f(x)/x, x \in Var(\sigma)]$ belongs to **Obs**, and it is easy to verify that for every refutable formula $\psi$, $\psi[f(x)/x, x \in Var(\sigma)]$ is also refutable. It follows easily that $\boldsymbol{t}$ is a countable collection of sets of form $Y \cup Z$, where $Y$ and $Z$ are as stipulated in Definition (32)a. To prove clause (b) of Definition (32), let $\mathcal{S} \in P$ and full assignment $h$ to $\mathcal{S}$ be given. By Lemma (29), let $\sigma \in SEQ$ and finite assignment $a : Var \to |\mathcal{S}|$ be such that $(\sigma, a)$ is a locking pair for $\Psi$, $\mathcal{S}$, and $P$. Then $(\mathcal{S}, a) \in X_\sigma$ and $\mathcal{S} \models \bigwedge \sigma[a]$, so $\mathcal{S} \models \Pi_\sigma[a]$. Since $h$ is onto $|\mathcal{S}|$, we can choose $f : Var(\sigma) \to Var$ such that $\mathcal{S} \models \Pi_{\sigma,f}[h]$. Hence, $\Pi_{\sigma,f}$ is the set in $\boldsymbol{t}$ that witnesses (32)b.

To finish the proof, we show that (32)c is satisfied. Let $\sigma \in SEQ$, $f : Var(\sigma) \to Var$, $\mathcal{U} \in P' \in \mathbf{P}$ with $P' \neq P$, and full assignment $g$ to $\mathcal{U}$ be given. Suppose for a contradiction that $\mathcal{U} \models \Pi_{\sigma,f}[g]$. Then we can choose full assignment $g'$ to $\mathcal{U}$ such that:

(96) $\mathcal{U} \models \Pi_\sigma[g']$.

Since $content(\sigma) \subseteq \Pi_\sigma$, there is an environment $e$ for $\mathcal{U}$ and $g'$ such that $\sigma \subset e$. We will show that for all $k \geq length(\sigma)$, $\emptyset \neq \Psi(e[k]) \subseteq P$. Since $e$ is for $\mathcal{U} \in P' \in \mathbf{P}$ with $P' \neq P$, this implies that $\Psi$ does not solve $P'$, contradicting our choice of $\Psi$. So let $k \geq length(\sigma)$ be given. Choose $\tau \in SEQ$ such that $e[k] = \sigma * \tau$. We must show:

(97) $\emptyset \neq \Psi(\sigma * \tau) \subseteq P$.

For this purpose, we establish:

(98) Claim: There is $(\mathcal{S}, a) \in X_\sigma$ such that $\mathcal{S} \models \exists \bar{x} \bigwedge(\sigma * \tau)[a]$, where $\bar{x}$ contains the variables in $Var(\tau) - domain(a)$.

*Proof of (98):* For a contradiction, suppose that for all $(\mathcal{S}, a) \in X_\sigma$, $\mathcal{S} \models \forall\bar{x}\neg\bigwedge(\sigma{*}\tau)[a]$. Let $(\mathcal{S}, a) \in X_\sigma$ be given. Since $Var(\sigma) \subseteq domain(a)$, $\mathcal{S} \models ((\neg\bigwedge\sigma) \vee (\forall\bar{x}\neg\bigwedge\tau))[a]$. With the fact that $\mathcal{S} \models \bigwedge\sigma[a]$, this implies that $\mathcal{S} \models \forall\bar{x}\neg\bigwedge\tau[a]$. By Lemma (92), $\forall\bar{x}\neg\bigwedge\tau$ is refutable. Hence we infer that $\forall\bar{x}\neg\bigwedge\tau \in \Pi_\sigma$, and it follows from (96) that $\mathcal{U} \models \forall\bar{x}\neg\bigwedge\tau[g']$. However, this is impossible since $e$ was chosen to be for $\mathcal{U}$ and $g'$, and $\sigma{*}\tau \subset e$. ▮

We now use (98) to prove (97). For any $(\mathcal{S}, a) \in X_\sigma$, $(\sigma, a)$ is a locking pair for $\Psi$, $\mathcal{S}$, and $P$. Hence, by Claim (98) and clause (c) of Definition (28), $\emptyset \neq \Psi(\sigma{*}\tau) \subseteq P$. ■

## 8.6. Proof of Theorem (59)

(59) Theorem: A problem of form $(T, \{\theta_0 \ldots \theta_n\})$ is solvable if and only if for every $i \leq n$, $\theta_i$ is equivalent in $T$ to the existential closure of any of the following kinds of formulas.

(a) a formula built from **Obs** using only conjunctions and disjunctions,

(b) a refutable formula, or

(c) a formula of form $\chi \wedge \varphi$, where $\chi$ is built from **Obs** using only conjunctions and disjunctions and $\varphi$ is refutable.

*Proof:* Let **P** be a problem of form $(T, \{\theta_0 \ldots \theta_n\})$. Call a sentence *special* if it is the existential closure of a formula of form $\bigwedge D \wedge \varphi$ where $D$ is a finite subset of **Obs**, or of a refutable formula, or of a formula of form $\bigwedge D \wedge \varphi$, where $D$ is a finite subset of **Obs** and $\varphi$ is a refutable formula. By Convention (3) and Lemma (91)b, it suffices to show that **P** is solvable if and only if for every $i \leq n$, $\theta_i$ is equivalent in $T$ to a disjunction of special sentences. For the right-to-left direction, let $i \leq n$ be given, and suppose that $\theta_i$ is equivalent in $T$ to $\psi_0 \vee \ldots \vee \psi_m$, where $\psi_0 \ldots \psi_m$ are special sentences. Then every member of $MOD(T \cup \theta_i)$ is a model of $\psi_p$ for some $p \leq m$. Moreover, for all $j \leq n$ with $j \neq i$ and for all $p \leq m$, no model of $MOD(T \cup \theta_j)$ is a model of $\psi_p$. It follows easily from Proposition (38) that if for all $i \leq n$, $\theta_i$ is equivalent in $T$ to a disjunction of special sentences, then $\mathbf{P}$ is solvable.

For the left-to-right direction, suppose that $\mathbf{P}$ is solvable. Let $i \leq n$ be given. By Proposition (40), let $\boldsymbol{t}$ be a tip-off for $MOD(T \cup \{\theta_i\})$ in $(T, \{\theta_0 \ldots \theta_n\})$. By Definition (32), for all $j \leq n$ with $j \neq i$, every member of $\boldsymbol{t}$ is unsatisfiable in every $\mathcal{U} \in MOD(T \cup \{\theta_j\})$. Since the $\theta_j$'s partition the models of $T$, it follows that $T \cup \pi \models \theta_i$ for all $\pi \in \boldsymbol{t}$. By compactness, for all $\pi \in \boldsymbol{t}$ there is finite $Y \subseteq \pi$ with $T \cup Y \models \theta_i$. From this, the fact that a conjunction of refutable formulas is refutable [Lemma (91)a], and Definition (32)a, we infer:

(99) For every $\mathcal{S} \in MOD(T \cup \{\theta_i\})$ there is a special $\chi_{\mathcal{S}} \in \mathcal{L}_{sen}$ such that:

(a) $\mathcal{S} \models \chi_{\mathcal{S}}$.

(b) $T \cup \{\chi_{\mathcal{S}}\} \models \theta_i$.

Let $\Sigma = \{\chi_{\mathcal{S}} \,|\, \mathcal{S} \in MOD(T \cup \{\theta_i\})\}$. We show that $\theta_i$ is equivalent over $T$ to some finite disjunction of members of $\Sigma$. By (99)b, for every disjunction $\rho$ of members of $\Sigma$, $T \cup \{\rho\} \models \theta_i$. Hence, it suffices to show that for some such $\rho$, $T \cup \{\theta_i\} \models \rho$. For a reductio suppose that for every disjunction $\rho$ of members of $\Sigma$, $T \cup \{\theta_i\} \not\models \rho$. Then for every finite $\Delta \subseteq \Sigma$, $T \cup \{\theta_i\} \cup \{\neg\varphi \,|\, \varphi \in \Delta\}$ is satisfiable. Hence by the compactness and Löwenheim-Skolem theorems there is (countable) $\mathcal{S} \in MOD(T \cup \{\theta_i\})$ such that $\mathcal{S} \models \{\neg\varphi \,|\, \varphi \in \Sigma\}$. But this implies that there is $\mathcal{S} \in MOD(T \cup \{\theta_i\})$ such that $\mathcal{S} \not\models \chi_{\mathcal{S}}$, contradicting (99)a. ■

## 8.7. Proof of Proposition (67)

(67) Proposition: Let solvable problem $\mathbf{P}$ of form $(T, \{P_0, P_1, \ldots\})$ be given. Then for all $P \in \mathbf{P}$, there is an enumeration $\{X_i \,|\, i \in N\}$ of sets of existential

closures of formulas of form $\bigwedge D \wedge \varphi$, where $D$ is a (possibly empty) finite subset of **Obs** and $\varphi$ a refutable formula, such that $P = \bigcup_{i \in N} MOD(T \cup X_i)$.

*Proof:* Let $P \in \mathbf{P}$ be given. By Proposition (40), let $\{\pi_i \,|\, i \in N\}$ be an enumeration of the members of a tip-off for $P$ in $\mathbf{P}$. For all $i \in N$ denote by $X_i$ the set of existential closures of all formulas of form $\bigwedge D$, with $D$ a nonempty, finite subset of $\pi_i$. By Definition (32) and Lemma (91)a, and since valid formulas are refutable, it suffices to show that $P = \bigcup_{i \in N} MOD(T \cup X_i)$. To show the left-to-right inclusion, let $\mathcal{S} \in P$ be given. By Definition (32)b let $i \in N$ be such that $\pi_i$ is satisfiable in $\mathcal{S}$. Hence, $T \cup X_i$ is satisfiable in $\mathcal{S}$. So $P \subseteq \bigcup_{i \in N} MOD(T \cup X_i)$. For the other direction, suppose for a contradiction that $\bigcup_{i \in N} MOD(T \cup X_i) - P \neq \emptyset$. Let structure $\mathcal{S}$ and $i \in N$ be such that $\mathcal{S} \in MOD(T \cup X_i) - P$. Since $\bigcup \mathbf{P} = MOD(T)$, $\mathcal{S}$ belongs to some $P' \in \mathbf{P}$ with $P' \neq P$. By Proposition (40) let $\boldsymbol{t}$ be a tip-off for $P'$ in $\mathbf{P}$, and let $\pi \in \boldsymbol{t}$ be such that $\pi$ is satisfiable in $\mathcal{S}$. Without loss of generality we may suppose that $\pi_i$ and $\pi$ share no free variables. It is clear that $T \cup \pi_i \cup \pi$ is unsatisfiable, since otherwise some model of $T$ satisfies members from distinct tip-offs, contradicting Definition (32). Summarizing:

(100) (a) all variables that occur free in $\pi_i$ do not occur free in $\pi$;

(b) $\pi$ is satisfiable in $\mathcal{S}$;

(c) $T \cup \pi_i \cup \pi$ is unsatisfiable.

By (100)c and compactness, let nonempty, finite $D \subseteq \pi_i$ and finite $D' \subseteq \pi$ be such that:

(101) $T \cup D \cup D'$ is unsatisfiable.

Denote by $\varphi$ the existential closure of $\bigwedge D$ and by $\varphi'$ the universal closure of $\neg \bigwedge D'$. It follows from (100)a and (101) that:

(102) $T \models \varphi \rightarrow \varphi'$.

We deduce from (100)b that $\mathcal{S} \not\models \varphi'$, which with (102) and $\mathcal{S} \models T$ implies that $\mathcal{S} \models \neg\varphi$. Since $\varphi \in X_i$ this contradicts the hypothesis that $\mathcal{S} \in MOD(X_i)$. ■

## 8.8. Proof of Proposition (79)

(79) THEOREM: Every solvable problem is solvable strongly efficiently.

Proof of the theorem relies on two lemmas. Here is the first.

(103) LEMMA: Let solvable problem $\mathbf{P}$ and scientist $\Psi$ that solves $\mathbf{P}$ be given. Then $\Psi$ is not dominated on $\mathbf{P}$ if and only if for all $\sigma \in SEQ$, if $\sigma$ is for $\mathbf{P}$ then there exists $P \in \mathbf{P}$ and environment $e$ for $\mathbf{P}$ such that:

(a) $e$ extends $\sigma$, and

(b) for every $k \geq length(\sigma)$, $\emptyset \neq \Psi(e[k]) \subseteq P$.

*Proof:* Suppose that $\Psi$ is not dominated on $\mathbf{P}$. Let $\sigma$ be for $\mathbf{P}$, and suppose for a contradiction that:

(104) for all $P \in \mathbf{P}$ and for all environments $e$ for $P$ that extend $\sigma$, there is $k \geq length(\sigma)$ such that either $\Psi(e[k])$ is not defined or $\Psi(\sigma) = \emptyset$ or $\Psi(e[k]) \not\subseteq P$.

Since some environment for $\mathbf{P}$ extends $\sigma$ and $\Psi$ solves $\mathbf{P}$, there is least $k_0 \in N$ with the following property:

(105) There is $P \in \mathbf{P}$ and environment $e$ for $P$ such that $e$ extends $\sigma$ and $\emptyset \neq \Psi(e[k]) \subseteq P$ for all $k \geq k_0$.

It follows from (104) and (105) that:

(106) $k_0 > \mathit{length}(\sigma)$.

Let scientist $\Psi'$ satisfy the following conditions, for all $\tau \in \mathit{SEQ}$. If $\sigma \subseteq \tau \subseteq e[k_0]$ then $\Psi'(\tau) = P$. If $\Psi(\tau)$ is defined but either $\sigma \not\subseteq \tau$ or $\tau \not\subseteq e[k_0]$, then $\Psi'(\tau) = \Psi(\tau)$.

It is immediate that $\Psi'$ solves $\mathbf{P}$. Let environment $e'$ for $P' \in \mathbf{P}$ be given. If $e'$ does not extend $\sigma$ then $\mathit{SP}(\Psi', e', P') = \mathit{SP}(\Psi, e', P')$ by the definition of $\Psi'$. If $e'$ extends $\sigma$ and $e' \neq e$, then $\mathit{SP}(\Psi, e', P') \geq k_0$ by the definition of $k_0$, and $\mathit{SP}(\Psi', e', P') \leq \mathit{SP}(\Psi, e', P')$ by the definition of $\Psi'$. Finally from (105), (106), and the definition of $\Psi'$ it is easy to see that $\mathit{SP}(\Psi', e, P') \leq \mathit{length}(\sigma) < k_0 = \mathit{SP}(\Psi, e, P')$. This contradicts the hypothesis. So we have shown that if $\Psi$ is not dominated on $\mathbf{P}$, then conditions (a) and (b) are satisfied.

For the opposite suppose that conditions (a) and (b) are satisfied. Let scientist $\Psi'$ solve $\mathbf{P}$, and let environment $e$ for $P \in \mathbf{P}$ be such that $\mathit{SP}(\Psi', e, P) < \mathit{SP}(\Psi, e, P)$ (if there is no such $\Psi'$, $P$ and $e$, then there is nothing left to prove). Since $\mathit{SP}(\Psi', e, P) < \mathit{SP}(\Psi, e, P)$ there is $k_0 \in N$ such that:

(107) It is not true that $\emptyset \neq \Psi(e[k_0]) \subseteq P$ and $\emptyset \neq \Psi'(e[k_0]) \subseteq P$.

By hypothesis there exists $P' \in \mathbf{P}$ and environment $e'$ for $P'$ such that $e'$ extends $e[k_0]$ and $\emptyset \neq \Psi(e'[k]) \subseteq P'$ for all $k \geq k_0$. This, (107), and the fact that $\Psi$ solves $\mathbf{P}$ imply that $\mathit{SP}(\Psi, e', P') \leq k_0 < \mathit{SP}(\Psi', e', P')$. With Definition (76) we conclude that $\Psi$ is not dominated on $\mathbf{P}$, as required. ■

Here is the second lemma.

(108) Lemma: Suppose that scientist $\Psi$ solves problem $\mathbf{P}$. Then $\Psi$ solves $\mathbf{P}$ strongly efficiently if and only if for every $\sigma \in SEQ$, if $\sigma$ is for $\mathbf{P}$ then there exists $P \in \mathbf{P}$, $\mathcal{S} \in P$, and full assignment $h$ to $\mathcal{S}$ such that:

(a) $\mathcal{S} \models \bigwedge \sigma[h]$, and

(b) for every environment $e$ for $\mathcal{S}$ and $h$ that extends $\sigma$, for every $k \geq length(\sigma)$, $\emptyset \neq \Psi(e[k]) \subseteq P$.

*Proof:* Suppose that $\Psi$ solves $\mathbf{P}$ strongly efficiently. Let $\sigma \in SEQ$ be for $\mathbf{P}$. We show that conditions (a),(b) are satisfied for some $P \in \mathbf{P}$, $\mathcal{S} \in P$, and full assignment $h$ to $\mathcal{S}$. Plainly, $\Psi$ is not dominated on $\mathbf{P}$. So by Lemma (103) there is $P \in \mathbf{P}$ such that $\bigwedge \sigma$ is satisfiable in some member of $P$ and $\emptyset \neq \Psi(\sigma) \subseteq P$. Suppose that for all $P' \in \mathbf{P}$, if $P' \neq P$ then $\bigwedge \sigma$ is satisfiable in no member of $P'$. Let scientist $\Psi'$ have the following properties, for all $\tau \in SEQ$. If $\sigma \subseteq \tau$ then $\Psi'(\tau) = P$. If $\sigma \not\subseteq \tau$ and $\Psi(\tau)$ is defined, then $\Psi'(\tau) = \Psi(\tau)$. It is easy to see that $\Psi'$ solves $\mathbf{P}$, and that for every $P' \in \mathbf{P}$ and for every environment $e$ for $P'$:

(a) $SP(\Psi', e, P') \leq SP(\Psi, e, P')$, and

(b) if $e$ extends $\sigma$ and there is $k > length(\sigma)$ such that either $\Psi(e[k])$ is undefined or $\Psi(e[k]) = \emptyset$ or $\Psi(e[k]) \not\subseteq P'$, then $SP(\Psi', e, P') < SP(\Psi, e, P')$.

It follows from Definition (76) that $\emptyset \neq \Psi(e[k]) \subseteq P$ for every environment $e$ for $P$ that extends $\sigma$ and for every $k \geq length(\sigma)$. So, conditions (a),(b) of the lemma are satisfied. Thus, to conclude this direction of the proof, suppose that there is $P' \in \mathbf{P}$ with $P' \neq P$ such that $\bigwedge \sigma$ is satisfiable in some member of $P'$. Let $P' \in \mathbf{P}$ and environment $e'$ for $P'$ be such that $P' \neq P$ and $\sigma \subseteq e'$. Let scientist $\Psi'$ be defined as follows, for every $\gamma \in SEQ$. If $\gamma \not\subseteq e'$ then $\Psi'(\gamma) = \Psi(\gamma)$. If $\gamma \subset \sigma$ then $\Psi'(\gamma) = \emptyset$. If $\sigma \subseteq \gamma \subseteq e'$ then $\Psi'(\gamma) = P'$. Trivially, $\Psi'$ solves $\mathbf{P}$ and $length(\sigma) = SP(\Psi', e', P') < SP(\Psi, e', P')$. From this and Definition (78), we deduce that there is $\mathcal{S} \in P$ and full assignment $h$ to $\mathcal{S}$ such that $\mathcal{S} \models \bigwedge \sigma[h]$, and $\emptyset \neq \Psi(e[k]) \subseteq P$ for every environment $e$ for $\mathcal{S}$ and $h$ that extends $\sigma$ and for every $k \geq \mathit{length}(\sigma)$. Hence conditions (a),(b) of the lemma are satisfied.

Conversely, suppose that for all $\sigma \in \mathit{SEQ}$, if $\sigma$ is for $\mathbf{P}$ then conditions (a),(b) are satisfied for some $P \in \mathbf{P}$, $\mathcal{S} \in P$, and full assignment $h$ to $\mathcal{S}$. We show that $\Psi$ solves $\mathbf{P}$ strongly efficiently. Suppose that scientist $\Psi'$ solves $\mathbf{P}$. Suppose there is $P_0 \in \mathbf{P}$ and environment $e_0$ for $P_0$ such that $\mathit{SP}(\Psi', e_0, P_0) < \mathit{SP}(\Psi, e_0, P_0)$. (If there is no such $\Psi'$, $P_0$ and $e_0$, then there is nothing left to prove.) Let $k_0 \in N$ be such that $k_0 = \mathit{SP}(\Psi', e_0, P_0)$. Choose $\mathcal{S} \in P$ and full assignment $h$ to $\mathcal{S}$ such that $\mathcal{S} \models \bigwedge e_0[k_0][h]$ and for every environment $e$ for $\mathcal{S}$ and $h$ that extends $e_0[k_0]$, for every $k \geq k_0$, $\emptyset \neq \Psi(e[k]) \subseteq P$. Suppose that $P = P_0$. Then $e_0$ is for $P$, and $\mathit{SP}(\Psi, e_0, P_0) \leq k_0 = \mathit{SP}(\Psi', e_0, P_0)$, contradiction. Hence $P \neq P_0$. Since $\Psi'(e_0[k_0]) = P_0 \neq P$, we infer that for every environment $e$ for $\mathcal{S}$ and $h$ that extends $e_0[k_0]$, $\mathit{SP}(\Psi, e, P) \leq k_0 < \mathit{SP}(\Psi', e, P)$. With Definition (78) this completes the proof of the lemma. ■

*Proof of Theorem (79):* Let problem $\mathbf{P}$ be solvable, and let scientist $\Psi$ be as specified in the proof of Proposition (38). It follows easily that for every $\sigma \in \mathit{SEQ}$ for $\mathbf{P}$, we may choose $P \in \mathbf{P}$, $\mathcal{S} \in P$, and full assignment $h$ to $\mathcal{S}$ with the following properties:

(109) (a) $\mathcal{S} \models \bigwedge \sigma[h]$;

(b) $\Psi(e[k]) = P$, for every $k \geq \mathit{length}(\sigma)$, and every environment $e$ for $\mathcal{S}$ and $h$ that extends $\sigma$.

Relying on Lemma (108) we conclude from (109) that $\Psi$ solves $\mathbf{P}$ strongly efficiently. ■

## 8.9. Proof of Theorem (83)

(83) THEOREM: Suppose that **Sym** is limited to a binary predicate, two constants, and a unary function symbol. Let **Obs** $= \mathcal{L}_{basic}$. Then for every countable collection $\Sigma$ of scientists there is a problem **P** with the following properties.

(a) Every member of **P** is strongly elementary.

(b) **P** is solvable.

(c) No member of $\Sigma$ solves **P**.

*Proof:* Let $R$ be the predicate, $\overline{0}$, $a$ be the constants, and $s$ be the unary function symbol of **Sym**. We denote $n$ applications of $s$ to $\overline{0}$ by $\overline{n}$. Let $\delta \in \mathcal{L}_{sen}$ be true in a structure iff (a) $R$ is interpreted as a discrete total ordering of the entire domain, and (b) $s$ is interpreted as a one-to-one function [that is, the structure must satisfy $\forall x \forall y(sx = sy \rightarrow x = y)$]. Given $i \in N$, let $\theta_i^+$ be $\overline{i} = a \wedge \delta \wedge \exists x \forall y Rxy$, and let $\theta_i^-$ be $\overline{i} = a \wedge \delta \wedge \neg\exists x \forall y Rxy$.

Let $X \subseteq N$ be given. We denote by $\mathbf{P}_X$ the set $\{MOD(\theta_i^+) \mid i \in X\} \cup \{MOD(\theta_i^-) \mid i \notin X\}$. It is obvious that $\mathbf{P}_X$ is a solvable problem whose members are strongly elementary. So let countable collection $\Sigma$ of scientists be given. To finish the proof it suffices to show that there is $X \subseteq N$ such that no member of $\Sigma$ solves $\mathbf{P}_X$. Because there are uncountably many subsets of $N$, it thus suffices to show:

(110) If $X$ and $Y$ are distinct subsets of $N$ then $\mathbf{P}_X \cup \mathbf{P}_Y$ is not solvable.

To demonstrate (110), let $X, Y \subseteq N$ and $i \in N$ be such that $i \in X$ iff $i \notin Y$. Set $P_1 = MOD(\theta_i^+)$ and $P_2 = MOD(\theta_i^-)$. Observe that $\{P_1, P_2\} \subseteq \mathbf{P}_X \cup \mathbf{P}_Y$. Hence it suffices to show that $\{P_1, P_2\}$ is not solvable. By Proposition (40), for this purpose it suffices to show that:

(111) $P_2$ does not have a tip-off in $\{P_1, P_2\}$.

Let $\mathcal{S} = \langle N, \leq, 0, i, \mathit{successor}\rangle$, and let $\mathcal{T} = \langle Z, \leq, 0, i, \mathit{successor}\rangle$, where $Z$ is the set of integers, 0 interprets $\overline{0}$, $i$ interprets $a$, and the successor function interprets $s$. Then $\mathcal{S} \in P_1$ and $\mathcal{T} \in P_2$, and an easy adaptation of Example (37) yields (111). ■

## 8.10. Proof of Proposition (85)

(85) Proposition: Suppose that **Sym** is limited to the vocabulary of arithmetic (including $\overline{0}$ and a unary function symbol $s$) plus the additional constant $a$. Suppose also that **Obs** includes all identities (that is, formulas of form $t_1 = t_2$, for terms $t_1, t_2$). Then there is a problem **P** with the following properties.

(a) Every member of **P** is strongly elementary.

(b) **P** is solvable computably.

(c) Every computable scientist that solves **P** is dominated on **P**.

*Proof:* Let $n$ applications of $s$ to $\overline{0}$ be denoted $\overline{n}$. Let $Q$ be the finite set of axioms of Robinson's Arithmetic. By standard results [3, Ch. 14], let $\phi(x, y) \in \mathcal{L}_{\mathit{form}}$ have two free variables $x, y$, exclude $a$, and be such that for all $i \in N$, $Q \models \exists x \phi(x, \overline{i})$ iff $i \in \mathrm{W}_i^{\mathit{num}}$. Given $i \in N$, set $P_i^+ = \mathit{MOD}(Q \cup \{a = \overline{i}, \exists x \phi(x, \overline{i})\})$, and $P_i^- = \mathit{MOD}(Q \cup \{a = \overline{i}, \neg\exists x \phi(x, \overline{i})\})$. We claim that $\mathbf{P} = \{P_i^+ \mid i \in \mathrm{W}_i^{\mathit{num}}\} \cup \{P_i^- \mid i \notin \mathrm{W}_i^{\mathit{num}}\}$ witnesses the proposition.

It is immediate that the propositions of **P** are strongly elementary. To show that **P** is computably solvable, define computable $\psi : \mathit{SEQ} \to N$ as follows. Let $\sigma \in \mathit{SEQ}$ be given. Suppose that $i \in N$ is unique with $a = \overline{i} \in \mathit{content}(\sigma)$. Then if $i$ has not appeared in $\mathrm{W}_i^{\mathit{num}}$, within $\mathit{length}(\sigma)$ steps of its standard enumeration, $\psi(\sigma)$ equals an index for $Q \cup \{a = \overline{i}, \neg\exists x \phi(x, \overline{i})\}$; and if $i$ has appeared in $\mathrm{W}_i^{\mathit{num}}$ within $\mathit{length}(\sigma)$ steps of its standard enumeration, $\psi(\sigma)$ is an index for $Q \cup \{a = \bar{i}, \exists x \phi(x, \bar{i})\}$. For all other $\sigma \in SEQ$, $\psi(\sigma)$ is undefined. It is easy to verify that if $\psi$ underlies $\Psi$, then $\Psi$ solves **P**.

Finally, for a contradiction, suppose that computable $\psi : SEQ \to N$ underlies a scientist that solves **P** and is not dominated on **P**. Then Definition (76) is easily seen to imply the following.

(112) For all $i \in N$,

(a) if $i \in \mathrm{W}_i^{num}$ then $\psi(a = \bar{i})$ is an *r.e.* index for consistent $X \subseteq \mathcal{L}_{form}$ with $X \models \exists x \phi(x, \bar{i})$,

(b) if $i \notin \mathrm{W}_i^{num}$ then $\psi(a = \bar{i})$ is an *r.e.* index for consistent $X \subseteq \mathcal{L}_{form}$ with $X \models \neg\exists x \phi(x, \bar{i})$.

But (112) yields a decision procedure for the halting problem. ■

## 8.11. Proof of Theorem (87)

(87) THEOREM: Suppose that **Obs** is recursive. Then every solvable, *r.e.* problem of form $(T, \{\theta_0 \dots \theta_n\})$ is computably solvable.

*Proof:* Let solvable, *r.e.* problem of form $(T, \{\theta_0 \dots \theta_n\})$ be given. We denote the problem by $(T, \{\theta_0 \dots \theta_n\})$ and exhibit a computable scientist $\Psi$ that solves it. For this purpose, let $\boldsymbol{A}$ be a computable proof procedure with the following property.

Given $n \in N$, $\sigma, \tau \in SEQ$ with $\sigma \subseteq \tau$, and formula $\varphi$ with $\bigwedge \sigma \models \varphi$, if $\boldsymbol{A}$ generates $\varphi$ from $\bigwedge \sigma$ in at most $n$ steps of computation, then $\boldsymbol{A}$ generates $\varphi$ from $\bigwedge \tau$ in at most $n$ steps of computation.

To specify computable $\psi : SEQ \to N$ underlying $\Psi$, we rely on some terminology and notation. Let $t$ be an *r.e.* index for $T$. Call a formula "inferable" just in case it is built from **Obs** using only conjunctions and disjunctions. Let $\{\varphi_j \mid j \in N\}$ be a recursive enumeration of all formulas $\varphi$ such that:

(a) $\varphi$ is either refutable or the conjunction of an inferable formula with a refutable formula;

(b) for some $0 \le i \le n$, $\theta_i$ is equivalent in $T$ to the existential closure of $\varphi$.

Since **Obs** is recursive and $T$ is *r.e.*, it is easy to verify that such a recursive enumeration exists (and that the refutable subformulas can be effectively distinguished from the inferable ones).

Define computable $\psi : SEQ \to N$ as follows. Let $\sigma \in SEQ$ be given. Suppose there exists least $j_0 \in N$ such that either

(113) (a) $\varphi_{j_0}$ is refutable and $\bigwedge \sigma$ does not imply $\neg\varphi_{j_0}$ in *length*$(\sigma)$ steps of computation (using $\boldsymbol{A}$), or

(b) $\varphi_{j_0}$ is the conjunction of an inferable formula $\chi$ with a refutable formula $\xi$, $\bigwedge \sigma$ implies $\chi$ in at most *length*$(\sigma)$ steps of computation, and $\bigwedge \sigma$ does not imply $\neg\xi$ in *length*$(\sigma)$ steps of computation (again, using $\boldsymbol{A}$).

Then $\psi(\sigma)$ is an index for $T \cup \{\varphi_{j_0}\}$. Otherwise $\psi(\sigma)$ is undefined. (Using $t$ and $j_0$, it is clear that an index for $T \cup \{\varphi_{j_0}\}$ can be constructed uniformly effectively.)

Suppose that $\psi$ underlies $\Psi$. To prove that $\Psi$ solves $(T, \{\theta_0 \ldots \theta_n\})$, let $i_0 \le n$, $\mathcal{S} \in MOD(T \cup \{\theta_{i_0}\})$, full assignment $h$ to $\mathcal{S}$, and environment $e$ for $\mathcal{S}$ and $h$ be given. By Theorem (59), there is least $j_0 \in N$ such that $\mathcal{S} \models \varphi_{j_0}[h]$. First we show:

(114) For cofinitely many $k$, $\psi(e[k])$ is an index for $T \cup \{\varphi_{j_0}\}$.

Let $j < j_0$ be given. Then $\mathcal{S} \not\models \varphi_j[h]$.

*Case 1:* $\varphi_j$ is refutable. Then by Definition (30) and the choice $\boldsymbol{A}$ of proof procedure, there is $n \in N$ such that $\boldsymbol{A}$ shows that $\bigwedge e[k] \models \neg\varphi_j$ for all $k > n$. Hence, for cofinitely many $k$, $\psi(e[k])$ is not an index for $T \cup \{\varphi_j\}$.

*Case 2:* $\varphi_j$ is the conjunction $\chi \wedge \xi$ of an inferable formula $\chi$ with a refutable formula $\xi$. Then either (a) $\mathcal{S} \not\models \chi[h]$ or (b) $\mathcal{S} \not\models \xi[h]$. In case (a), by the definition of "inferable," $\bigwedge e[k] \not\models \chi$ for all $k \in N$, and in case (b) $\bigwedge e[k] \models \neg\xi$ for cofinitely many $k$ (as in Case 1). Hence, for cofinitely many $k$, $\psi(e[k])$ is not an index for $T \cup \{\varphi_j\}$.

In light of (113) and the foregoing cases, to prove (114) it remains only to show that:

(115) (a) If $\varphi_{j_0}$ is refutable then for cofinitely many $k$, $\bigwedge e[k]$ does not imply $\neg\varphi_{j_0}$ in $k$ steps of computation (using $\boldsymbol{A}$), and

(b) if $\varphi_{j_0}$ is the conjunction of an inferable formula $\chi$ with a refutable formula $\xi$, then for cofinitely many $k$, $\bigwedge e[k]$ implies $\chi$ in at most $k$ steps of computation, and $\bigwedge e[k]$ does not imply $\neg\xi$ in $k$ steps of computation (again, using $\boldsymbol{A}$).

*Case 1:* $\varphi_{j_0}$ is refutable. Then because $\mathcal{S} \models \varphi_{j_0}[h]$, for all $k \in N$, $\bigwedge e[k] \not\models \neg\varphi_{j_0}$.

*Case 2:* $\varphi_{j_0}$ is the conjunction $\chi \wedge \xi$ of an inferable formula $\chi$ with a refutable formula $\xi$. Then because $\mathcal{S} \models \varphi_{j_0}[h]$, (a) for all $k \in N$, $\bigwedge e[k] \not\models \neg\xi$, and (b) for cofinitely many $k$, $\bigwedge e[k]$ implies $\chi$ in at most $k$ steps of computation (by the definition of "inferable" and the choice of $\boldsymbol{A}$).

This proves (115), hence (114). So it remains only to show that $T \cup \{\varphi_{j_0}\}$ is consistent and implies $T \cup \{\theta_{i_0}\}$.

Since $\mathcal{S} \models \varphi_{j_0}[h]$, and $\mathcal{S} \in MOD(T)$, it follows that $T \cup \{\varphi_{j_0}\}$ is consistent. Finally, since the $\theta_i$'s partition the models of $T$, and the existential closure of $\varphi_{j_0}$ is equivalent in $T$ to one of the $\theta_i$'s, it follows that $T \cup \{\varphi_{j_0}\} \models T \cup \{\theta_{i_0}\}$. ■

# References

[1] D. Angluin. Inductive inference of formal languages from positive data. *Information and Control*, 45:117–135, 1980.

[2] L. Blum and M. Blum. Toward a mathematical theory of inductive inference. *Information and Control*, 28:125–155, 1975.

[3] G. Boolos and R. Jeffrey. *Computability and Logic (Third Edition)*. Cambridge University Press, Cambridge, England, 1989.

[4] D. S. Bridges. *Computability: A Mathematical Sketchbook*. Springer-Verlag, Berlin, 1994.

[5] C. C. Chang and H. J. Keisler. *Model Theory (2nd Edition)*. North Holland, Amsterdam, 1977.

[6] G. Chierchia and S. McConnell-Ginet. *Meaning and Grammar: An Introduction to Semantics (2nd edition)*. MIT Press, Cambridge MA, 2000.

[7] W. Craig. On axiomatizability within a system. *Journal of Symbolic Logic*, 18:30–32, 1953.

[8] H. D. Ebbinghaus, J. Flum, and W. Thomas. *Mathematical Logic, Second Edition*. Springer-Verlag, Berlin, 1994.

[9] H. Enderton. *A Mathematical Introduction to Logic*. Academic Press, New York, 1972.

[10] H. Gaifman and M. Snir. Probabilities over rich languages. *Journal of Symbolic Logic*, 47:495–548, 1982.

[11] W. Hodges. *Model Theory*. Cambridge University Press, Cambridge, England, 1993.

[12] H. J. Keisler. Fundamentals of model theory. In Jon Barwise, editor, *Handbook of Mathematical Logic*, pages 47–104. North Holland, Amsterdam, 1977.

[13] K. Kelly and C. Glymour. Theory Discovery from Data with Mixed Quantifiers. *Journal of Philosophical Logic*, 19:1–33, 1990.

[14] A. Levy. *Basic Set Theory*. Springer-Verlag, Berlin, 1979.

[15] M. Machtey and P. Young. *An Introduction to the General Theory of Algorithms*. North-Holland, New York, 1978.

[16] Eric Martin and Daniel Osherson. *Elements of scientific inquiry*. M.I.T. Press, 1998.

[17] K. Wexler and P. Culicover. *Formal Principles of Language Acquisition*. M.I.T. Press, Cambridge MA, 1980.