Inductive Inquiry via Theory Revision[1]

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1. Introduction

The achievements of modern science are stupendous but what does this imply about the scientists responsible for them? Here are two opposing reactions to the accomplishments of these fine women and men.

(a) How clever!
(b) What luck!

Response (a) tacitly invokes a criterion of good inductive thinking and suggests that scientists have so far scored well. Response (b) denies this, and perhaps even denies that there is a distinction between good and bad scientific reasoning (see [9]). The issue is not the veracity and scope of current scientific doctrine. Even if modern doctrine is both true and momentous it could be the result of fortune and happenstance. In particular, human genetic endowment might be tuned to physical reality, predisposing even "wild guesses" towards accuracy. In this case, scientific success is due more to the quirks of evolutionary history than to scientists' problem-solving ability (see [6, Ch. 1]).

We might try to frame the opposition between (a) and (b) in terms of counterfactual claims of the form: Had physical laws been so-and-so instead of what they actually are, then scientists would have figured them out anyway. That would make them seem clever. One difficulty with such a formulation is ambiguity surrounding reference to "scientists." In counterfactual circumstances, are they just like actual scientists, or do they themselves operate according to the different laws so-and-so? To skirt the latter question, we can focus on the input-output function that a scientist embodies. Then it can be asked whether this very input-output function, applied to the data issuing from a different set of physical laws, would have converged to an accurate conjecture about those laws. (The function, being abstract, does not interact with physical law.)

Put this way, the choice between (a) and (b) hinges on the extent to which scientists' inductive strategies are general purpose. Can the strategies be used to solve a wide class of inductive problems, or just the one problem that scientists were fortunate to find in their path? The answer, of course, depends on facts about human psychology that are presently unknown. There are nonetheless issues that can be addressed at the present juncture. In particular, it would be helpful to possess an example of a "general purpose" scheme for inductive inquiry. What would such a thing look like?

It is at this point that we wish to stick our (tiny) oar into the discussion. Suppose that the concept of "scientist" is interpreted in the extensional way envisioned in Section 2.4 of the second essay; and suppose that "inductive problems" are sets of propositions, as discussed there in Section 2.2. Then it might be possible to provide at least one clear example of a general purpose scientific strategy.

By "general purpose," is not meant "able to solve all problems." There are no general purpose scientists in such a sense because unsolvable problems [like (41) of the essay #2] lie outside the competence of every scientist. Nor can we hope (more modestly) for a single scientist that solves all solvable problems. To see why, suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$, $\mathbf{Sym} = \emptyset$, and let $P_1 = \{N\}$, $P_2 = \{\emptyset \neq D \subseteq N \mid D \text{ finite}\}$. In Exercise (47) of the second essay, you were invited to demonstrate that the problem $\{P_1, P_2\}$ is not solvable. But each of the problems $\{P_1\}$, $\{P_2\}$ is trivially solvable. (Since each contains but a single proposition, it suffices to conjecture that one proposition on every datum.) This example can be adapted to prove:

(1) Proposition: There is a pair of solvable problems whose union is not solvable.[1]

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[1] Since the proposition invokes no hypotheses about **Obs** or **Sym**, it is intended to hold for any such choices. Small adjustments to the foregoing example suffice to show this. Proposition (1) is known as a "non-union" result. Theorems of this kind were first made prominent in the numerical paradigm of inductive inference. See [4].

If $\mathbf{P}_1$, $\mathbf{P}_2$ is such a solvable pair then no scientist solves both since he would then solve $\mathbf{P}_1 \cup \mathbf{P}_2$, which is impossible.

It follows that a general purpose scientist must in reality be a parameterized family of scientists. Each solvable problem would be solved by setting the parameter in the right way. But what kind of parameter are we talking about? The parameter could be interpreted in such a way as to deprive the family of its interest. For example, it would not be instructive if the parameter amounted to specifying a whole scientist. We already know that the family of all scientists is general purpose in the sense of solving every solvable problem! (For solvable problem $\mathbf{P}$, it suffices to set the parameter to some scientist that solves $\mathbf{P}$.)

Our goal is therefore to exhibit a family of scientists whose free parameter is just a fragment of a larger scheme for inductive inquiry. For this purpose, we shall consider scientists whose successive conjectures can be interpreted as the fruit of *hypothesis revision* starting from an initial scientific theory. Among other results, it will be shown that initial theories can be used to parameterize a single scheme of inquiry that solves a broad class of problems. Specifically, for every solvable problem of form $(T, \{\theta_0 \ldots \theta_n\})$, the scientist that results from passing the parameter T to the scheme will be seen to solve $(T, \{\theta_0 \ldots \theta_n\})$.[2] Moreover, the scheme in question will be built around a form of hypothesis revision that appears to meet some basic conditions of *rational theory change*.

If we keep these promises, the results might clarify the kind of facts needed to motivate response (a) above. At least, it will be seen how one kind of scientist could succeed in a wide class of potential realities by relying on a sensible strategy of hypothesis selection. Whether any human scientists in fact approximate such a strategy then becomes a well defined empirical issue.

So much for the big picture. Are you ready for the details? We proceed as follows. The next section introduces a class of *revision functions*, in other words, operations that

[2] For $(T, \{\theta_0 \ldots \theta_n\})$, see Definition (55) in essay #2.

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convert one theory into another as a function of the available evidence. Revision functions are harnessed for the purpose of inquiry in Section 3, and basic facts are proved. In Section 4 we explore the idea of a parameterized family of scientists based on theory revision. Our principal results are stated there. Efficient inquiry via revision is taken up in Section 5. Revision of a theory in light of data is well defined whether or not the theory is deductively closed. The impact of closure on successful inquiry via revision is the topic of Section 6. Section 7 considers iterated revision of the starting theory under the impact of the latest datum (in place of repeated revision of the starting theory under accumulating data). Proofs can be found in Section 8.

2. Theory revision

Our overall goal is to exhibit a parameterized class of scientists that operate via rational theory revision. As a first step, we introduce the kind of theory revision that will animate our scientists. Our ideas are drawn in part from the voluminous literature on *belief revision*, a large fragment of which has been masterfully summarized and integrated in [17].[3]

There is a terminological disagreement with the larger literature that can be set aside here. In Section 1 of the first essay we distinguished between belief and acceptance, and reserved the latter term for the kind of theoretical commitments that figure in scientific inquiry. What others call "belief revision functions" would therefore be more aptly called "functions for the revision of accepted theories" in the present context. As a compromise (and for brevity), we simply drop the qualifier "belief," referring instead to "revision functions."

[3]Our approach has been particularly influenced by [16], which generalizes [3]. We set things up somewhat differently, however. In particular, it will be convenient to define revision directly without the usual excursion through "contraction."

2.1. Revision functions

We shall attempt to exhibit inquiry within our paradigm as a process of rational revision of a starting theory in the light of data. In order to give substance to this idea, we conceive a scientist as initiating inquiry with a set X of formulas that represent his provisional theory prior to examining the environment. The arrival of data σ will then serve to modify X according to some fixed scheme of revision. For now, deductive closure is not imposed on theories; they are arbitrary subsets of $\mathcal{L}_{form}$. The effect on inquiry of deductive closure is the topic of Section 6, below.

The potential data that confront theories will be associated with the set SEQ, introduced in Section 2.3.6 of essay #2. Recall that the latter set represents all possible information that can be presented to scientists in our paradigm. A consequence of limiting attention to SEQ is that data are always assumed to be logically consistent. Let us recall from the earlier discussion that SEQ depends on the choice **Obs**. (Spefically, SEQ is the set of consistent finite sequences over **Obs**.)

As discussed in the first essay (Section 4.1), there is typically more than one way to revise a theory in the face of data that contradict it. Any particular revision strategy may be understood as determining a function that maps pairs of the form (X, σ), where $X \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$, into new theories. In other words, revision is a function with domain $pow(\mathcal{L}_{form}) \times SEQ$ and co-domain $pow(\mathcal{L}_{form})$. We typically denote a revision function by the symbol $\dotplus$. The result of applying $\dotplus$ to a pair in its domain is denoted by $X \dotplus \sigma$, and signifies the impact of the data σ on the theory X according to the revision scheme $\dotplus$.

Not just any function from $pow(\mathcal{L}_{form}) \times SEQ$ to $pow(\mathcal{L}_{form})$ counts as a legitimate scheme of revision, however. For one thing, we require that the data appear in the successor theory. This is because in our paradigm, Nature can never misrepresent what's true in her choice of structure. The data don't lie. So it makes sense to incorporate the data into the revised theory. In addition, the successor theory must include the "inoffensive" parts of the starting theory. To make the latter idea clear, we rely on the following definition.

(2) Definition: Let $B \subseteq \mathcal{L}_{form}$ and $\sigma \in \mathit{SEQ}$ be given. A formula $\varphi \in B$ is called *σ-innocent (with respect to B)* just in case for all $A \subseteq B$, if $A \cup \mathit{content}(\sigma)$ is consistent then $A \cup \{\varphi\} \cup \mathit{content}(\sigma)$ is consistent. We denote by $\mathit{Inn}(B, \sigma)$ the set of all σ-innocent formulas in B.

In other words, φ is σ-innocent with respect to B just in case φ figures in no non-redundant proof of $\neg \bigwedge \sigma$ that includes just formulas from B. Using propositional logic to illustrate, let $B = \{p, p \to q, p \vee r\}$ and $\sigma = \neg q$. Then $\mathit{Inn}(B, \sigma) = \{p \vee r\}$.

When revising B to make $\mathit{content}(\sigma)$ part of the new theory, the suppression of members of $\mathit{Inn}(B, \sigma)$ seems capricious; so the revision should leave behind at least $\mathit{Inn}(B, \sigma)$. On the other hand, enough formulas must be removed from B to ensure the consistency of the successor theory when the data are added. Finally, the new theory should include all of σ but nothing else that goes beyond B. These boundary conditions are the only ones we impose on legitimate revision.

(3) Definition: A mapping $\dotplus$ from $\mathit{pow}(\mathcal{L}_{form}) \times \mathit{SEQ}$ to $\mathit{pow}(\mathcal{L}_{form})$ is a *revision function* just in case for all $B \subseteq \mathcal{L}_{form}$,

(a) $\mathit{Inn}(B, \sigma) \cup \mathit{content}(\sigma) \subseteq B \dotplus \sigma \subseteq B \cup \mathit{content}(\sigma)$;

(b) $B \dotplus \sigma$ is consistent.

The following lemma is immediate.

(4) Lemma: Let $B \subseteq \mathcal{L}_{form}$ and $\sigma \in \mathit{SEQ}$ be such that $B \not\models \neg \bigwedge \sigma$. Then for all revision functions $\dotplus$, $B \dotplus \sigma = B \cup \mathit{content}(\sigma)$.

In other words, if new data are consistent with the current theory B, they are just dumped in without touching anything else. Once again, this policy is justified by the credibility of the data, which has been a central feature of our paradigm. If the data are inconsistent with B, they are dumped in nonetheless, at the price of displacing some formulas of B

The following Proposition (easily verified, but not essential to the sequel) gives some idea of the liberality of Definition (3) compared to other accounts of revision, e.g., [1]. To state the result, let us call a mapping $\oplus$ from $pow(\mathcal{L}_{form}) \times SEQ$ to $pow(\mathcal{L}_{form})$ a *partial meet revision* just in case for all $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$, $B \oplus \sigma$ is the union of $content(\sigma)$ with the intersection of some class of $\subseteq$-maximal subsets of B that are consistent with $content(\sigma)$.[4]

(5) PROPOSITION: The class of revision functions [defined in (3)] properly includes the class of partial meet revisions.

2.2. Special kinds of revision

We've seen that Definition (3) embraces a broad class of revision functions. Indeed, the class is broad enough to embrace some questionable policies. For example, draining the successor theory of every formula except the innocent ones (plus the data), has a scorched earth feel to it. So we are led to define more respectable subsets. A familiar idea (discussed in [2]) is to effect minimal change in the original theory, in the sense of abandoning no more than is necessary to avoid contradiction with new data. One way of capturing this idea is the following.

(6) DEFINITION: A revision function $\dotplus$ is *maxichoice* just in case for every $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$, no subset of $B \cup content(\sigma)$ that strictly includes $B \dotplus \sigma$ is consistent.

[4] See [11] for discussion of the partial meet idea.

Thus, maxichoice revision functions keep a $\subseteq$-maximal subset of the original theory that is consistent with the data.

Although maxichoice revision is pleasingly conservative, it is still compatible with disorderly selection of theories. For example, suppose that $\sigma, \tau \in SEQ$ are distinct, and that $content(\sigma)$ is logically equivalent to $content(\tau)$. It is possible for $\dot{+}$ to be maxichoice yet for $(B \dot{+} \sigma) \cup (B \dot{+} \tau)$ to be inconsistent. Another kind of bad behavior was described in Section 4.2 of essay #2. Let us repeat the earlier discussion. As before, we use propositional logic for simplicity. Suppose that a revision function behaves as shown here:

(7)

Theory T_1	:	$\{\ q,\ \neg p,\ (q \wedge r) \rightarrow p\ \}$
Datum	:	r
Chosen revision R_1	:	$\{\ q,\ \neg p,\ r\ \}$.
Rejected revision R_2	:	$\{\ q,\ (q \wedge r) \rightarrow p,\ r\ \}$.

Thus, faced with datum r, the revision function retrenches Theory T_1 by throwing out the conditional $(q \wedge r) \rightarrow p$ before adding in r. This yields the successor theory R_1. A different option would have been to throw out $\neg p$ from T_1 before adding r, yielding the successor R_2. The revision function has thereby revealed a preference for R_1 compared to R_2. (If this way of phrasing the matter strikes you as too anthropomorphic, think of the revision function as representing the preferences of a flesh-and-blood scientist.)

Now suppose that the same revision function behaves as summarized here:

(8)

Theory T_2	:	$\{\ r,\ \neg p,\ (q \wedge r) \rightarrow p\ \}$
Datum	:	q
Chosen revision R_2	:	$\{\ q,\ (q \wedge r) \rightarrow p\ ,\ r\ \}$.
Rejected revision R_1	:	$\{\ q,\ \neg p,\ r\ \}$.

We see that faced with datum q and Theory T_2, the revision function retrenches T_2 by throwing out $\neg p$ before adding q. This yields the revision R_2. A different option would have been to throw out $(q \wedge r) \rightarrow p$ from T_2 before adding q, yielding R_1. So now we must conclude that there is a preference for R_2 over R_1.

Nothing prevents a maxichoice revision function from behaving as indicated in (7) and (8). Yet the two preferences are inconsistent since the revision function prefers the successor theory R_1 to R_2 in the first situation but R_2 to R_1 in the second situation. This seems muddled inasmuch as the different starting theories give no reason to invert one's opinion about the relative acceptability of the potential successors. To preclude such choices, we define a subclass of revision functions with a more orderly approach to theory selection.

(9) DEFINITION: A revision function $\dotplus$ is *definite* just in case there is a strict total ordering $\prec$ of $pow(\mathcal{L}_{form})$ such that for all $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$, $B \dotplus \sigma$ is the $\prec$-least consistent subset of $B \cup content(\sigma)$ that contains $Inn(B, \sigma) \cup content(\sigma)$.

Thus, definite revision is built upon a prior ordering of all potential theories. The ordering can be interpreted in terms of preference, with earlier theories being prefered to later ones. Revision proceeds by choosing the most preferred theory that can legitimately serve as successor [in other words, the earliest theory that does not run afoul of Definition (3)]. You can see that the inconsistent preferences illustrated in (7) and (8) cannot be exhibited by definite revision.

It remains to show that definite revision functions exist [it is not evident that the needed ordering can be constructed in such a way as to satisfy Definition (3)]. In fact, the following proposition shows that there are revision functions that are simultaneously maxichoice and definite.

(10) PROPOSITION: There are revision functions that are both maxichoice and definite.

Such revision functions are not formal curiosities. The proof of the proposition (given in Section 8.1) shows that the class of revision functions that are simultaneously maxichoice and definite is well populated.

Scientists that embody maxichoice, definite revision functions satisfy many formal criteria of rationality. They have transitive and connected preferences among theories, and revise their theories so as to minimize change and maximize theory-acceptability.

Such is our model of theory-revision, and of rational theory-revision in particular. We now begin to explore its use in inductive inquiry. First, let it be emphasized that there are many alternative accounts of revision, and that the inductive powers of any of them could be investigated in place of ours. Although we have examined variants of the foregoing definitions, we do not know what fraction of the results reported below would survive transplant to the various models reviewed in [17]. The model developed above is motivated by its simplicity and generality. Also, it yields striking results about inductive inquiry. (We hope you will soon agree.)

3. Inquiry via revision

3.1. From revision to science

In the present section we consider how theory revision can be harnassed for the purpose of inquiry. The matter is simple in essence. Given a set B of formulas and a revision operator $\dotplus$, the function $\lambda\sigma\,.\,B \dotplus \sigma$ maps SEQ into the power set of $\mathcal{L}_{form}$.[5] Intuitively, $\lambda\sigma\,.\,B \dotplus \sigma$ converts data into theory; data are represented by SEQ and theories by sets of formulas. With such a function we associate the scientist that maps $\sigma \in SEQ$ into the class of models

[5] Following standard usage, λ in the expression "$\lambda\sigma\,.\,B \dotplus \sigma$" indicates that σ serves as argument to the function, whereas B and $\dotplus$ are fixed.

of $B \dotplus \sigma$, and call such scientists *revision-based* (the official definition is given in Section 3.3 below). Scientists of this kind are special in two ways. First, their outputs are not arbitrary propositions, but must instead be definable by a subset of $\mathcal{L}_{form}$. In other words, each conjecture must have the form $MOD(B)$ for some $B \subseteq \mathcal{L}_{form}$ instead of being just any collection of structures. Second, revision-based scientists are not arbitrary mappings from *SEQ* to the definable propositions. Instead, they are constrained by Definition (3) of revision. Additional constraint arises if we require the $\dotplus$ in $\lambda\sigma \,.\, B \dotplus \sigma$ to denote a restricted kind of revision function, such as maxichoice and definite. We will see later that revision-based scientists employing maxichoice definite revision functions are canonical for a broad class of problems; any solvable problem in the class is solved by a scientist of this kind.

In the preceding discussion we distinguished two special properties of revision-based scientists. The first requires output propositions to be elementary classes of structures. Isolating just this property yields the set of *linguistic* scientists. They are "linguistic" in the sense that the propositions they announce are expressible in our first-order language $\mathcal{L}$. Thus, the revision-based scientists are a subset of the linguistic scientists, and information about the competence of the latter also provides insight about the former. So we start our discussion with linguistic scientists. Revision-based scientists are introduced subsequently, in Section 3.3.

3.2. Linguistic scientists

Recall from Section 2.4 of the second essay that scientists issue classes of structures in response to members of *SEQ*. When the classes are definable from a subset of $\mathcal{L}_{form}$, the scientists are called "linguistic." Officially:

(11) Definition: Scientist Ψ is *linguistic* just in case there is $\psi : SEQ \rightarrow pow(\mathcal{L}_{form})$ such that for all $\sigma \in SEQ$, $\Psi(\sigma)$ is defined iff $\psi(\sigma)$ is defined, and when both are

defined $\Psi(\sigma) = MOD(\psi(\sigma))$. In this case, we say that ψ *underlies* Ψ.

To lighten the notation we sometimes identify linguistic scientists with the functions $\psi : SEQ \rightarrow pow(\mathcal{L}_{form})$ that underlie them. In other words, if Ψ is linguistic, we allow ourselves to write $\Psi(\sigma) = B$, for $B \subseteq \mathcal{L}_{form}$, in place of the circumlocution: "$\psi(\sigma) = B$, where ψ underlies Ψ."

Are linguistic scientists a canonical form of inquiry? Of course not. Suppose that proposition P includes no isomorphically closed class of structures (for example, P might be a singleton, consisting of a single structure). Then no problem that includes P is solvable by linguistic scientist since no nonempty subset of P can be picked out by a set of formulas. [This is because $MOD(B)$ is isomorphically closed, for any $B \subseteq \mathcal{L}_{form}$.] There are plenty of solvable problems that include such emaciated propositions. Indeed, we can remove from each proposition of any solvable problem all structures but one, and the resulting problem will still be solvable.[6] To illustrate, since $\{P_f, P_b\}$ of Example (24)b of (essay #2) is solvable, so is $\{\{(\omega, <)\}, \{(\omega^*, <)\}\}$ (assuming $\mathbf{Obs} = \mathcal{L}_{basic}$).[7] No linguistic scientist can solve the latter problem because no elementary class of structures is included in $\{(\omega, <)\}$ [or in $\{(\omega^*, <)\}$].

On the other hand, linguistic scientists turn out to be well adapted to problems of form $(T, \{P_0, P_1, \ldots\})$. Recall from Definition (65) of essay #2 that problem $\mathbf{P}$ has this form just in case $\mathbf{P} = \{P_0, P_1, \ldots\}$ and $\bigcup \mathbf{P} = MOD(T)$. It is noteworthy that linguistic scientists can solve any solvable problem of form $(T, \{P_0, P_1, \ldots\})$. We record this fact in the next proposition, which follows immediately from Proposition (67) of the second essay.

[6] There is a more general fact worth stating. If $\mathbf{P} = \{P_i \mid i \in N\}$ and $\mathbf{Q} = \{Q_i \mid i \in N\}$ are problems such that $P_i \subseteq Q_i$ for all i, then $\mathbf{P}$ is solvable if $\mathbf{Q}$ is.

[7] Recall that ω denotes the set of natural numbers ordered naturally, and ω^* denotes the natural numbers ordered backwards, as $\cdots < 3 < 2 < 1 < 0$.

(12) Proposition: Every solvable problem of the form $(T, \{P_0, P_1, \ldots\})$ is solved by a linguistic scientist. Indeed, the linguistic scientist may be chosen so that $range(\Psi) \subseteq pow(\mathcal{L}_{sen})$.

The coda to the proposition means that for problems of the form $(T, \{P_0, P_1, \ldots\})$, no generality is lost by depriving linguistic scientists of the right to announce open formulas.[8]

3.3. Scientists based on revision

We now pass from linguistic scientists to their revision-based subset.

(13) Definition: Scientist Ψ is *revision-based* just in case there is $B \subseteq \mathcal{L}_{form}$ and revision function $\dot{+}$ with the following property. For all $\sigma \in SEQ$, $\Psi(\sigma) = MOD(B \dot{+} \sigma)$.

Thus, faced with data σ, a revision-based scientist conjectures the proposition whose structures satisfy the successor theory $B \dot{+} \sigma$. As before, we allow ourselves to identify revision-based scientists with the functions $\lambda\sigma \,.\, B \dot{+} \sigma$ that underlie them.

Since revision functions are total functions, so are revision-based scientists. Note also that the revision-based scientists are a proper subset of the linguistic scientists. It is easy to see that the inclusion is proper since linguistic scientists may be undefined on some data. Even among the total linguistic scientists, some are not revision-based. Definition (3) implies, for example, that no revision-based scientist issues a conjecture that contradicts his data whereas this is possible for a linguistic scientist.

[8] This holds only for problems of form $(T, \{P_0, P_1, \ldots\})$. In the general case, it is not difficult to exhibit an elementary class P such that $P = MOD(B)$ implies $B \not\subseteq \mathcal{L}_{sen}$. For such a proposition P, the degenerate problem $\{P\}$ is solvable by a linguistic scientist but not by any linguistic scientist with range in $pow(\mathcal{L}_{sen})$.

The model of revision developed in Section 2 above invites us to view empirical inquiry as the process of revising one's background theory by evidence that accumulates in the environment. We take the revision-based scientists to give formal substance to this idea. Moreover, if we choose the revision function $\dot{+}$ to be definite maxichoice, the revision-based scientist $\lambda\sigma\,.\,B \dot{+} \sigma$ has some claim to the title "rational," as we observed just after Proposition (10). As a consequence, we shall be particularly interested in the competence of scientists of this form. We note that our enterprise would be vitiated if the starting theory B could be taken to be inconsistent, for there is nothing commendable in starting from a theory that can be shown false prior to examining data. So our positive results will only concern starting theories B that are consistent. When we use the term "maxichoice definite scientist" we mean a revision-based scientist whose starting theory is consistent and whose revision function is maxichoice and definite.

The behavior of maxichoice definite scientists is distinctive. They don't issue conjectures that contradict the data, for example, and they don't revisit a conjecture that was abandoned at an earlier stage of inquiry. Also, their preferences for theories are consistent in the sense discussed in Section 2.2. It would be nice to characterize the class of maxichoice definite scientists in such terms (namely, as the class of scientists who never issue conjectures that contradict the data, etc.), but we have not pursued this question. When we examine parameterized families of maxichoice definite scientists, moreover, the constraints on behavior become yet stricter.

Section 4 below is devoted to characterizing the competence of revision-based scientists with respect to problems of the form $(T, \{\theta_0 \dots \theta_n\})$. As a preliminary, the remainder of the present section brings out various facts that emerge at a more general level of analysis. Thus, we step back from problems of the special form $(T, \{\theta_0 \dots \theta_n\})$ or $(T, \{P_0, P_1, \dots\})$, and consider the entire class of problems. In this more general setting, we hope to accomplish three things. First, we show that the maxichoice revision functions are canonical for the class of problems that can be solved by revision-based scientists. That is, every problem that can be solved in the revision-based way can be solved by a revision-based scientist that relies on a maxichoice function $\dot{+}$ (Section 3.4).[9] Next, we consider the interaction of computability and revision (Section 3.5). Finally, we define a partial order on the competence of revision-based scientists (Section 3.6).

These three topics have both mathematical and conceptual interest, but they are more technical and "theory internal" than the sequel. So you will be forgiven if you skip the remainder of the present section and proceed directly to Section 4.

3.4. The inductive power of revision

In the present subsection we establish that a single, maxichoice revision function is sufficient to define successful, revision-based scientists. The matter may be stated as follows.

(14) THEOREM: Suppose that **Obs** $= \mathcal{L}_{basic}$. There is a maxichoice revision function $\dot{+}$ with the following property. Let problem **P** be such that for some $Y \subseteq \mathcal{L}_{form}$ and revision function $\dot{\oplus}$, $\lambda\sigma \,.\, Y \dot{\oplus} \sigma$ solves **P**. Then there is a consistent $X \subseteq \mathcal{L}_{form}$ such that $\lambda\sigma \,.\, X \dot{+} \sigma$ solves **P**.

Thus, one size fits all. A single maxichoice revision function allows the solution of any problem that is solved through the use of revision functions.[10] The theorem is proved in Section 8.2. We believe that the hypothesis **Obs** $= \mathcal{L}_{basic}$ is stronger than needed, in particular, that it can be weakened to the supposition that **Obs** is closed under negation. The only demonstration of this claim known to us, however, is arduous and unrevealing.

[9] We do not know whether this maxichoice revision-based scientist can also be chosen to be definite.
[10] We do not know whether this maxichoice revision function can be taken to be definite.

In contrast, a simpler argument (given in Section 8.3) suffices to show that the maxichoice scientist $\lambda\sigma\,.\,X \dot{+} \sigma$ evoked in Theorem (14) solves its problem efficiently.[11] Officially:

(15) THEOREM: The revision-based scientist $\lambda\sigma\,.\,X \dot{+} \sigma$ of Theorem (14) solves **P** efficiently.

Theorems (14) and (15) do not imply that every revision function is adapted to feasible problems, only that some of them are. In fact, not even the definite maxichoice revision functions can all be made to work successfully. This is revealed by the following result.

(16) THEOREM: Suppose that **Sym** is limited to a binary predicate and countably many constants and **Obs** $= \mathcal{L}_{basic}$. Then there exists a definite maxichoice revision function $\dot{+}$, and propositions P_1, P_2 that meet the following conditions.

(a) $\{P_1, P_2\}$ is solvable and each of P_1, P_2 is closed under elementary equivalence;
(b) for all $B \subseteq \mathcal{L}_{form}$, $\lambda\sigma\,.\,B \dot{+} \sigma$ does not solve $\{P_1, P_2\}$.

Clause (a) ensures that both P_1 and P_2 can be conjectured by a revision-based scientist. To conjecture P_i $(i = 1, 2)$, it suffices to issue a set of sentences that defines a nonempty subset of P_i; by clause (a), this is possible. Despite the nameability of P_1, P_2, there are definite maxichoice revision functions that can't be used to solve $\{P_1, P_2\}$. The theorem is proved in Section 8.4.

3.5. Scientists based on computable revision

From Definition (81) of the second essay, it follows that a revision-based scientist $\lambda\sigma\,.\,B \dot{+} \sigma$ is computable if and only if there is computable $\psi : SEQ \to N$ such that for all $\sigma \in SEQ$,

[11] Efficiency, you recall, involves non-dominated use of data, as defined in Section 6.1 of essay #2.

$B \dotplus \sigma = \mathrm{W}^{form}_{\psi(\sigma)}$. In this subsection we consider the consequences of requiring revision-based scientists to be computer simulable. First, consider some good news. Recall from Definition (86) of essay #2 that a problem of form $(T, \{\theta_0 \ldots \theta_n\})$ is called "*r.e.*" just in case T is a recursively enumerable set of sentences. We have:

(17) Proposition: Suppose that **Obs** is closed under negation. For every solvable *r.e.* problem $(T, \{\theta_0 \ldots \theta_n\})$, there is consistent $X \subseteq \mathcal{L}_{form}$ with $X \models T$, and definite maxichoice revision function $\dotplus$ such that $\lambda\sigma \,.\, X \dotplus \sigma$ is computable and solves $(T, \{\theta_0 \ldots \theta_n\})$.

In other words, the solvable *r.e.* problems $(T, \{\theta_0 \ldots \theta_n\})$ fall within the inductive competence of computable revision. The proof is given in Section 8.5.

In contrast to Proposition (17), it will now be seen that theory revision and computability do not always mix well. The reason is the consistency requirement expressed in Definition (3)b, which makes revision hard to calculate. The next proposition shows that even some problems that are computably solvable are outside the reach of computable revision. Indeed, each proposition in such a problem can be taken to be strongly elementary, thus ensuring that its propositions can be denoted by a single sentence.[12] In sum:

(18) Proposition: Suppose that **Sym** is limited to the vocabulary of arithmetic (including $\overline{0}$ and a unary function symbol s) plus an additional constant. Also suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$. Then there is a problem **P** with the following properties.

(a) Every member of **P** is strongly elementary.

(b) **P** is computably solvable.

[12] Recall from Definition (82) of essay #2 that a proposition is strongly elementary just in case it has the form $MOD(\theta)$ for some $\theta \in \mathcal{L}_{sen}$.

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(c) some revision-based scientist solves **P**.

(d) For every $B \subseteq \mathcal{L}_{form}$ and revision function $\dotplus$, if $\lambda\sigma\,.\,B \dotplus \sigma$ is computable, then it fails to solve **P**.

For the proof, see Section 8.6.

The foregoing result prompts the search for interesting subclasses of problems that can be solved by computable, revision-based scientists. It also suggests the need to define weaker senses of "computer simulable" so as to attenuate the impact of the consistency requirement expressed in Definition (3)b. We shall not pursue these projects here, however, preferring instead to work out the inductive logic of theory revision in the simpler context of potentially ineffective functions.

3.6. Comparing the competence of revision-based scientists

A useful partial order on the competence of revision-based scientists may be defined as follows. We say that revision function $\dot{\oplus}$ "subsumes" revision function $\dotplus$ just in case for all $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$, $B \dot{\oplus} \sigma \supseteq B \dotplus \sigma$. Definition (6) implies immediately that the maxichoice revision functions are maximal elements of the subsumes relation. That is, every revision function is subsumed by some maxichoice revision function, and no maxichoice function is subsumed "properly" by any other function.

Subsumption between revision functions is related to scientific competence in the following way.

(19) LEMMA: Suppose that revision functions $\dotplus$ and $\dot{\oplus}$ are such that $\dot{\oplus}$ subsumes $\dotplus$. Let problem **P** and $B \subseteq \mathcal{L}_{form}$ be given. If $\lambda\sigma\,.\,B \dotplus \sigma$ solves **P** then $\lambda\sigma\,.\,B \dot{\oplus} \sigma$ does also.

The lemma is easy to verify, and defines a clear sense in which maxichoice revision functions represent the most powerful scientific method based on hypothesis revision. If a problem is solvable by revision, then it is solvable by a revision-based scientist whose "$\dot{+}$" is maxichoice. Conversely, for a given starting point B, the weakest, revision-based scientist is the revision function that maps (B, σ) to $Inn(B, \sigma) \cup content(\sigma)$. The latter function may be called "full meet." The revision function it underlies is subsumed by every other revision function. So the lemma implies that if the full meet scientist solves **P** then so does every scientist of form $\lambda\sigma \,.\, B \dot{+} \sigma$. (The "full meet" terminology is adapted from [11].)

4. Theories as parameters

Now we are ready to make good on our promise about parameterized scientists (see page 4). It will be seen how to parameterize revision-based scientists with the theory T figuring in problems of the form $(T, \{\theta_0 \ldots \theta_n\})$. If the problem is solvable then our parameterized scientist will solve it.

You are probably thinking that our goal can be reached by constructing the right revision function $\dot{+}$, and then defining the family of scientists $\{\ \lambda\sigma \,.\, T \dot{+} \sigma \mid T \subseteq \mathcal{L}_{sen}\}$, where T is the free parameter. The revision function $\dot{+}$ would be chosen so that for any solvable problem $(T, \{\theta_0 \ldots \theta_n\})$, the scientist $\lambda\sigma \,.\, T \dot{+} \sigma$ would solve it. Matters won't be quite so simple. This is because we will shortly see problems $(T, \{\theta_0 \ldots \theta_n\})$ that are solved by no revision based scientist with starting theory T. This fact might seem discouraging, but it is surmounted by adding additional formulas to T. Specifically, it will be shown that scientists of form $\lambda\sigma \,.\, T \cup X \dot{+} \sigma$ have wide inductive powers for problems of form $(T, \{\theta_0 \ldots \theta_n\})$ provided that the right supplement X is chosen to extend T. Indeed, provided that **Obs** is closed under negation, a single supplement works for all solvable $(T, \{\theta_0 \ldots \theta_n\})$. In other words, there is $X \subseteq \mathcal{L}_{form}$ such that our desired family of scientists can be written as

$\{ \lambda\sigma \,.\, (T \cup X) \dot{+} \sigma \mid T \subseteq \mathcal{L}_{sen} \}$, where T is still the only free parameter.[13]

For the details, let us first consider a large class of theory extensions that lead revision-based scientists to failure. This will help motivate the choice of successful theory-supplement introduced subsequently.

4.1. Extensions of the background theory that are too modest

As noted above, to exploit theory revision for a problem $(T, \{\theta_0 \ldots \theta_n\})$, it is natural to consider scientists of form $\lambda\sigma \,.\, T \dot{+} \sigma$. Such scientists attempt to revise the background theory T in the face of data coming from the environment. Unfortunately, there are simple problems of this kind that lead all such scientists to failure. Indeed, the following proposition shows that failure also results if T is extended to any consistent subset of $\mathcal{L}_{sen}$. Recall that the latter set is limited to formulas without free variables.

(20) PROPOSITION: Suppose that **Sym** is limited to the binary predicate R. Suppose that **Obs** $= \mathcal{L}_{basic}$. Let $T = \{\exists x \forall y Rxy \leftrightarrow \neg \exists y \forall x Rxy\}$, and $\theta = \exists x \forall y Rxy$. Then $(T, \{\theta, \neg\theta\})$ is solvable, but for all revision functions $\dot{+}$ and all $B \subseteq \mathcal{L}_{sen}$ consistent with T, $\lambda\sigma \,.\, B \dot{+} \sigma$ does not solve $(T, \{\theta, \neg\theta\})$.

For the proof, see Section 8.7.

Proposition (20) reveals that revision-based scientists cannot be made to work unless their starting points extend the background theory T to include open formulas. The formulas may be considered an "inductive leap" ready to be abandoned in whole or part according to the data encountered. Their free variables embody hypotheses about which of the objects shown to the scientist have special properties involved in the problem under investigation.

[13] Recall from Definition (4) of essay #2 that **Obs** is closed under negation iff for every $\varphi \in$ **Obs**, there is $\psi \in$ **Obs** that is logically equivalent to $\neg\varphi$.

4.2. A universal method of inquiry based on revision

If we are willing to extend our background theories with open formulas, then revision can be used successfully for inquiry. Indeed, in case **Obs** is closed under negation, it is possible to choose a single extension and a single revision function that works for any solvable problem of the form $(T, \{\theta_0 \ldots \theta_n\})$. In this sense the next theorem exhibits a canonical method of inquiry. Its proof exhibits a theory-extender X and a definite maxichoice revision function $\dot{+}$ with the following property. Every solvable problem $(T, \{\theta_0 \ldots \theta_n\})$ is solved by the scientist $\lambda\sigma\,.\,(T \cup X) \dot{+} \sigma$. Moreover, this scientist has some claim to the title "rational." For, its starting theory is a consistent extension of the background theory T, and revision occurs on the basis of a definite maxichoice function.[14]

(21) Theorem: Suppose that **Obs** is closed under negation. There exists $X \subseteq \mathcal{L}_{form}$ and definite maxichoice revision function $\dot{+}$ such that for all consistent $T \subseteq \mathcal{L}_{sen}$, the following holds.

(a) $T \cup X$ is consistent.

(b) $\lambda\sigma\,.\,(T \cup X) \dot{+} \sigma$ solves every solvable problem of the form $(T, \{\theta_0 \ldots \theta_n\})$.

The theorem exhibits the parameterized family of scientists hoped for in Section 1 above. The family is $\lambda\sigma\,.\,(T \cup X) \dot{+} \sigma$, where T is the sole parameter. When confronted with problem $(T, \{\theta_0 \ldots \theta_n\})$, putting T in for the parameter guides the resulting scientist to success. Our "family" of scientists therefore looks more like a single scientist with the good sense to establish his starting theory in light of the inductive problem he faces. The inductive acumen of this single scientist is embodied in the fixed choice of revision function $\dot{+}$ and theory-supplement X.

[14] That is, the revision occurs on the basis of transitive, connected preferences for theories that inflict minimal change on current opinion (as discussed in Section 2.2).

So, what are the identities of the fixed $\dot{+}$ and X? They are revealed in the constructive proof of (21) given in Section 8.8. We cannot claim that the choices of $\dot{+}$ and X in the proof are the simplest possible, only that nothing simpler has yet come to mind.

4.3. Supplementing as a function of the problem

Theorem (21) shows that some revision functions can be used for successful inquiry, but it does not show that all of them can. It thus leaves open the possibility that our conception of revision function is so lax as to embrace revision that is useless for inquiry. This would be true if some revision functions were led to failure on a solvable problem $(T, \{P_0, P_1, \ldots\})$ no matter how T is extended. But such is not the case. It will now be shown that every revision function can be made to succeed on every solvable problem of the form $(T, \{P_0, P_1, \ldots\})$.

(22) THEOREM: Suppose that solvable problem $\mathbf{P}$ is of form $(T, \{P_0, P_1, \ldots\})$, and that **Obs** is closed under negation. Then there is a consistent extension $X \subseteq \mathcal{L}_{form}$ of T such that for every revision function $\dot{+}$, $\lambda\sigma \,.\, X \dot{+} \sigma$ solves $\mathbf{P}$.

In the present theorem we choose the theory-supplement X as a function of the problem to be solved. This is what allows all revision functions to solve the problem (if it is solvable at all). In Theorem (21), X was chosen once and for all. Only a single revision function is therefore invoked in the latter theorem. We also note the contrast with Theorem (16) of Section 3.4. There we exhibited a revision function $\dot{+}$ and a solvable problem $\mathbf{P}$ such that $\dot{+}$ could not be used to solve $\mathbf{P}$. But the problem $\mathbf{P}$ in the earlier theorem was not of form $(T, \{P_0, P_1, \ldots\})$. The present result affirms that all solvable problems of the latter form can be solved using any revision function starting with a suitably chosen theory. For the proof of Theorem (22), see Section 8.9.

Theorem (22) suggests that our definition of revision function is sufficiently strict, since it does not tolerate revision strategies that are inapt for science. The definition also appears to be sufficiently liberal since it embraces revision functions that are rational and multipurpose in the sense of Theorem (21). Let us therefore rejoice in the conviction that a sound and sufficient conception of revision has been achieved. Indeed, the moment for rejoicing is well chosen since our satisfaction will prove shortlived! In Section 5 we will encounter revision functions that cannot be used for the efficient solution of certain problems of form $(T, \{\theta, \neg\theta\})$. For now, however, all is well. Indeed, things get even better when we specialize our problems to those whose background theories are finitely axiomatizable.

4.4. Supplementing as a function of finitely axiomatizable theories

In this subsection and the next we look more closely at the way background theories can be successfully extended. The extension X of T forseen in Theorem (22) depends not just on T but more generally on the problem $(T, \{P_0, P_1, \ldots\})$. If we limit attention to problems of the form $(T, \{\theta_0 \ldots \theta_n\})$, assume that **Obs** is closed under negation, and add the hypothesis that T is finitely axiomatizable, then T can be successfully extended without concern for the partition imposed by $\theta_0 \ldots \theta_n$. This is the burden of the next proposition. In the succeeding subsection we show that the hypothesis of finite axiomatizability cannot be lifted.

Let us tell you frankly that questions involving finite axiomatizability are not central to the message of this essay. So the remainder of the present section (viz., 4.4 and 4.5) can be omitted on a first reading, and the discussion picked up in Section 5.

The proof of the following proposition is given in Section 8.10.

(23) Proposition: Suppose that **Obs** is closed under negation. Let $T \subseteq \mathcal{L}_{sen}$ be consistent and finitely axiomatizable. Then there is consistent $X \subseteq \mathcal{L}_{form}$ with

$X \models T$ such that for every revision function $\dotplus$, $\lambda\sigma \,.\, X \dotplus \sigma$ solves every solvable problem of form $(T, \{\theta_0 \ldots \theta_n\})$.

4.5. The inability to supplement independently of T

It would be nice to find a method of extending a theory T that allows all revision functions to succeed on solvable problems of form $(T, \{\theta_0 \ldots \theta_n\})$. In this case the required extension would not depend on the sentences $\theta_0 \ldots \theta_n$, but only on the background theory T. Fulfilling this desire would amount to lifting the hypothesis of finite axiomatizability from Proposition (23). Unfortunately, the next proposition shows this hope to be unrealizable.

(24) PROPOSITION: Suppose that **Sym** is limited to a binary predicate, a constant, and a unary function symbol. Suppose that **Obs** $= \mathcal{L}_{basic}$. Then there exists $T \subseteq \mathcal{L}_{sen}$ and definite maxichoice revision function $\dotplus$ such that for all $B \subseteq \mathcal{L}_{form}$, $\lambda\sigma \,.\, B \dotplus \sigma$ fails to solve some solvable problem of form $(T, \{\theta, \neg\theta\})$.

The proposition shows that in the general case, theory extensions must be chosen as a function of the problem to be solved. Otherwise, some revision functions will be led to needless failure. The proof is given in Section 8.11.

5. Efficient revision

One goal of the present essay is to represent inquiry as a process of rational theory revision. The theorems of the preceding section show that for a wide class of problems, revision-based scientists can be made to succeed whenever success is possible in principle. We now consider efficient inquiry in the same terms. Our question is whether and to what extent efficient inquiry can be carried out using revision functions.

First it will be shown that revision functions are compatible with efficient solvability for problems of the form $(T, \{P_0, P_1, \ldots\})$. That is, for every solvable problem of this form, some revision function can be made to work efficiently. In contrast, it will then be demonstrated that not all revision functions are suitable for efficient discovery. Indeed, even some maxichoice definite ones cannot be used to efficiently solve certain "easy" problems. The consequences of this finding for our theory of inquiry are then discussed.

Throughout the section we rely on the conception of efficient inquiry introduced in Definition (76) of essay #2. To recapitulate, scientist Ψ is *dominated* on problem **P** if a rival scientist succeeds at least as quickly as Ψ on all environments for **P**, and more quickly on some. Conversely, Ψ is *efficient* on **P** if no rival scientist dominates Ψ.

5.1. Inductive efficiency by revision is possible

Theorem (15) of Section 3.4 provides a sense in which theory revision is compatible with efficient solvability. Indeed, it was there shown that a sole maxichoice revision function suffices to efficiently solve every problem, provided only that the problem be solvable in the revision-based way. We now show that efficient inquiry can be carried out on a broad class of problems by revision-based scientists whose starting points extend the background theory.

(25) THEOREM: Suppose that **Obs** is closed under negation. For every solvable problem **P** of the form $(T, \{P_0, P_1, \ldots\})$ there is a consistent extension $X \subseteq \mathcal{L}_{form}$ of T and a definite maxichoice revision function $\dot{+}$ such that $\lambda\sigma \, . \, X \dot{+} \sigma$ solves **P** efficiently.

See Section 8.12 for the proof.

If attention is limited to problems of the form $(T, \{\theta_0 \ldots \theta_n\})$, Theorem (25) can be improved. We rely on the following definition, which isolates the revision functions that strive to hold onto a given theory T.

(26) Definition: Let revision function $\dot{+}$ and $T \subseteq \mathcal{L}_{sen}$ be given. We say that $\dot{+}$ is *T-preserving* just in case for all $\sigma \in \mathit{SEQ}$ and $X \subseteq \mathcal{L}_{form}$, if $T \cup \mathit{content}(\sigma)$ is consistent then $T \subseteq (T \cup X) \dot{+} \sigma$.

Thus, a T-preserving revision function maintains T in the successor theory if T is a subset of the starting theory and is consistent with the data σ. The following lemma records the fact that T-preserving revision functions exist. It is proved in Section 8.13.

(27) Lemma: For all $T \subseteq \mathcal{L}_{sen}$, there exists a T-preserving maxichoice definite revision function.

The following theorem shows that the T-preserving revision functions can be used to efficiently solve any solvable problem of form $(T, \{\theta_0 \ldots \theta_n\})$. See Section 8.14 for the proof.

(28) Theorem: Suppose that **Obs** is closed under negation. For every solvable problem **P** of the form $(T, \{\theta_0 \ldots \theta_n\})$ there is a consistent extension $X \subseteq \mathcal{L}_{form}$ of T with the following properties.

(a) For all revision functions $\dot{+}$, $\lambda\sigma \,.\, X \dot{+} \sigma$ solves **P**.

(b) For all T-preserving revision functions $\dot{+}$, $\lambda\sigma \,.\, X \dot{+} \sigma$ solves **P** efficiently.

Further improvement is possible in the case of finitely axiomatizable theories. The following corollary is proved in Section 8.15.

(29) COROLLARY: Suppose that **Obs** is closed under negation. Let $T \subseteq \mathcal{L}_{sen}$ be consistent and finitely axiomatizable. Let solvable problem of form $(T, \{\theta_0 \ldots \theta_n\})$ be given. Then there is consistent extension $X \subseteq \mathcal{L}_{form}$ of T such that for every revision function $\dot{+}$, $\lambda\sigma \,.\, X \dot{+} \sigma$ efficiently solves $(T, \{\theta_0 \ldots \theta_n\})$.

5.2. Inductive efficiency by revision is not inevitable

Theorem (25) shows that there are enough revision functions to carry out efficient inquiry on problems of form $(T, \{P_0, P_1, \ldots\})$. Moreover, just the maxichoice definite subclass is sufficient. It would be pleasing to report the converse fact as well, namely, that every revision function can be used efficiently. Theorem (22) would be reinforced thereby, since the latter states that every revision function can be used to solve any solvable problem of form $(T, \{P_0, P_1, \ldots\})$. Unfortunately, there is no such counterpart to Theorem (22). Some revision functions cannot be brought to efficiently solve certain solvable problems. Worse, the guilty revision functions can be taken to be maxichoice definite and the recalcitrant problem has a particularly simple form. Here is a precise statement of the situation.

(30) THEOREM: Suppose that **Sym** is limited to countably many constants and **Obs** $= \mathcal{L}_{basic}$. Then there is a problem of form $(T, \{\theta, \neg\theta\})$ and a definite maxichoice revision function $\dot{+}$ with the following properties.

(a) T is recursive.

(b) $(T, \{\theta, \neg\theta\})$ is solvable.

(c) For all $B \subseteq \mathcal{L}_{form}$, if $\lambda\sigma \,.\, B \dot{+} \sigma$ solves $(T, \{\theta, \neg\theta\})$ then $\lambda\sigma \,.\, B \dot{+} \sigma$ is dominated on $(T, \{\theta, \neg\theta\})$.

For the proof, see Section 8.16.

Although the maxichoice definite revision functions possess strong credentials of rationality, their properties do not guarantee that they can be used for efficient inquiry. This is what the foregoing theorem shows. To state the matter more generally, let us call a revision function $\dot{+}$ "efficient" just in case for every solvable problem $\mathbf{P}$ of the form $(T, \{\theta, \neg\theta\})$ there is consistent $X \subseteq \mathcal{L}_{form}$ such that $\lambda\sigma \,.\, X \dot{+} \sigma$ solves $\mathbf{P}$ efficiently. Theorems (15) and (30) show that some but not all definite maxichoice revision functions are efficient.

The inefficient revision functions are in some way defective, since it is not rational to conduct inquiry in needlessly dilatory fashion. In particular, Theorem (25) shows that inefficient revision can always be replaced by an efficient method of theory change, hence there would seem to be no justification for using the inefficient function. The question thus arises as to additional conditions that can be imposed on revision functions in view of guaranteeing efficiency. Such conditions should be intuitively justifiable in terms of modifying theories in a rational way, and also pick out the efficient subset of functions.

One promising class of revision functions may be defined as follows.

(31) Definition:

(a) Given $X \subseteq \mathcal{L}_{form}$, let $sentence(X)$ denote $X \cap \mathcal{L}_{sen}$, that is, the formulas of X without free variables.

(b) We say that revision function $\dot{+}$ is *sentence preserving* just in case for all $\sigma \in SEQ$ and $X \subseteq \mathcal{L}_{form}$, if $sentence(X) \cup content(\sigma)$ is consistent then $sentence(X) \subseteq X \dot{+} \sigma$.

A corollary to the proof of Lemma (27) reveals the existence of maxichoice definite, sentence preserving revision functions. From the proof of Theorem (28) we easily obtain:

(32) Corollary: Suppose that **Obs** is closed under negation. For every solvable problem $\mathbf{P}$ of the form $(T, \{\theta_0 \ldots \theta_n\})$ there is a consistent extension $X \subseteq \mathcal{L}_{form}$ of T

with the following properties.

(a) For all revision functions $\dotplus$, $\lambda\sigma\,.\,X \dotplus \sigma$ solves **P**.

(b) For all sentence preserving revision functions $\dotplus$, $\lambda\sigma\,.\,X \dotplus \sigma$ solves **P** efficiently.

Does Corollary (32) justify equating the maxichoice definite, sentence preserving revision functions as "rational?" The difficulty with this suggestion is the status of sentence preservation. Whereas the maxichoice definite revision functions appeal to intuitions about conservative revision and consistent preferences over theories (as discussed in Section 2.2), it is not clear what can be said on behalf of Definition (31), other than that it yields efficient inquiry.

6. Closure

Recall that the scientist's theories are interpreted as arbitrary subsets of formulas, not necessarily closed under deduction. The present section explores the impact on revision-based inquiry of requiring the starting theory B of scientist $\lambda\sigma\,.\,B \dotplus \sigma$ to be closed under some fragment of logic. We shall see that closed starting points are consistent with revision-based success. However, even a weak form of closure perturbs Theorem (22). That is, not every revision function can be used to solve every solvable problem of the form $(T, \{P_0, P_1, \ldots\})$ if the starting point is required to include its own logical consequences.

We begin our discussion with some remarks about the epistemological issues surrounding deductive closure of theories. The following notation will be used. Given $B \subseteq \mathcal{L}_{form}$, we use $Cn(B)$ to denote the set of logical consequences of B. Thus, $B \subseteq Cn(B)$, and B is closed under Cn if and only if $B = Cn(B)$.

6.1. Closure operators

Within the literature on belief revision, two conceptions of belief states have emerged. According to the "coherence" view, belief states are deductively closed, and the agent is committed in the same way to statements whose conviction is rooted directly in experience as she is to their logical consequences. The coherence approach is elaborated and defended in [11]. In contrast, belief states within the "foundationalist" approach contain only those formulas which the agent has extra-logical reason to believe, and thus are not in general deductively closed. The foundationalist approach is developed in [10, 13, 15, 14, 20] and in work cited there. For a particulary illuminating discussion of the issues, see [17, Ch. 1].

Both coherence and foundationalism advocate logical integrity in the following sense. A rational agent confronted with information that her belief state implies a falsehood is required to abandon some beliefs in order to remove the implication. However, foundationalism provides more guidance than coherence regarding what to abandon and what to conserve. To see this, let ϕ, ψ be logically independent formulas, and consider $A_1 = \{\phi, \psi\} \dotplus \neg\phi$ versus $A_2 = Cn(\{\phi, \psi\}) \dotplus \neg\phi$. For every revision function $\dotplus$, $A_1 = \{\neg\phi, \psi\}$. In contrast, A_2 can vary widely. Thus, for one choice of $\dotplus$, A_2 includes ψ but not $\psi \rightarrow \phi$, whereas the situation is reversed for another choice. Giving such liberty to revision functions leads some of them to mischief, as will be seen shortly.

Closure under Cn versus no closure at all can be viewed as extremes along a gradient. In order to consider intermediate points we rely on the following definition.

(33) Definition: By a *closure operator* is meant any function cl to and from subsets of $\mathcal{L}_{form}$ such that for all $B \subseteq \mathcal{L}_{form}$: $B \subseteq cl(B) \subseteq Cn(B)$.

At one end, we have the identity closure operator cl defined by $cl(B) = B$ for every $B \subseteq \mathcal{L}_{form}$. At the other end is deductive consequence. When theories are closed under a closure operator cl, then revision-based scientists have the form $\lambda\sigma\,.\,cl(B) \dot{+} \sigma$, for some $B \subseteq \mathcal{L}_{form}$ and revision function $\dot{+}$. If B and $\dot{+}$ are chosen astutely, revision-based scientists can still be made to work under these conditions. Indeed, examination of the proof of Theorem (21) shows that it can be strengthened to the following.

(34) THEOREM: Suppose that **Obs** is closed under negation. There exists $X \subseteq \mathcal{L}_{form}$ and definite maxichoice revision function $\dot{+}$ such that for all closure operators cl and all consistent $T \subseteq \mathcal{L}_{sen}$, the following holds.

(a) $T \cup X$ is consistent.

(b) $\lambda\sigma\,.\,cl(T \cup X) \dot{+} \sigma$ solves every solvable problem of the form $(T, \{\theta_0 \dots \theta_n\})$.

The foregoing theorem provides a sense in which closure does no harm to revision-based inquiry. Closure nonetheless complicates matters inasmuch as closed starting points require the revision function to be chosen with care if inquiry is to succeed. This is explained next.

6.2. Discovery under coherence-like starting points

Theorem (22) shows that every revision function can be used to solve any solvable problem of the form $(T, \{P_0, P_1, \dots\})$. Requiring belief states to be deductively closed, however, perturbs this result. In fact, the following theorem shows that it is enough to close belief states under the single rule: ψ, ϕ / $\psi \rightarrow \phi$. Belief state $B \subseteq \mathcal{L}_{form}$ is closed under the latter rule just in case $\psi, \phi \in B$ implies $\psi \rightarrow \phi \in B$. Notice how weak and innocent this condition appears to be!

(35) THEOREM: Let **Sym** consist of a binary predicate, and suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$. Then there exists a problem of form $(T, \{\theta, \neg\theta\})$ (with T finite), and maxichoice revision function $\dot{+}$ such that:

(a) $(T, \{\theta, \neg\theta\})$ is solvable;

(b) for all $B \subseteq \mathcal{L}_{form}$, if B is closed under ψ, ϕ / $\psi \rightarrow \phi$, then $\lambda\sigma \,.\, B \dot{+} \sigma$ does not solve $(T, \{\theta, \neg\theta\})$.

For the proof, see Section 8.17.

6.3. Remarks on closure and successful revision

Let us consider what conclusions may be drawn from Theorem (35). The theorem does not show that deductive closure of theories is incompatible with scientific success via theory revision. After all, Theorem (34) assures us that some revision function succeeds no matter what closure operator is in force. Rather, the lesson of Theorem (35) seems to bear on the issue of commitment to individual assertions. That the theory of an ideally rational agent should be deductively closed seems difficult to challenge. However, such an ideal agent may nonetheless be more attached to her background theory T and to its extension X than to their nontrivial, logical consequences. Indeed, this kind of situation arises whenever we discover an implausible consequence of our strong beliefs, and suffer doubt about which to abandon. Often our desire is to conserve as much as possible of the original beliefs, with no attempt to measure how large a fragment of their consequences will be lost. With this in mind, Theorem (35) suggests that when an agent faces contradictory data, she should seek to salvage a subset of her starting theory and then let the logical consequences fall where they may. For, as the theorem shows, a revision function that does not adopt this policy may be doomed to fail on some solvable problems, starting from any theory that is deductively closed.

Alternatively, if closed starting theories have high epistemological priority, then further conditions should be placed on revision functions. The conditions should ensure that all conforming revision functions can be used successfully on solvable problems, starting from closed starting points. Otherwise, one is left in the uncomfortable position of insisting that starting theories be deductively closed, while admitting that some acceptable means of revising those theories lead to needless scientific failure.

Another observation may serve to put a finer point on the distinction between coherence and foundationalist approaches to representing the epistemological state of an idealized scientist Ψ. Suppose Ψ solves problem $\mathbf{P}$, and consider the $P \in \mathbf{P}$ that is implied cofinitely often by Ψ's conjectures on some environment e for P. It is tempting to say that Ψ ends up *knowing* that P is true in this situation because (a) Ψ *believes* P (at least, he consistently announces P at the "end" of the game), (b) P is *true* in the world giving rise to e, and (c) it is *not just lucky* that Ψ ends up believing P in E since Ψ embodies a *reliable* strategy that succeeds in *any* environment for any proposition in $\mathbf{P}$. Thus, the inductive logic built around our paradigm of discovery is well suited to a *reliabilist* account of knowledge (for discussions of reliabilism, see [18, 12]). One well-known difficulty with reliabilism, however, is deductive closure. For, a procedure that can reliably discover the truth of both the sentences φ and θ may not be reliable for discovering the truth of their conjunction.[15] In response, it is natural to define the *known* sentences at a given stage of inquiry to be the deductive closure of whatever sentences are true, and believed to be so by dint of the application of a reliable process. So, once again we see a distinction between a "core" set of known sentences versus the result of closing them via deduction. Theorem (35) underlines the importance of making this kind of distinction.

[15] This is clearest in a probabilistic setting since the discovery-rates for the two sentences might be stochastically independent, thereby lowering the discovery rate for both together. But even if a scientific strategy has perfect reliability for each of φ and θ it does not follow that the same is true of their conjunction.

7. Iterated revision

7.1. Another conception of inquiry

The scientists discussed so far in this essay revise their starting theory by confronting it anew with each initial segment of the environment. An alternative conception allows starting theories to evolve with time, confronting only the latest datum at any given stage of inquiry. To explain, let $B \subseteq \mathcal{L}_{form}$ be a starting theory. Faced with data $\langle \delta_1, \delta_2 \rangle \in SEQ$, the scientist $\lambda\sigma \,.\, B \dotplus \sigma$ conjectures $B \dotplus \delta_1$ at the first step of inquiry and $B \dotplus \langle \delta_1, \delta_2 \rangle$ at the second. In contrast, an iterated model of inquiry conceives the scientist as first conjecturing $B \dotplus \delta_1$, as before, but then conjecturing $(B \dotplus \delta_1) \dotplus \delta_2$. In general, at step $n+1$ of iterated inquiry, the theory facing the $(n+1)$th datum has already been modified n times. To formalize this idea we define the "iterated extension" of a revision function.

(36) Definition: Let revision function $\dotplus$ be given. The *iterated extension of* $\dotplus$ is defined to be the function $\dotplus_{ie} : pow(\mathcal{L}_{form}) \times SEQ \rightarrow pow(\mathcal{L}_{form})$ such that for all $B \subseteq \mathcal{L}_{form}$, $\sigma \in SEQ$ and $k \in N$,

$$B \dotplus_{ie} \sigma = \begin{cases} B & \text{if } \sigma = \emptyset, \\ [B \dotplus_{ie} (\beta_0 \ldots \beta_{k-1})] \dotplus \beta_k & \text{if } \sigma = (\beta_0 \ldots \beta_k). \end{cases}$$

Iterative revision resembles memory-limitation within the numerical paradigm, discussed in Section 2.5 of the first essay. In both cases the latest datum plays a special role in hypothesis selection, whereas earlier data are preserved only through their impact on the hypothesis of the preceding stage.

Iterated revision represents in a natural manner the influence of earlier hypotheses on the choice of later ones. This is because the fragment of the starting theory that survives long enough to confront a non-initial datum depends on decisions about what to preserve in the face of earlier data. In contrast, non-iterative revision allows the following kind of cut-and-paste operation. Let Ψ_1 and Ψ_2 be revision-based scientists with the same starting point, B. Then the following scientist Ψ_3 is also revision-based. For all $\sigma \in \mathit{SEQ}$:

$$\Psi_3(\sigma) = \begin{cases} \Psi_1(\sigma) & \text{if } \mathit{length}(\sigma) < 1000; \\ \Psi_2(\sigma) & \text{otherwise.} \end{cases}$$

Scientist Ψ_3 is revision-based because revision functions $\dotplus$ impose no constraint on the relation between $B \dotplus \sigma$ and $B \dotplus (\sigma * \tau)$. [See Definition (3).]

We now consider what can and cannot be achieved using iterative revision. Our first two propositions show that iterative revision is compatible with successful inquiry, and even with efficient inquiry. However, it will then be demonstrated that some revision functions cannot be used in iterative fashion to solve certain simple problems, no matter what starting point is chosen.

7.2. Achievements of iterative revision

The following two propositions show that iterative revision can be used successfully for inquiry. In particular (and similarly to before), a sole revision function and theory-extender suffice to solve every solvable problem of form $(T, \{\theta_0 \dots \theta_n\})$.

(37) Proposition: Suppose that **Obs** is closed under negation. There exists $X \subseteq \mathcal{L}_{form}$ and definite maxichoice revision function $\dotplus$ such that for all consistent $T \subseteq \mathcal{L}_{sen}$, the following holds.

(a) $T \cup X$ is consistent.

(b) $\lambda\sigma\,.\,(T \cup X) \dotplus_{ie} \sigma$ solves every solvable problem of the form $(T, \{\theta_0 \ldots \theta_n\})$.

The proof is given in Section 8.18. We see that Proposition (37) is the iterative counterpart to Theorem (21). The analogy extends to efficient inquiry insofar as the next proposition shows that iterative revision can be used to efficiently solve every solvable problem of form $(T, \{P_0, P_1, \ldots\})$. [Compare Theorem (25).] The proof is given in Section 8.19.

(38) Proposition: Suppose that **Obs** is closed under negation. For every solvable problem **P** of the form $(T, \{P_0, P_1, \ldots\})$ there is a consistent extension $X \subseteq \mathcal{L}_{form}$ of T and a definite maxichoice revision function $\dotplus$ such that $\lambda\sigma\,.\,X \dotplus_{ie} \sigma$ solves **P** efficiently.

Despite the achievements recorded in the foregoing propositions we shall now provide a sense in which hypothesis revision is less successful in the iterative context than before.

7.3. Iterative revision is not robust

Given $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$, Lemma (4) shows that $B \dotplus \sigma \models \bigwedge \sigma$. In contrast, it is easy to construct examples to show that $B \dotplus_{ie} \sigma$ may contradict $\bigwedge \sigma$, although it must imply σ's last member. Such license to ignore past data gives iterative revision so much flexibility that some revision functions can no longer be used to solve simple problems. The infirmity afflicts even the class of maxichoice definite functions. This is shown by the following theorem, which may be contrasted with Theorem (22). To state our result, let **two** be the sentence $\exists xy[x \neq y \wedge \forall z(z = x \vee z = y)]$, asserting that there are exactly two individuals. It is easy to verify that $(\emptyset, \{\mathbf{two}, \neg\mathbf{two}\})$ is solvable. However:

(39) Theorem: Suppose that $\mathbf{Sym} = \emptyset$ and $\mathbf{Obs} = \mathcal{L}_{basic}$. Then there exists a definite maxichoice revision function $\dotplus$ such that for all $B \subseteq \mathcal{L}_{form}$, $\lambda\sigma\,.\,B \dotplus_{ie} \sigma$ does not

solve $(\emptyset, \{\mathbf{two}, \neg\mathbf{two}\})$.

See Section 8.20 for the proof. Theorem (22) and Proposition (39) imply the existence of a maxichoice, definite revision function $\dot{+}$ such that $\dot{+}_{ie} \neq \dot{\oplus}$ for every iterated extension $\dot{\oplus}$ of a revision function. In this sense, iterated and non-iterated theory revision yield different approaches to scientific discovery.

7.4. Iterative scientists under deductive closure

Having examined deductive closure of belief states in Section 6, and iterated belief revision in the present section, it is natural to wonder about the consequences of combining the two features of inquiry.[16] To give substance to this idea requires closing each belief state that emerges at successive steps of iterated revision. It is not enough to close only the starting belief state since its successors may be left open by the revision process. We therefore formalize the matter as follows.

(40) Definition: Let revision function $\dot{+}$ be given. The *closed iterated extension of* $\dot{+}$ is defined to be the function $\dot{+}^{Cn}_{ie} : pow(\mathcal{L}_{form}) \times SEQ \rightarrow pow(\mathcal{L}_{form})$ such that for all $B \subseteq \mathcal{L}_{form}$, $\sigma \in SEQ$ and $k \in N$,

$$B \dot{+}^{Cn}_{ie} \sigma = \begin{cases} B & \text{if } \sigma = \emptyset, \\ Cn(B \dot{+}^{Cn}_{ie} (\beta_0 \ldots \beta_{k-1})) \dot{+} \beta_k & \text{if } \sigma = (\beta_0 \ldots \beta_k). \end{cases}$$

It will now be seen that iterative revision under closure, like its unclosed counterpart, suffers from the existence of maxichoice definite revision functions unable to solve simple problems. As before, let **two** be the sentence $\exists xy[x \neq y \wedge \forall z(z = x \vee z = y)]$. Then we have the following parallel to Theorem (39). It is proved in Section 8.21.

[16]The present subsection is not central to our discussion, and may be omitted on a first reading.

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(41) Proposition: Suppose that **Sym** $= \emptyset$ and **Obs** $= \mathcal{L}_{basic}$. Then there exists a definite maxichoice revision function $\dot{+}$ such that for all $B \subseteq \mathcal{L}_{form}$, $\lambda\sigma \,.\, B \dot{+}^{Cn}_{ie} \sigma$ does not solve $(\emptyset, \{\mathbf{two}, \neg\mathbf{two}\})$.

8. Proofs

8.1. Proof of Proposition (10)

(10) Proposition: There are revision functions that are both maxichoice and definite.

In the present subsection we establish the existence of definite maxichoice revision functions, thereby proving Proposition (10). For this purpose we rely on a construction that will be pivotal in later proofs. Here and elsewhere we rely on elementary facts about ordinal numbers (see [8, 19] for background).

(42) Definition: Let nonnull ordinal κ be given. Let enumeration $\mathbf{S} = \{S_\alpha \mid 0 < \alpha < \kappa\}$ of members of $pow(\mathcal{L}_{form})$ also be given. We specify a function $\dot{+}_{\mathbf{S}} : pow(\mathcal{L}_{form}) \times SEQ \to pow(\mathcal{L}_{form})$. Let $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$ be given. Then $B \dot{+}_{\mathbf{S}} \sigma$ is defined by induction over $\alpha < \kappa$. Specifically, for each $\alpha < \kappa$ we define the set Y^α, and put $B \dot{+}_{\mathbf{S}} \sigma = \bigcup_{\alpha<\kappa} Y^\alpha$. The inductive definition of Y^α is as follows.

Set $Y^0 = Inn(B, \sigma) \cup content(\sigma)$. Let nonnull ordinal $\alpha < \kappa$ be given, and suppose that Y^β is defined for all $\beta < \alpha$. Then $Y^\alpha = S_\alpha$ if $S_\alpha \subseteq B \cup content(\sigma)$ and $(\bigcup_{\beta<\alpha} Y^\beta) \cup S_\alpha$ is consistent; otherwise $Y^\alpha = \emptyset$.

Intuitively, $B \dot{+}_{\mathbf{S}} \sigma$ is constructed as follows. At stage 0 you dump $Inn(B, \sigma) \cup content(\sigma)$ into $B \dot{+}_{\mathbf{S}} \sigma$. At stage 1 you examine S_1. If $S_1 \subseteq B \cup content(\sigma)$ and $S_0 \cup S_1$ is consistent, then you dump S_1 into $B \dotplus_{\mathbf{S}} \sigma$; otherwise, you leave $B \dotplus_{\mathbf{S}} \sigma$ as it was after the preceding stage. Keep going in this way. Denote by S the portion of $B \dotplus_{\mathbf{S}} \sigma$ constructed at stages $0, 1, 2 \ldots$. At stage ω, you examine S_ω. Suppose that $S_\omega \subseteq B \cup \mathit{content}(\sigma)$ and $S \cup S_\omega$ is consistent. Then you dump S_ω into $B \dotplus_{\mathbf{S}} \sigma$; otherwise, leave $B \dotplus_{\mathbf{S}} \sigma$ as S. Keep going ...

The important fact about this construction is that it yields a definite maxichoice revision function, provided only that the enumeration $\{S_\alpha \,|\, 0 < \alpha < \kappa\}$ contains all singleton sets $\{\varphi\}$, $\varphi \in \mathcal{L}_{\mathit{form}}$. Precisely:

(43) Proposition: Let ordinal κ be given. Let enumeration $\mathbf{S} = \{S_\alpha \,|\, 0 < \alpha < \kappa\}$ of members of $\mathit{pow}(\mathcal{L}_{\mathit{form}})$ contain all singleton sets. Then $\dotplus_{\mathbf{S}}$ is a definite maxichoice revision function.

Proof: Let $B \subseteq \mathcal{L}_{\mathit{form}}$ and (consistent) $\sigma \in \mathit{SEQ}$ be given. Clearly, $\mathit{Inn}(B, \sigma) \cup \mathit{content}(\sigma) \subseteq B \dotplus_{\mathbf{S}} \sigma$. An easy application of compactness shows that $B \dotplus_{\mathbf{S}} \sigma$ is consistent. Moreover, because $\{\chi\} \in \mathbf{S}$ for all $\chi \in \mathcal{L}_{\mathit{form}}$, the construction in Definition (42) ensures that for all $\chi \in (B \cup \mathit{content}(\sigma)) - (B \dotplus_{\mathbf{S}} \sigma)$, $(B \dotplus_{\mathbf{S}} \sigma) \cup \{\chi\}$ is inconsistent. This implies easily that $\dotplus_{\mathbf{S}}$ is a maxichoice revision function.

Define the following strict total order $\prec^\#$ on $\mathit{pow}(\mathcal{L}_{\mathit{form}})$. Given $X, Y \subseteq \mathcal{L}_{\mathit{form}}$, $X \prec^\# Y$ iff the following condition holds.

> There is nonnull $\alpha < \kappa$ such that $S_\alpha \subseteq X$, $S_\alpha \not\subseteq Y$, and for all nonnull $\beta < \alpha$, $S_\beta \subseteq X$ iff $S_\beta \subseteq Y$.

Indeed, $\prec^\#$ is a strict total ordering of $\mathit{pow}(\mathcal{L}_{\mathit{form}})$ because $\mathbf{S}$ contains all singleton sets. We need to show that $B \dotplus_{\mathbf{S}} \sigma$ is the $\prec^\#$-least consistent subset of $B \cup \mathit{content}(\sigma)$ that contains $\mathit{Inn}(B, \sigma) \cup \mathit{content}(\sigma)$. Let Y^α, $\alpha < \kappa$, be the subsets of $B \cup \mathit{content}(\sigma)$ defined from B and σ as in Definition (42). For a contradiction suppose that $C \subseteq \mathcal{L}_{\mathit{form}}$ is such that:

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(44) (a) $\mathit{Inn}(B, \sigma) \cup \mathit{content}(\sigma) \subseteq C$,
(b) $C \prec^{\#} \bigcup_{\alpha<\kappa} Y^{\alpha}$,
(c) $C \subseteq B \cup \mathit{content}(\sigma)$,
(d) C is consistent.

By (44)b, let nonnull $\alpha < \kappa$ be such that $S_\alpha \subseteq C$, $S_\alpha \not\subseteq \bigcup_{\gamma<\kappa} Y^{\gamma}$, and for all nonnull $\beta < \alpha$, $S_\beta \subseteq C$ iff $S_\beta \subseteq \bigcup_{\gamma<\kappa} Y^{\gamma}$. By (44)c, $S_\alpha \subseteq B \cup \mathit{content}(\sigma)$. By (44)a,d, $\mathit{Inn}(B, \sigma) \cup \mathit{content}(\sigma) \cup \bigcup\{S_\beta \mid S_\beta \subseteq C \text{ and } 0 < \beta < \alpha\}$ is consistent. These facts, along with the construction in Definition (42) imply that $S_\alpha \subseteq Y^{\alpha}$, contradiction. ■

So, there are plenty of definite maxichoice revision functions.

8.2. Proof of Theorem (14)

(14) Theorem: There is a maxichoice revision function $\dotplus$ with the following property. Let problem $\mathbf{P}$ be such that for some $Y \subseteq \mathcal{L}_{\mathit{form}}$ and revision function $\dot{\oplus}$, $\lambda\sigma \,.\, Y \dot{\oplus} \sigma$ solves $\mathbf{P}$. Then there is a consistent $X \subseteq \mathcal{L}_{\mathit{form}}$ such that $\lambda\sigma \,.\, X \dotplus \sigma$ solves $\mathbf{P}$.

To prove the theorem we rely on an evident fact, namely: for a revision-based scientist to solve a problem $\mathbf{P}$, it must be possible to produce sets of formulas that imply the propositions of $\mathbf{P}$ while remaining consistent with the incoming data. The following definition states this condition precisely.

(45) Definition: Problem $\mathbf{P}$ is *feasible* just in case for all $P \in \mathbf{P}$, and all $\sigma \in \mathit{SEQ}$ for P, there is $Y \subseteq \mathcal{L}_{\mathit{form}}$ such that $\emptyset \neq \mathit{MOD}(Y \cup \mathit{content}(\sigma)) \subseteq P$.

We will now demonstrate two things. First, if a problem is not feasible then it is not solvable by a revision-based scientist. Second, there is a maxichoice revision function $\dotplus$ such that if a problem is both solvable and feasible then $\dotplus$ suffices to solve the problem in a revision-based way. First, revision-based solvability implies feasibility:

(46) PROPOSITION: Suppose that problem $\mathbf{P}$ is not feasible. Then for all $B \subseteq \mathcal{L}_{form}$ and revision functions $\dotplus$, $\lambda\sigma \,.\, B \dotplus \sigma$ does not solve $\mathbf{P}$.

Proof: Let problem $\mathbf{P}$, $P \in \mathbf{P}$, and $\sigma \in \mathit{SEQ}$ for P be such that for all $Y \subseteq \mathcal{L}_{form}$ either $\mathit{MOD}(Y \cup \mathit{content}(\sigma)) = \emptyset$ or $\mathit{MOD}(Y \cup \mathit{content}(\sigma)) \not\subseteq P$. Then for all $\tau \in \mathit{SEQ}$:

(47) for all $Y' \subseteq \mathcal{L}_{form}$, either

$$\mathit{MOD}(Y' \cup \mathit{content}(\sigma * \tau)) = \emptyset, \quad \text{or}$$

$$\mathit{MOD}(Y' \cup \mathit{content}(\sigma * \tau)) \not\subseteq P.$$

Let $X \subseteq \mathcal{L}_{form}$, revision function $\dotplus$, and environment e for P with $\sigma \subseteq e$ be given. Then (47) implies immediately that $\lambda\sigma \,.\, X \dotplus \sigma$ does not solve P in e. ■

The second part of the proof of Theorem (14) is formulated as follows.

(48) PROPOSITION: There is a maxichoice revision function $\dotplus$ with the following property. For every solvable, feasible problem $\mathbf{P}$ there is consistent $X \subseteq \mathcal{L}_{form}$ such that $\lambda\sigma \,.\, X \dotplus \sigma$ solves $\mathbf{P}$.

Thus, the class of scientists of form $\lambda\sigma . X \dot{+} \sigma$ with X consistent and $\dot{+}$ maxichoice is a canonical strategy for the feasible problems.

The usual kind of completion argument shows the following.

(49) LEMMA: For every revision function $\dot{\oplus}$ there is a maxichoice revision function $\dot{+}$ with the following property. For all $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$, $B \dot{\oplus} \sigma \subseteq B \dot{+} \sigma$.

(50) DEFINITION: A linguistic scientist Λ is *consistent* just in case for all $\sigma \in SEQ$, $\Lambda(\sigma) \cup content(\sigma)$ is consistent.

We also have:

(51) LEMMA: Every solvable, feasible problem is solved by a consistent linguistic scientist.

Proof: Let scientist Ψ solve feasible problem $\mathbf{P}$. By feasibility, for every $P \in \mathbf{P}$ and for every $\sigma \in SEQ$ for P, choose $Y(\sigma, P) \subseteq \mathcal{L}_{form}$ such that (a) $\emptyset \neq MOD(Y(\sigma, P)) \subseteq P$ and (b) $Y(\sigma, P) \cup content(\sigma)$ is consistent. Define linguistic scientist Λ such that for all $\sigma \in SEQ$, $\Lambda(\sigma) = MOD(Y(\sigma, P))$ if σ is for P and $\emptyset \neq \Psi(\sigma) \subseteq P$; otherwise, $\Lambda(\sigma) = MOD(\sigma)$. It is clear that Λ solves $\mathbf{P}$, and that Λ is consistent. ■

Proof of Proposition (48): By (49), let maxichoice revision function $\dot{+}$ have the following property. For all $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$, if $\sigma \neq \emptyset$ and $B \cap \{\bigwedge \sigma \rightarrow \varphi \,|\, \varphi \in \mathcal{L}_{form}\}$ is consistent with $content(\sigma)$, then $B \cap \{\bigwedge \sigma \rightarrow \varphi \,|\, \varphi \in \mathcal{L}_{form}\} \subseteq B \dot{+} \sigma$. Let solvable and feasible problem $\mathbf{P}$ be given. There is nothing to prove if $\mathbf{P} = \emptyset$, so choose $P_0 \in \mathbf{P}$. By feasibility, there is $Y_1 \subseteq \mathcal{L}_{form}$ such that $\emptyset \neq MOD(Y_1) \subseteq P_0$. Let $Y_0 \subseteq \mathcal{L}_{form}$ be the result of doubling the index of every variable appearing in Y_1. Then also $\emptyset \neq MOD(Y_0) \subseteq P_0$. So we may choose $\mathcal{S}_0 \in P_0$ and full assignment h with $\mathcal{S}_0 \models Y_0[h]$. Let environment e_0 be for $\mathcal{S}_0$ and h. Then we have:

(52) $\mathcal{S}_0 \models Y_0 \cup \mathit{content}(e_0)[h]$.

Since $\mathbf{P}$ is solvable and feasible, by Lemma (51) there is linguistic scientist Λ that solves $\mathbf{P}$ and satisfies:

(53) for all $\sigma \in \mathit{SEQ}$, $\Lambda(\sigma)$ is consistent with $\mathit{content}(\sigma)$.

Define $X_0 = \{\bigwedge \sigma \rightarrow \varphi \,|\, \emptyset \neq \mathit{content}(\sigma) \subseteq \mathit{content}(e_0)$ and $\varphi \in Y_0\}$. Define $X_1 = \{\bigwedge \sigma \rightarrow \varphi \,|\, \mathit{content}(\sigma) \not\subseteq \mathit{content}(e_0)$ and $\varphi \in \Lambda(\sigma)\}$. We take $X = X_0 \cup X_1$. Observe that $\mathcal{S}_0 \models X_0[h]$ because by (52) $\mathcal{S}_0 \models \varphi[h]$ for all $\varphi \in Y_0$. Also $\mathcal{S}_0 \models X_1[h]$ because by (52) again $\mathcal{S}_0 \not\models \bigwedge \sigma[h]$ for all $\sigma \in \mathit{SEQ}$ with $\mathit{content}(\sigma) \not\subseteq \mathit{content}(e_0)$. So X is consistent. It remains to show that $\lambda\sigma \,.\, X \dotplus \sigma$ solves $\mathbf{P}$. Let environment e for $P_1 \in \mathbf{P}$ be given. There are two cases.

Case 1: $\mathit{content}(e) = \mathit{content}(e_0)$. Let $k > 0$ be given. By (52), $Y_0 \cup \mathit{content}(e[k])$ is consistent. So the definition of $\dotplus$ and X implies that $X \dotplus e[k] = \{\bigwedge e[k] \rightarrow \varphi \,|\, \varphi \in Y_0\} \cup \mathit{content}(e[k]) \models Y_0$. Since e is for $\mathcal{S}_0$, e is for both P_0 and P_1, and we infer from the assumption that $\mathbf{P}$ is solvable that $P_0 = P_1$. Since $\emptyset \neq \mathit{MOD}(Y_0) \subseteq P_0$, it follows that $\lambda\sigma \,.\, X \dotplus \sigma$ solves P_1 in e.

Case 2: $\mathit{content}(e) \neq \mathit{content}(e_0)$. Let $k_0 > 0$ be least such that $\mathit{content}(e[k_0]) \not\subseteq \mathit{content}(e_0)$. Then by (53) and the definition of $\dotplus$ and X, for all $k \geq k_0$, $X \dotplus e[k] = \{\bigwedge e[k] \rightarrow \varphi \,|\, \varphi \in \Lambda(e[k])\} \cup \mathit{content}(e[k])$, hence $X \dotplus e[k] \models \Lambda(e[k])$. Since Λ solves P_1 on e, so does $\lambda\sigma \,.\, X \dotplus \sigma$. ■

8.3. Proof of Theorem (15)

(15) THEOREM: Suppose that **Obs** $= \mathcal{L}_{basic}$. There is a maxichoice revision function $\dot{+}$ with the following property. Let problem **P** be such that for some $Y \subseteq \mathcal{L}_{form}$ and revision function $\dot{\oplus}$, $\lambda\sigma \,.\, Y \dot{\oplus} \sigma$ solves **P**. Then there is a consistent $X \subseteq \mathcal{L}_{form}$ such that $\lambda\sigma \,.\, X \dot{+} \sigma$ solves **P** efficiently.

Proof: Let problem **P** satisfy the hypothesis of the theorem. Then **P** is solvable and feasible. By Theorem (79) of the second essay, we can assume that the linguistic scientist Λ considered in the proof of Theorem (14) solves **P** strongly efficiently. Let revision function $\dot{+}$ and $X \subseteq \mathcal{L}_{form}$ be the revision function and formula-set defined in the proof of Proposition (48). Using Lemma (108) of essay #2, and the fact that Λ solves **P** strongly efficiently, we infer easily from the proof of Proposition (48) that $\lambda\sigma \,.\, X \dot{+} \sigma$ solves **P** strongly efficiently. Hence, $\lambda\sigma \,.\, X \dot{+} \sigma$ solves **P** efficiently. ■

8.4. Proof of Theorem (16)

(16) THEOREM: Suppose that **Sym** is limited to a binary predicate and countably many constants and **Obs** $= \mathcal{L}_{basic}$. Then there exists a definite maxichoice revision function $\dot{+}$, and propositions P_1, P_2 that meet the following conditions.

(a) $\{P_1, P_2\}$ is solvable and each of P_1, P_2 is closed under elementary equivalence (hence, $\{P_1, P_2\}$ is feasible).

(b) for all $B \subseteq \mathcal{L}_{form}$, $\lambda\sigma \,.\, B \dot{+} \sigma$ does not solve $\{P_1, P_2\}$.

Proof: Let R be the binary predicate of **Sym**. Let $\{\overline{n} \,|\, n \in N\}$ enumerate the constants of **Sym**. Let $\mathcal{S}$ be the structure with domain N that interprets $\overline{n}$ as n, for all $n \in N$, and such that $R^{\mathcal{S}}$ is the usual total order on N. Denote by θ a constant-free sentence such that $\mathcal{S} \models \theta$, and $\mathcal{U} \models \neg\theta$ for all finite structures $\mathcal{U}$. Define P_1 to be the set of all structures elementary equivalent to $\mathcal{S}$, P_2 the set of all finite structures. It is immediate that each of P_1, P_2 are closed under elementary equivalence. Moreover, it is straightforward to show that $\{P_1, P_2\}$ has tip-offs, so Proposition (38) of the second essay implies that $\{P_1, P_2\}$ is solvable.

Now we define $\dot{+}$. Fix an environment e for $\mathcal{S}$. For all $k \in N$, let $\{S_{(\omega\times k)+i} \,|\, i \in N,\ k+i \neq 0\}$ be an enumeration of all finite $D \subseteq \mathcal{L}_{form}$ such that $D \cup \mathit{content}(e[k])$ is consistent and $D \cup \mathit{content}(e[k]) \models \theta$. Fix an enumeration $\{\varphi_i \,|\, i \in N\}$ of $\mathcal{L}_{form}$, and set $S_{\omega^2+i} = \{\varphi_i\}$ for all $i \in N$. Let $\mathbf{S} = \{S_\alpha \,|\, 0 < \alpha < \omega^2 + \omega\}$, and let $\dot{+}$ be constructed from $\mathbf{S}$ as in Definition (42). So $\dot{+}$ is definite maxichoice by Proposition (43). Let $B \subseteq \mathcal{L}_{form}$ be given, and suppose that $\lambda\sigma \,.\, B \dot{+} \sigma$ solves P_1 in e. To verify (16)b, it suffices to show that for some environment e' for P_2, $\lambda\sigma \,.\, B \dot{+} \sigma$ does not solve P_2 in e'. For this purpose, we need the following fact.

(54) FACT: Let consistent and finite $D \subseteq \mathcal{L}_{form}$ be given. Then there is $\mathcal{U} \in P_2$ and full assignment h to $\mathcal{U}$ such that $\mathcal{U} \models \{\beta \in \mathcal{L}_{basic} \,|\, D \models \beta\}[h]$.

Proof of Fact (54): Let consistent and finite $D \subseteq \mathcal{L}_{form}$ be given. Let $X_1 = D \cup \{\overline{0} = \overline{n} \,|\, \overline{n}$ does not appear in $D\}$. Then X_1 is consistent. Let $w_0, w_1 \ldots$ be an enumeration without repetition of the cofinitely many variables that do not appear in D. Then $X_2 = X_1 \cup \{w_n = \overline{n} \,|\, n \in N\}$ is consistent. Let X_3 be a maximally consistent extension of X_2 in $X_2 \cup \mathcal{L}_{basic}$. By a familiar construction due to Leon Henkin (used for proving the existence of models [7]) it is clear that there is finite structure $\mathcal{U}$ and full assignment h to $\mathcal{U}$ such that $\mathcal{U} \models (X_3 \cap \mathcal{L}_{basic})[h]$. So, $\mathcal{U}$ and h satisfy Fact (54). ∎

Since $\lambda\sigma \,.\, B \dot{+} \sigma$ solves P_1 in e, $\mathit{MOD}(B \dot{+} e[k]) \subseteq P_1$ for some $k \in N$. Hence there is smallest $k_0 \in N$ such that for some $Y \subseteq B$, $Y \cup \mathit{content}(e[k_0])$ is consistent and $Y \cup$

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$\mathit{content}(e[k_0]) \models \theta$. By compactness, let $i_0 \in N$ (with $i_0 \neq 0$ if $k_0 = 0$) be smallest such that:

(55) (a) $S_{(\omega\times k_0)+i_0} \subseteq B$ and $S_{(\omega\times k_0)+i_0} \cup \mathit{content}(e[k_0])$ is consistent;

(b) $S_{(\omega\times k_0)+i_0} \cup \mathit{content}(e[k_0]) \models \theta$.

By (55)a and Fact (54), let structure $\mathcal{U} \in P_2$ and full assignment h to $\mathcal{U}$ be such that $\mathcal{U} \models \{\beta \in \mathcal{L}_{basic} \mid S_{(\omega\times k_0)+i_0} \cup \mathit{content}(e[k_0]) \models \beta\}[h]$. Let e' be an environment for $\mathcal{U}$ and h that extends $e[k_0]$. It follows from (55)a and the choice of e' that for all $k \geq k_0$, there is $Y \subseteq B$ such that $Y \cup \mathit{content}(e'[k])$ is consistent and $S_{(\omega\times k_0)+i_0} \subseteq Y$. This with the choice of k_0, i_0 implies for all $k \geq k_0$, $S_{(\omega\times k_0)+i_0} \subseteq B \dotplus e'[k]$. With (55)b we infer that for all $k \geq k_0$, $B \dotplus e'[k] \models \theta$. However, since $|\,\mathcal{U}\,|$ is finite, $\mathcal{U} \not\models \theta$ by hypothesis. Hence for all $k \geq k_0$, $B \dotplus e'[k] \not\subseteq P_2$. Hence $\lambda\sigma\,.\,B \dotplus \sigma$ does not solve P_2 in e', as required. ■

8.5. Proof of Proposition (17)

(56) PROPOSITION: (17) PROPOSITION: Suppose that **Obs** is closed under negation. For every solvable *r.e.* problem $(T, \{\theta_0 \ldots \theta_n\})$, there is consistent $X \subseteq \mathcal{L}_{form}$ with $X \models T$, and definite maxichoice revision function $\dotplus$ such that $\lambda\sigma\,.\,X \dotplus \sigma$ is computable and solves $(T, \{\theta_0 \ldots \theta_n\})$.

We need the following lemma.

(57) LEMMA: Let $B \subseteq \mathcal{L}_{form}$, $\sigma \in \mathit{SEQ}$, and revision function $\dotplus$ be given. Suppose there is $C \subseteq B$ such that: (a) for every $\chi \in C$, $\chi \models \neg \bigwedge \sigma$, and (b) $B - C \cup \mathit{content}(\sigma)$ is consistent. Prove that $B \dotplus \sigma = (B - C) \cup \mathit{content}(\sigma)$.

Proof: Condition (a) implies that $Inn(B, \sigma) \cap C = \emptyset$. With Condition (b) it follows that $Inn(B, \sigma) = B - C$. The exercise follows immediately. ■

Proof of Proposition (17): Let T_0 be an *r.e.* axiomatisation of T. By Corollary (61) essay #2, let refutable formula $\varphi_0(\bar{x}_0)$ (with free variables $\bar{x}_0$) be such that $T \models \exists \bar{x}_0 \varphi_0(\bar{x}_0) \rightarrow \theta_0$ and $T \cup \{\varphi_0(\bar{x}_0)\}$ is consistent. Let $\{\varphi_i(\bar{x}_i) \,|\, i \in N\}$ be a recursive enumeration of all refutable formulas $\varphi(\bar{x})$ such that for some $i \leq n$, $T \models \exists \bar{x} \varphi(\bar{x}) \rightarrow \theta_i$. Note that the enumeration starts off with the formula $\varphi_0(\bar{x}_0)$ chosen to be consistent with T. Set $X = T_0 \cup \{\varphi_0(\bar{x}_0) \vee \ldots \vee \varphi_i(\bar{x}_i) \,|\, i \in N\}$. Since $T_0 \cup \{\varphi_0(\bar{x}_0)\}$ is consistent, so is X.

Now we define $\dot{+}$. Let partial function $f : SEQ \rightarrow N$ be defined as follows. Let $\sigma \in SEQ$ be given. If there is no $i \in N$ such that $T \cup \{\varphi_0(\bar{x}_0) \vee \ldots \vee \varphi_i(\bar{x}_i)\} \cup content(\sigma)$ is consistent, then $f(\sigma)$ is undefined; otherwise, $f(\sigma)$ is the least $i \in N$ such that $T \cup \{\varphi_0(\bar{x}_0) \vee \ldots \vee \varphi_i(\bar{x}_i)\} \cup content(\sigma)$ is consistent. Since T is decidable and since $\{\varphi_i(\bar{x}_i) \,|\, i \in N\}$ is a recursive enumeration, f is a partial recursive function. Let $S_0 = T_0$ and for all $i \in N$, let $S_{i+1} = \{\varphi_0(\bar{x}_0) \vee \ldots \vee \varphi_i(\bar{x}_i)\}$. Let $\mathbf{S} = \{S_i \,|\, i < \omega\}$, and let $\dot{+}$ be constructed from $\mathbf{S}$ as in Definition (42). So $\dot{+}$ is maxichoice and definite by Proposition (43). It is easy to verify that for all $\sigma \in SEQ$, the following holds. If $f(\sigma)$ is undefined then $X \dot{-} \neg \bigwedge \sigma = T_0$.

If $f(\sigma)$ is defined then $X \dot{-} \neg \bigwedge \sigma = T_0 \cup \{\varphi_0(\bar{x}_0) \vee \ldots \vee \varphi_i(\bar{x}_i) \,|\, i \geq f(\sigma)\}$. Since f is partial recursive and since T_0 is *r.e.*, it follows immediately that $\lambda \sigma \,.\, X \dot{+} \sigma$ is computable.

Now we show that for every revision function $\dot{+}$ (whether or not it is computable), $\lambda \sigma \,.\, X \dot{+} \sigma$ solves $(T, \{\theta_0 \ldots \theta_n\})$. This suffices to complete the proof. Let revision function $\dot{+}$, $j \leq n$, $\mathcal{S} \in MOD(T \cup \{\theta_j\})$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given. We must show that:

(58) for cofinitely k, $X \dot{+} e[k] \models T \cup \{\theta_j\}$.

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By Theorem (59) of the second essay, let i_0 be least with $\mathcal{S} \models \varphi_{i_0}(\bar{x}_{i_0})[h]$. So, because the θ_i's are mutually exclusive in T, we have:

(59) $T \models \varphi_{i_0}(\bar{x}_{i_0}) \rightarrow \theta_j$.

By the choice of i_0, $\mathcal{S} \models T \cup \{\varphi_0(\bar{x}_0) \vee \ldots \vee \varphi_{i_0}(\bar{x}_{i_0})\}[h]$, and for all $i < i_0$, $\mathcal{S} \models \neg\varphi_i(\bar{x}_i)[h]$. Since the φ_i's are $\forall$ formulas, and since h is onto $|\,\mathcal{S}\,|$, it follows that there is $k_0 \in N$ such that for all $k \geq k_0$:

(60) (a) for all $i < i_0$, $\varphi_0(\bar{x}_0) \vee \ldots \vee \varphi_i(\bar{x}_i) \models \neg \bigwedge e[k]$, and

(b) $T_0 \cup \{\varphi_0(\bar{x}_0) \vee \ldots \vee \varphi_i(\bar{x}_i) \,|\, i \geq i_0\} \not\models \neg \bigwedge e[k]$.

With Lemma (57), this implies that for all $k \geq k_0$, $X \dotplus e[k] = T_0 \cup \{\varphi_0(\bar{x}_0) \vee \ldots \vee \varphi_i(\bar{x}_i) \,|\, i \geq i_0\} \cup content(e[k])$. Hence for all $k \geq k_0$, $T_0 \cup \{\varphi_0(\bar{x}_0) \vee \ldots \vee \varphi_{i_0}(\bar{x}_{i_0})\} \cup content(e[k]) \subseteq X \dotplus e[k]$. With (60)a, we infer that for all $k \geq k_0$, $X \dotplus e[k] \models T \cup \{\varphi_{i_0}(\bar{x}_{i_0})\}$. This with (59) yields (58). ■

8.6. Proof of Proposition (18)

(18) Proposition: Suppose that **Sym** is limited to the vocabulary of arithmetic (including $\overline{0}$ and a unary function symbol s) plus an additional constant. Also suppose that **Obs** $= \mathcal{L}_{basic}$. Then there is a problem **P** with the following properties.

(a) Every member of **P** is strongly elementary.

(b) **P** is computably solvable.

(c) some revision-based scientist solves **P**.

(d) For every $B \subseteq \mathcal{L}_{form}$ and revision function $\dotplus$, if $\lambda\sigma \, . \, B \dotplus \sigma$ is computable, then it fails to solve **P**.

Proof: In view of Proposition (48), clauses (a) and (b) of the present proposition entail and (c). So we prove parts (a), (b), and (d).

Let a be the additional constant of **Sym**. For $n \in N$, let $\overline{n}$ be the result of n applications of s to $\overline{0}$. Let Q be the seven axioms of Robinson's arithmetic (see [5, Ch. 14]) and let $\boldsymbol{N}$ be the standard model of arithmetic. It is well known that every recursively enumerable subset of N is weakly representable in Q by a formula which defines it in $\boldsymbol{N}$. So, let $Y \subseteq N$ be a non-recursive, recursively enumerable set and let $\varphi(x) \in \mathcal{L}_{form}$ have one free variable x, exclude a, and be such that for all $n \in N$, both

(61) $Q \models \varphi(\overline{n})$ iff $n \in Y$,

and

(62) $\boldsymbol{N} \models \varphi(\overline{n})$ iff $n \in Y$.

For $n \in N$, if $n \in Y$ then set proposition P_n equal to $MOD(\bigwedge Q \wedge a = \overline{n})$; otherwise, if $n \notin Y$, set proposition P_n equal to $MOD(\bigwedge Q \wedge a = \overline{n} \wedge \neg\varphi(a))$. It follows immediately from (62) that for all $n \in N$, $P_n \neq \emptyset$.

Let $\mathbf{P} = \{P_n \,|\, n \in N\}$. The verification of (18)a is immediate. For (18)b, we describe a computable scientist Ψ that solves **P**. Let $\sigma \in SEQ$ be given. If for all $n \in N$, $\bigwedge \sigma \not\models a = \overline{n}$, then $\Psi(\sigma)$ is undefined. Otherwise, let $n \in N$ be least such that $\bigwedge \sigma \models a = \overline{n}$. Then $\Psi(\sigma)$ is an index for $\{\bigwedge Q \wedge a = \overline{n} \wedge \neg\varphi(a)\}$ if n does not appear in some standard enumeration of Y within $length(\sigma)$ steps; otherwise, $\Psi(\sigma)$ is an index for $\{\bigwedge Q \wedge a = \overline{n}\}$. It is easy to see that Ψ solves every member of $\mathbf{P}$.

For (18)d, we proceed as follows. Let T be the set of $\sigma \in SEQ$ such that $\boldsymbol{N}$ satisfies the set of a-free formulas contained in $content(\sigma)$. It is easy to verify that T is recursively enumerable. For a contradiction, suppose that $B \subseteq \mathcal{L}_{form}$ and revision function $\dot{+}$ are such that $\lambda\sigma\,.\,B \dot{+} \sigma$ is computable and solves $\mathbf{P}$. Let $n \in N$ be given with $n \in Y$, and suppose that the first member of nonvoid $\sigma \in T$ is $a = \overline{n}$. Then, by (61), $\bigwedge \sigma \models \bigwedge Q \rightarrow \varphi(a)$, so Definition (3)b implies that $B \dot{+} \sigma \not\models \bigwedge Q \wedge \neg\varphi(a)$. This shows:

(63) If $n \in Y$, then for all $\sigma \in T$ that begin with $a = \overline{n}$, $B \dot{+} \sigma \not\models \bigwedge Q \wedge \neg\varphi(a)$.

Now suppose that $n \notin Y$. Let $\mathcal{S} \in P_n$ be given such that $\boldsymbol{N}$ is the a-free reduct of $\mathcal{S}$ [such an $\mathcal{S}$ exists by (62)]. Let e be an environment for $\mathcal{S}$ that begins with $a = \overline{n}$. Then, since $\lambda\sigma\,.\,B \dot{+} \sigma$ solves P_n, there is $k \in N$ such that $B \dot{+} e[k] \models \bigwedge Q \wedge \neg\varphi(a)$. Moreover, $\boldsymbol{N}$ satisfies every a-free formula of $content(e[k])$. This shows:

(64) If $n \notin Y$, then there is $\sigma \in T$ that begins with $a = \overline{n}$ and is such that $B \dot{+} \sigma \models \bigwedge Q \wedge \neg\varphi(a)$.

However, in view of the recursive enumerability of T and the computability of $\lambda\sigma\,.\,B \dot{+} \sigma$, the conjunction of (63) and (64) yields a positive test for the complement of Y, which contradicts the hypothesis that Y is not recursive. ■

8.7. Proof of Proposition (20)

(20) Proposition: Suppose that **Sym** is limited to the binary predicate R. Suppose that **Obs** $= \mathcal{L}_{basic}$. Let $T = \{\exists x \forall y Rxy \leftrightarrow \neg\exists y \forall x Rxy\}$, and $\theta =$

$\exists x \forall y Rxy$. Then $(T, \{\theta, \neg\theta\})$ is solvable, but for all revision functions $\dotplus$ and all $B \subseteq \mathcal{L}_{sen}$ consistent with T, $\lambda\sigma \,.\, B \dotplus \sigma$ does not solve $(T, \{\theta, \neg\theta\})$.

Proof: Corollary (58) of essay #2 shows that $(T, \{\theta, \neg\theta\})$ is solvable. Choose revision function $\dotplus$, and let $B \subseteq \mathcal{L}_{sen}$ be consistent with T. Because $B \cup T$ is a consistent set of closed formulas, we may choose structure $\mathcal{S}$ and full assignment h such that $\mathcal{S} \models B \cup T[h]$. Suppose that $\mathcal{S} \models \neg\theta$ (the argument is parallel for the case $\mathcal{S} \models \theta$). We show that $\lambda\sigma \,.\, B \dotplus \sigma$ does not solve $MOD(T \cup \{\theta\})$. Let $\sigma \in SEQ$ be given, and suppose that $\mathcal{S} \models \bigwedge \sigma[h]$. Then there is an interpretation of B and $\bigwedge \sigma$ that does not satisfy θ. Along with Definition (3)a this proves:

(65) For all $\sigma \in SEQ$, if $\mathcal{S} \models \bigwedge \sigma[h]$, then $B \dotplus \sigma \not\models \theta$.

By the choice of T and the fact that $\mathcal{S} \models T \cup \{\neg\theta\}$, $\mathcal{S} \models \exists y \forall x Rxy$. Let $\mathcal{U}$ be such that $|\,\mathcal{U}\,| = |\,\mathcal{S}\,|$ and for all $x, y \in |\,\mathcal{U}\,|$, $(x, y) \in R^{\mathcal{U}}$ if and only if $(y, x) \in R^{\mathcal{S}}$. Then it can be seen that:

(66) (a) $\mathcal{U} \models T \cup \{\theta\}$.
(b) For all $\sigma \in SEQ$, $\mathcal{U} \models \bigwedge \sigma[h]$ if and only if $\mathcal{S} \models \bigwedge \sigma[h]$.

Let e be an environment for $\mathcal{U}$ and h. By (65) and (66)b, for all $k \in N$, $B \dotplus e[k] \not\models \theta$. So by (66)a, $\lambda\sigma \,.\, B \dotplus \sigma$ does not solve $MOD(T \cup \{\theta\})$ in e. ■

8.8. Proof of Theorem (21)

(21) THEOREM: Suppose that **Obs** is closed under negation. There exists $X \subseteq \mathcal{L}_{form}$ and definite maxichoice revision function $\dotplus$ such that for all consistent $T \subseteq \mathcal{L}_{sen}$, the following holds.

(a) $T \cup X$ is consistent.

(b) $\lambda\sigma \,.\, (T \cup X) \dot{+} \sigma$ solves every solvable problem of the form $(T, \{\theta_0 \ldots \theta_n\})$.

Proof: First we define X, then show it to satisfy (21)a. Next we define $\dot{+}$, then show it to satisfy (21)b.

8.8.1. Definition of X.

Let $\{(\varphi^1_n(\boldsymbol{x}), \varphi^2_n(\boldsymbol{x})) \mid n \in N\}$ enumerate all pairs of refutable formulas having all free variables present in the sequence of variables $\boldsymbol{x}$. For every $n \in N$ we fix an enumeration $\{\boldsymbol{v}^i_n \mid i \in N\}$ of all sequences of variables such that the following properties hold.

(67) (a) For all $i \in N$, $\boldsymbol{v}^i_n$ is of length equal to the length of $\boldsymbol{x}$ in $\varphi^1_n(\boldsymbol{x})$ (and hence of $\boldsymbol{x}$ in $\varphi^2_n(\boldsymbol{x})$).

(b) No variable is repeated in $\boldsymbol{v}^0_n$.

(c) For all $m \in N$, if $n \neq m$ then $\boldsymbol{v}^0_n$ and $\boldsymbol{v}^0_m$ share no variables.

For all $n, i \in N$, we use the following abbreviations.

H_n is: $\exists \boldsymbol{x} \varphi^1_n(\boldsymbol{x}) \leftrightarrow \neg \exists \boldsymbol{x} \varphi^2_n(\boldsymbol{x})$.

F^i_n is: $\bigvee_{r \leq i} [\, \varphi^1_n(\boldsymbol{v}^r_n) \vee \varphi^2_n(\boldsymbol{v}^r_n)\,]$.

Now we set $X = \{H_n \rightarrow F^i_n \mid n, i \in N\}$.

8.8.2. Proof of (21)a.

Let consistent $T \subseteq \mathcal{L}_{sen}$ be given. Let structure $\mathcal{S}$ be such that $\mathcal{S} \models T$. We exhibit an assignment h to $\mathcal{S}$ such that $\mathcal{S} \models X[h]$. (It is not assumed that h will end up being onto

$|\mathcal{S}|$.) For all $n \in N$ such that $\mathcal{S} \models H_n$, if $p = \mathit{length}(\boldsymbol{v}_n^0)$ and if variables $x_1 \ldots x_p$ are such that $\boldsymbol{v}_n^0 = (x_1 \ldots x_p)$, then choose $s_1 \ldots s_p \in |\mathcal{S}|$ with $\mathcal{S} \models \varphi_n^1 \vee \varphi_n^2[s_1/x_1 \ldots s_p/x_p]$, and set $h(x_i) = s_i$ for all i, $0 < i \leq p$. This condition on h can be satisfied because of (67). For all variables x such that $h(x)$ is not defined by the foregoing, $h(x)$ may be selected arbitrarily. Hence for all $n \in N$, if $\mathcal{S} \models H_n$ then $\mathcal{S} \models F_n^0[h]$. Since for all $n, i \in N$, $F_n^0 \models F_n^i$, this proves that $\mathcal{S} \models X[h]$.

8.8.2.1. Definition of $\dot{+}$. Fix an enumeration $\{\theta_i \,|\, i > 0\}$ of $\mathcal{L}_{sen}$, an enumeration $\{\varphi_i \,|\, i \in N\}$ of X, and an enumeration $\{\psi_i \,|\, i \in N\}$ of $\mathcal{L}_{form}$. For all $i > 0$, set $S_i = \{\theta_i\}$, and for all $i \in N$, set $S_{\omega+i} = \{\varphi_i\}$, and $S_{(\omega\times 2)+i} = \{\psi_i\}$. Set $\mathbf{S} = \{S_\alpha \,|\, 0 < \alpha < \omega \times 3\}$, and let $\dot{+}_{\mathbf{S}}$ be constructed from $\mathbf{S}$ as in Definition (42). So $\dot{+}_{\mathbf{S}}$ is definite and maxichoice by Proposition (43). As an immediate consequence we have the following.

(68) Let $T \subseteq \mathcal{L}_{sen}$ and $\sigma \in \mathit{SEQ}$ be such that $T \cup \mathit{content}(\sigma)$ is consistent. Then $T \subseteq (T \cup X) \dot{+}_{\mathbf{S}} \sigma$.

8.8.2.2. Proof of (21)b. Let solvable problem of form $(T, \{\theta_0 \ldots \theta_n\})$ be given. Let $m \leq n$ also be given. By Corollary (61) of essay #2, there is $n \in N$ such that $T \models \theta_m \leftrightarrow \exists \boldsymbol{x} \varphi_n^1(\boldsymbol{x})$ and $T \models \neg\theta_m \leftrightarrow \exists \boldsymbol{x} \varphi_n^2(\boldsymbol{x})$. Hence $T \models H_n$. Let $\mathcal{S} \in \mathit{MOD}(T \cup \{\theta_m\})$ and full assignment h to $\mathcal{S}$ be given. (The case $\mathcal{S} \in \mathit{MOD}(T \cup \{\neg\theta_m\})$ is parallel.) Then we may choose the least $i \in N$ such that $\mathcal{S} \models \varphi_n^1(\boldsymbol{v}_n^i)[h]$. So $\mathcal{S} \models H_n \rightarrow F_n^i[h]$. Let e be an environment for $\mathcal{S}$ and h. In view of Definition (3)b (which ensures consistency), to complete the proof it suffices to show:

(69) (a) for cofinitely many k, $(T \cup X) \dot{+}_{\mathbf{S}} e[k] \models H_n$;

(b) for cofinitely many k, $(T \cup X) \dot{+}_{\mathbf{S}} e[k] \models H_n \rightarrow F_n^i$;

(c) for cofinitely many k, $(T \cup X) \dot{+}_{\mathbf{S}} e[k] \models \neg F_n^{i-1} \wedge \neg \varphi_n^2(\boldsymbol{v}_n^i)$.

Claim (69)c follows immediately from the choice of i and the choice of e. Claim (69)a follows immediately from (68), the choice of e, and the fact that $T \models H_n$. So it remains to demonstrate (69)b.

Let $j \in N$ be least such that $\varphi_j = H_n \rightarrow F_n^i$. Let X_0 be the set of all φ_p, $p < j$ such that $\mathcal{S} \models \varphi_p[h]$. By our choice of e, and the fact that $\mathcal{S} \models T$ and $\mathcal{S} \models H_n \rightarrow F_n^i[h]$, we have:

(70) For all $k > 0$, $T \cup X_0 \cup \{H_n \rightarrow F_n^i\} \not\models \neg \bigwedge e[k]$.

Since the pairs $\{(\varphi_n^1(\boldsymbol{x}), \varphi_n^2(\boldsymbol{x})) \mid n \in N\}$ are refutable [see Definition (30) of essay #2]. Lemma (91)b of essay #2]. implies that for all i, n, F_n^i is refutable. Since h is a full assignment to $\mathcal{S}$ we may choose $k_0 > 0$ such that:

(71) For all $p < j$, if $\varphi_p = H_{n'} \rightarrow F_{n'}^{i'}$ then the following holds: $\mathcal{S} \models F_{n'}^{i'}[h]$ iff $F_{n'}^{i'} \not\models \neg \bigwedge e[k_0]$.

Let $k \geq k_0$ be given. Consider $Y^{\omega+j}$ in the construction of $(T \cup X) \dot{+}_{\mathbf{S}} e[k]$ via Definition (42). By (68), (71), and the choice of k, $Y^\beta \subseteq T \cup X_0$ for all nonnull $\beta < \omega + j$. So by (70), $Y^\beta \cup \{H_n \rightarrow F_n^i\} \not\models \neg \bigwedge e[k]$ for all nonnull $\beta < \omega + j$. Hence, $H_n \rightarrow F_n^i \in Y^{\omega+j} \subseteq (T \cup X) \dot{+}_{\mathbf{S}} \neg \bigwedge e[k]$, verifying (69)b. ■

8.9. Proof of Theorem (22)

(22) THEOREM: Suppose that solvable problem **P** is of form $(T, \{P_0, P_1, \ldots\})$, and that **Obs** is closed under negation. Then there is a consistent extension $X \subseteq \mathcal{L}_{form}$ of T such that for every revision function $\dot{+}$, $\lambda\sigma \,.\, X \dot{+} \sigma$ solves **P**.

Proof: By Proposition (40) of the second essay, and our hypothesis on **Obs**, for all $j \in N$ let $\boldsymbol{t}_j$ be a tip-off for P_j in $\mathbf{P}$ that satisfies the condition stated in Lemma (33) of essay #2, By the countability of tip-offs, let $\bigcup_{j \in N} \boldsymbol{t}_j$ be enumerated as $\{\pi_i \,|\, i \in N\}$. Without loss of generality we may assume that each π_i is consistent with T. For all $i \in N$ fix an enumeration $\{\varphi_i^n \,|\, n \in N\}$ of the refutable formulas in π_i. Then set:

$$X = T \cup \{(\varphi_0^0 \wedge \ldots \wedge \varphi_0^{n_0}) \vee \ldots \vee (\varphi_i^0 \wedge \ldots \wedge \varphi_i^{n_i}) \,|\, i, n_0 \ldots n_i \in N\}.$$

We show that X satisfies the claim of the theorem.

Note that $T \cup \{\varphi_0^n \,|\, n \in N\}$ is consistent by hypothesis. This implies that X is consistent.

Let revision function $\dotplus$, $j \in N$, $\mathcal{S} \in P_j$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given. To finish the proof we must show that:

(72) for cofinitely k, $\emptyset \neq MOD(X \dotplus e[k]) \subseteq P_j$.

By Definition (32) of essay #2, let i_0 be least with $\mathcal{S} \models \pi_{i_0}[h]$. For $i < i_0$, $\mathcal{S} \not\models \pi_i[h]$, so for each $i < i_0$ we may choose $c(i) \in N$ such that:

(73) (a) $\mathcal{S} \models (\varphi_i^0 \wedge \ldots \wedge \varphi_i^{c(i)-1})[h]$;

(b) $\mathcal{S} \not\models \varphi_i^{c(i)}[h]$.

Since for all $i < i_0$, $\varphi_i^{c(i)}$ is refutable, and since h is onto $|\,\mathcal{S}\,|$, it follows from (73)b that there is $k_0 > 0$ such that:

(74) for all $k \geq k_0$ and for all $i < i_0$, $content(e[k]) \cup \{\varphi_i^{c(i)}\}$ is inconsistent.

Let $Y = \{(\varphi_0^0 \wedge \ldots \wedge \varphi_0^{n_0}) \vee \ldots \vee (\varphi_i^0 \wedge \ldots \wedge \varphi_i^{n_i}) \,|\, i < i_0,\ n_0 \geq c(0) \ldots n_i \geq c(i)\}$. We deduce from (74) that:

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(75) for all $k \geq k_0$ and for all $\varphi \in Y$, $content(e[k]) \cup \{\varphi\}$ is inconsistent.

We deduce from (73)a and the fact that $\mathcal{S} \models \pi_{i_0}[h]$ that:

(76) for all $k \in N$, $content(e[k]) \cup (X - Y)$ is consistent.

Then (75), (76), and Lemma (57) imply that:

(77) for all $k \geq k_0$, $X \dot{+} e[k] = content(e[k]) \cup (X - Y)$.

Now note that $Z = \{(\varphi_0^0 \wedge \ldots \wedge \varphi_0^{c(0)}) \vee \ldots \vee (\varphi_{i_0-1}^0 \wedge \ldots \wedge \varphi_{i_0-1}^{c(i_0-1)}) \vee (\varphi_{i_0}^0 \wedge \ldots \wedge \varphi_{i_0}^n) \,|\, n \in N\} \subseteq X - Y$. Moreover, (74) implies that $content(e[k]) \cup Z \models \{\varphi_{i_0}^n \,|\, n \in N\}$ for all $k \geq k_0$. This with (77) implies that $X \dot{+} e[k] \models T \cup \{\varphi_{i_0}^n \,|\, n \in N\}$ for all $k \geq k_0$. Hence $X \dot{+} e[k] \models T \cup \pi_{i_0}$ for all $k \geq k_0$. This with Definition (32) of essay #2, and the hypothesis that $\mathcal{S} \models \pi_{i_0}[h]$ implies (72). ■

8.10. Proof of Proposition (23)

(23) Proposition: Suppose that **Obs** is closed under negation. Let $T \subseteq \mathcal{L}_{sen}$ be consistent and finitely axiomatizable. Then there is consistent $X \subseteq \mathcal{L}_{form}$ with $X \models T$ such that for every revision function $\dot{+}$, $\lambda\sigma\,.\,X \dot{+} \sigma$ solves every solvable problem of form $(T, \{\theta_0 \ldots \theta_n\})$.

We need:

(78) Lemma: Let $B \subseteq \mathcal{L}_{form}$, $\sigma \in SEQ$ and revision function $\dot{+}$ be given. Suppose that for all $\psi, \chi \in B$, either $\psi \models \chi$, or $\chi \models \psi$. Show that $B \dot{+} \sigma = (B - Y) \cup content(\sigma)$, where $Y = \{\psi \in B \,|\, \psi \not\models \neg \bigwedge \sigma\}$.

Proof: Trivially, $Inn(B, \sigma) \cap Y = \emptyset$. So it suffices to show that $B - Y \not\models \neg \bigwedge \sigma$. Suppose otherwise. By compactess there is finite $D \subseteq B - Y$ such that $D \models \neg \bigwedge \sigma$. By hypothesis there is $\psi \in D$ such that for all $\chi \in D$, $\psi \models \chi$. Hence $\psi \models \neg \bigwedge \sigma$, so $\psi \in Y$, contradiction. ■

Proof of Proposition (23): First we define X, using some notation. Fix an enumeration $\{\varphi_m \,|\, m \in N\}$ of all refutable formulas. Let $(n_1 \ldots n_k) \in N^k$ be a strictly increasing sequence of k numbers. By induction, we associate a formula χ_{n_i} with each n_i appearing in $(n_1 \ldots n_k)$. It will turn out that χ_{n_i} is a disjunction over $\{\varphi_m \,|\, m \in N\}$, and that φ_{n_i} is the disjunct of highest index in χ_{n_i}.

Basis step: χ_{n_1} is $\varphi_0 \vee \ldots \vee \varphi_{n_1}$. Observe that χ_{n_1} is a disjunction over $\{\varphi_m \,|\, m \in N\}$, and that φ_{n_1} is the disjunct of highest index in χ_{n_1}.

Induction step: Let $0 < i < k$ be given. Then $n_{i+1} > n_i$. Suppose that χ_{n_i} has been defined to be a disjunction over $\{\varphi_m \,|\, m \in N\}$ such that φ_{n_i} is the disjunct of highest index in χ_{n_i}. Let χ' be the result of suppressing φ_{n_i} in χ_{n_i}. Then $\chi_{n_{i+1}}$ is $\chi' \vee \varphi_{n_i+1} \vee \ldots \vee \varphi_{n_{i+1}}$. Observe that $\chi_{n_{i+1}}$ is a disjunction over $\{\varphi_m \,|\, m \in N\}$, and that $\varphi_{n_{i+1}}$ is the disjunct of highest index in $\chi_{n_{i+1}}$.

Finally, we associate the formula $\chi_{n_1} \wedge \ldots \wedge \chi_{n_k}$ with $(n_1 \ldots n_k)$. For example, the formula associated with $(2, 5, 7)$ is:

$$(\varphi_0 \vee \varphi_1 \vee \varphi_2) \wedge (\varphi_0 \vee \varphi_1 \vee \varphi_3 \vee \varphi_4 \vee \varphi_5) \wedge (\varphi_0 \vee \varphi_1 \vee \varphi_3 \vee \varphi_4 \vee \varphi_6 \vee \varphi_7).$$

Let $t \in \mathcal{L}_{sen}$ axiomatize T. Then we define X to be the set of all consistent formulas of form $t \wedge \chi$, where χ is any formula associated with some nonempty, increasing sequence of natural numbers.

The key to our construction is the following fact.

(79) FACT: For every $\delta, \delta' \in X$, $\delta \models \delta'$ or $\delta' \models \delta$.

Proof of Fact (79): Let γ and γ' be distinct, nonempty, finite, increasing sequences of numbers. Then there is $\lambda \in N^{<\omega}$ and $n \in N$ such that exactly one of the following holds:

(a) $\lambda{*}n$ is an initial segment of γ and for all $m \le n$, $\lambda{*}m$ is not an initial segment of γ';

(b) $\lambda{*}n$ is an initial segment of γ' and for all $m \le n$, $\lambda{*}m$ is not an initial segment of γ.

Suppose λ and n satisfy (a); the other case is exactly parallel. Let formula χ be associated with γ, and formula χ' be associated with γ'. Then it can be seen that the first $\mathit{length}(\lambda)$ conjuncts in χ are the same as the first $\mathit{length}(\lambda)$ conjuncts in χ', and that all of the disjuncts appearing in the $\mathit{length}(\lambda) + 1$st conjunct of χ appear as disjuncts in each of the conjuncts of χ' that come after the first $\mathit{length}(\lambda)$ ones. So $\chi \models \chi'$. Hence $t \wedge \chi \models t \wedge \chi'$, and all $\delta \in X$ are of the form $t \wedge \chi$. ▮

We now show that X satisfies the claim of the proposition. Since every member of X is consistent, it follows from Fact (79) that X is consistent. Let there be given solvable problem of form $(T, \{\theta_0 \ldots \theta_n\})$, revision function $\dotplus$, and $i \le n$. To finish the proof we show that $\lambda\sigma\,.\,X \dotplus \sigma$ solves $MOD(T \cup \{\theta_i\})$. Let $\mathcal{S} \in MOD(T \cup \{\theta_i\})$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given. By our hypothesis on **Obs** and Corollary (61) of essay #2, there is $\theta \in \mathcal{L}_{sen}$ such that θ is the existential closure of a refutable formula and:

(80) $T \models \theta_i \leftrightarrow \theta$.

Since h is onto $|\,\mathcal{S}\,|$, this implies the existence of $m_0 \in N$ such that the existential closure of φ_{m_0} is θ, and $\mathcal{S} \models \varphi_{m_0}[h]$. From Lemma (78) and Fact (79) we deduce that:

(81) for all $k \in N$, $X \dotplus e[k]$ includes the set of all formulas in X that are consistent with $\bigwedge e[k]$.

Denote by γ the increasing sequence of integers that ends with m_0 and such that for all $m \leq m_0$, m occurs in γ if and only if $\mathcal{S} \models \varphi_m[h]$. Denote by χ the formula associated with γ. It is immediate that $t \wedge \chi \in X$, and that for all $k \in N$, $t \wedge \chi$ is consistent with $\bigwedge e[k]$. We can thus deduce from (81) that:

(82) for all $k \in N$, $t \wedge \chi$ belongs to $X \dotplus e[k]$.

Since h is onto $|\,\mathcal{S}\,|$, there is $k_0 \in N$ such that for all $k \geq k_0$ and for all $m \leq m_0$, if $m \notin \mathit{content}(\gamma)$ then $\bigwedge e[k] \models \neg\varphi_m$. From the definition of χ this can be seen to imply that for all $k \geq k_0$ and for all $m \in \mathit{content}(\gamma)$, $\mathit{content}(e[k]) \cup \{\chi\} \models \varphi_m$. In particular:

(83) for all $k \geq k_0$, $\mathit{content}(e[k]) \cup \{\chi\} \models \varphi_{m_0}$.

From (80), (82), (83), and the definition of φ_{m_0} we deduce that for all $k \geq k_0$, $X \dotplus e[k] \models T \cup \{\theta_i\}$. ■

8.11. Proof of Proposition (24)

(24) Proposition: Suppose that **Sym** is limited to a binary predicate, a constant, and a unary function symbol. Suppose that **Obs** $= \mathcal{L}_{basic}$. Then there exists $T \subseteq \mathcal{L}_{sen}$ and definite maxichoice revision function $\dotplus$ such that for all $B \subseteq \mathcal{L}_{form}$, $\lambda\sigma\,.\,B \dotplus \sigma$ fails to solve some solvable problem of form $(T, \{\theta, \neg\theta\})$.

Proof: Let R be the binary predicate, $\overline{0}$ the constant, s the unary function symbol of **Sym**. For $n \in N$, let $\overline{n}$ be the result of n applications of s to $\overline{0}$.

Let $\{\chi_i \,|\, i > 0\}$ enumerate $\mathcal{L}_{form}$, and for all $i > 0$ set $S_i = \{\chi_i\}$. Let $\mathbf{S} = \{S_i \,|\, 0 < i < \omega\}$, and let $\dot{+}_{\mathbf{S}}$ be constructed from $\mathbf{S}$ as in Definition (42). So $\dot{+}_{\mathbf{S}}$ is definite and maxichoice by Proposition (43). Fix a nonrecursive subset E of N, and define T to be:

$$\begin{array}{l} \{\, (\forall y R\overline{2n}y) \rightarrow \overline{2n} = \overline{2m+1} \,|\, n \in N \text{ and } m \in E \,\} \quad \cup \\ \{\, (\forall y R\overline{2n}y) \rightarrow \overline{2n} \neq \overline{2m+1} \,|\, n \in N \text{ and } m \notin E \,\} \end{array}$$

Let $B \subseteq \mathcal{L}_{form}$ be given. By our assumption on **Obs** and Corollary (58) of essay #2, for every universal sentence θ, $(T, \{\theta, \neg\theta\})$ is solvable. So there would be nothing left to prove without assuming:

(84) For every universal sentence θ, $\lambda\sigma \,.\, B \dot{+}_{\mathbf{S}} \sigma$ solves $(T, \{\theta, \neg\theta\})$.

We use (84) to derive a contradiction. Specifically, an environment e for T will be exhibited such that for infinitely many k, there exists $n, m \in N$ with:

$$(85) \qquad \begin{array}{l} B \dot{+}_{\mathbf{S}} e[k] \models (\forall y R\overline{2n}y) \wedge \overline{2n} \neq \overline{2m+1} \text{ if } m \in E, \\ B \dot{+}_{\mathbf{S}} e[k] \models (\forall y R\overline{2n}y) \wedge \overline{2n} = \overline{2m+1} \text{ if } m \notin E. \end{array}$$

This will imply that for infinitely many k, $B \dot{+}_{\mathbf{S}} e[k] \not\models T$. So, trivially, $\lambda\sigma.B \dot{+}_{\mathbf{S}} \sigma$ does not solve $(T, \{\forall x(x = x), \neg\forall x(x = x)\})$, contradicting (84).

Fix an enumeration $\{\alpha_i \,|\, i \in N\}$ of all atomic formulas. To exhibit the promised environment e we will build by induction on $i \in N$ a sequence $\{\tau_i \,|\, i \in N\}$ of members of SEQ and a sequence $\{n_i \,|\, i \in N\}$ such that for all $i \in N$:

(86) (a) for all $j < i$, $\tau_j \subseteq \tau_i$;

(b) exactly one of α_i or $\neg\alpha_i$ occurs in τ_i;

(c) $\bigwedge \tau_i \models \neg\forall y R\overline{2i}y$;

(d) $B \dot{+}_{\mathbf{S}} \tau_i \models \forall y R\overline{2n_i}y$;

(e) for some $m \in E$, $B \dot{+}_{\mathbf{S}} \tau_i \models \overline{2n_i} \neq \overline{2m+1}$, or for some $m \notin E$, $B \dot{+}_{\mathbf{S}} \tau_i \models \overline{2n_i} = \overline{2m+1}$.

Then we will set $e = \bigcup_{i \in N} \tau_i$. It follows from (86)a,b that e is an environment, which by (86)c is for T. It follows from (86)d,e that (85) is satisfied. Let $i \in N$ be given. Suppose that τ_j and n_j have been defined for all $j < i$ and satisfy (86) for $i = j$. We define τ_i and n_i that satisfy (86). Choose $n_i \neq i$ such that $\overline{2n_i}$ does not appear in τ_j for any $j < i$. We may choose an environment d for $T \cup \{\neg\forall y R\overline{2i}y, \forall y R\overline{2n_i}y\}$ that extends τ_j for all $j < i$. By (84) we deduce the existence of $k_0 \in N$ with:

(87) (a) for all $j < i$, $d[k_0]$ extends τ_j;

(b) exactly one of α_i, $\neg\alpha_i$ occurs in $d[k_0]$;

(c) $\bigwedge d[k_0] \models \neg\forall y R\overline{2i}y$;

(d) $B \dot{+}_{\mathbf{S}} d[k_0] \models \forall y R\overline{2n_i}y$.

By (87)d and compactness, let finite $D \subseteq (B \dot{+}_{\mathbf{S}} d[k_0]) \cap B$ be such that $D \cup \{\bigwedge d[k_0]\} \models \forall y R\overline{2n_i}y$. Denote by p the greatest integer such that $\chi_p \in D$, and set $D' = \{\chi_0 \ldots \chi_p\} \cap (B \dot{+}_{\mathbf{S}} d[k_0]) \cap B$. So:

(88) $D \subseteq D' \subseteq B \dot{+}_{\mathbf{S}} d[k_0]$.

Since E is non recursive there is $m \in E$ with $D' \cup \{\bigwedge d[k_0]\} \not\models \overline{2n_i} = \overline{2m+1}$, or there is $m \notin E$ with $D' \cup \{\bigwedge d[k_0]\} \not\models \overline{2n_i} \neq \overline{2m+1}$. Choose such an m, and set $\tau_i = d[k_0]*(\overline{2n_i} \neq \overline{2m+1})$ if $m \in E$, $\tau_i = d[k_0]*(\overline{2n_i} = \overline{2m+1})$ if $m \notin E$. From (87)a-c we infer (86)a-c immediately. By (88), the definition of $\dot{+}_{\mathbf{S}}$ and the choice of m we obtain: $D' \subseteq (B \dot{+}_{\mathbf{S}} \tau_i) \cap B$, hence $B \dot{+}_{\mathbf{S}} \tau_i \models \forall y R \overline{2n_i} y$, verifying (86)d. By Lemma (4), $B \dot{+}_{\mathbf{S}} \tau_i \models (\overline{2n_i} \neq \overline{2m+1})$ if $m \in E$, and $B \dot{+}_{\mathbf{S}} \tau_i \models (\overline{2n_i} = \overline{2m+1})$ if $m \notin E$, verifying (86)e. ■

8.12. Proof of Theorem (25)

It will be convenient to prove something a little stronger, namely:

> (25) STRENGTHENED THEOREM: Suppose that **Obs** is closed under negation. For every solvable problem $\mathbf{P}$ of the form $(T, \{P_0, P_1, \ldots\})$ there is a consistent extension $X \subseteq \mathcal{L}_{form}$ of T and a definite maxichoice revision function $\dot{+}$ such that $\lambda\sigma \,.\, X \dot{+} \sigma$ solves $\mathbf{P}$ *strongly* efficiently.[17]

Proof: Let solvable problem $\mathbf{P} = (T, \{P_0, P_1, \ldots\})$ be given. Recall from the proof of Theorem (22) the enumeration $\{\pi_i \,|\, i \in N\}$ and the enumeration $\{\varphi_i^n \,|\, n \in N\}$ of π_i. Recall also the definition of $X \subseteq \mathcal{L}_{form}$. For all $i \in N$, set $S_{i+1} = T \cup \{(\varphi_0^0 \wedge \ldots \wedge \varphi_0^{n_0}) \vee \ldots \vee (\varphi_i^0 \wedge \ldots \wedge \varphi_i^{n_i}) \,|\, n_0 \ldots n_i \in N\}$. Fix an enumeration $\{\psi_i \,|\, i \in N\}$ of $\mathcal{L}_{form}$, and set $S_{\omega+i} = \{\psi_i\}$ for all $i \in N$. Let $\mathbf{S} = \{S_\alpha \,|\, 0 < \alpha < \omega \times 2\}$, and let $\dot{+}$ be constructed from $\mathbf{S}$ as in Definition (42). So, $\dot{+}$ is definite maxichoice by Proposition (43). The proof of Theorem (22) shows that $\lambda\sigma \,.\, X \dot{+} \sigma$ solves $\mathbf{P}$. To complete the proof of the theorem we show that $\lambda\sigma \,.\, X \dot{+} \sigma$ solves $\mathbf{P}$ efficiently.

Let $\sigma \in SEQ$ be for $\mathbf{P}$. By Definition (32) of essay #2, there is least $i_0 \in N$ such that $T \cup \pi_{i_0} \cup content(\sigma)$ is consistent. Let $P \in \mathbf{P}$, $\mathcal{S} \in P$, and full assignment h to $\mathcal{S}$ be such

[17] Strong efficiency was introduced in Definition (78) of essay #2.

that $\mathcal{S} \models \pi_{i_0} \cup \mathit{content}(\sigma)[h]$. Let $\tau \in \mathit{SEQ}$ extend σ and be such that $\mathcal{S} \models \bigwedge \tau[h]$. As an immediate consequence of our choice of $\dotplus$, we have:

$$S_{i_0+1} \subseteq X \dotplus \tau.$$

Moreover by compactness, for all $i < i_0$ we may choose $c(i) \in N$ such that $T \cup \mathit{content}(\sigma) \cup \{\varphi_i^0 \wedge \ldots \wedge \varphi_i^{c(i)}\}$ is inconsistent. Since $\{(\varphi_0^0 \wedge \ldots \wedge \varphi_0^{c(0)}) \vee \ldots \vee (\varphi_{i_0-1}^0 \wedge \ldots \wedge \varphi_{i_0-1}^{c(i_0-1)}) \vee (\varphi_{i_0}^0 \wedge \ldots \wedge \varphi_{i_0}^n) \,|\, n \in N\} \subseteq S_{i_0+1}$, this implies that $S_{i_0} \cup \mathit{content}(\sigma) \models \{\varphi_{i_0}^0 \wedge \ldots \wedge \varphi_{i_0}^n \,|\, n \in N\}$. With $S_{i_0} \subseteq X \dotplus \tau$ we deduce that $X \dotplus \tau \models T \cup \pi_{i_0}$ which implies via Definition (32) of essay #2 (and for the case $\sigma = 0$, the fact that X is consistent) that $\emptyset \neq \mathit{MOD}(X \dotplus \tau) \subseteq P$. It follows immediately from Lemma (108) of essay #2 that $\lambda\sigma\,.\,X \dotplus \sigma$ solves $\mathbf{P}$ strongly efficiently. ■

8.13. Proof of Lemma (27)

> (27) Lemma: For all $T \subseteq \mathcal{L}_{sen}$, there exists a T-preserving definite maxichoice revision function.

Proof: Let $T \subseteq \mathcal{L}_{sen}$ be given. Fix an enumeration $\{\varphi_i \,|\, i > 0\}$ of T and an enumeration $\{\psi_i \,|\, i \in N\}$ of $\mathcal{L}_{form} - T$. For all $i > 0$, set $S_i = \{\varphi_i\}$. For all $i \in N$, set $S_{\omega+i} = \{\psi_i\}$. Let $\mathbf{S} = \{S_\alpha \,|\, 0 < \alpha < \omega \times 2\}$, and let $\dotplus$ be constructed from $\mathbf{S}$ as in Definition (42). So, $\dotplus$ is definite maxichoice by Proposition (43). For all $\sigma \in \mathit{SEQ}$ and $X \subseteq \mathcal{L}_{form}$, if $T \cup \mathit{content}(\sigma)$ is consistent then $\mathit{Inn}(T \cup X, \sigma) \cup T \cup \mathit{content}(\sigma)$ is consistent. This together with the definition of $\dotplus$ implies immediately that for all $\sigma \in \mathit{SEQ}$, if $T \cup \mathit{content}(\sigma)$ is consistent then $\mathit{Inn}(T \cup X, \sigma) \cup \mathit{content}(\sigma) \cup T \subseteq T \dotplus \sigma$. Hence $\dotplus$ is T-preserving. ■

8.14. Proof of Theorem (28)

Actually, we'll prove something a little stronger, namely:

> (28) Strengthened Theorem: Suppose that **Obs** is closed under negation. For every solvable problem **P** of the form $(T, \{\theta_0 \dots \theta_n\})$ there is a consistent extension $X \subseteq \mathcal{L}_{form}$ of T with the following properties.
>
> (a) For all revision functions $\dot{+}$, $\lambda\sigma \,.\, X \dot{+} \sigma$ solves **P**.
>
> (b) For all T-preserving revision functions $\dot{+}$, $\lambda\sigma \,.\, X \dot{+} \sigma$ solves **P** *strongly* efficiently.[18]

Proof: Let solvable problem **P** of the form $(T, \{\theta_0 \dots \theta_n\})$ be given. By Corollary (61) of essay #2, let refutable formula $\varphi_0(\bar{x}_0)$ (with free variables $\bar{x}_0$) be such that $T \models \exists \bar{x}_0 \varphi_0(\bar{x}_0) \rightarrow \theta_0$ and $T \cup \{\varphi_0(\bar{x}_0)\}$ is consistent. Let $\{\varphi_i(\bar{x}_i) \,|\, i \in N\}$ be an enumeration of all refutable formulas $\varphi(\bar{x})$ such that for some $j \leq n$, $T \models \exists \bar{x} \varphi(\bar{x}) \rightarrow \theta_j$. Note that the enumeration starts off with the formula $\varphi_0(\bar{x}_0)$ chosen to be consistent with T. For all $i \in N$, denote by ψ_i the formula $\varphi_0(\bar{x}_0) \vee \dots \vee \varphi_i(\bar{x}_i)$. Set $X = T \cup \{\psi_i \,|\, i \in N\}$. Since $T \cup \{\varphi_0(\bar{x}_0)\}$ is consistent, so is X. We show that X satisfies clauses (a) and (b) of the theorem.

We prove part (a). Let revision function $\dot{+}$, $j \leq n$, $\mathcal{S} \in MOD(T \cup \{\theta_j\})$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given. We must show that:

(89) for cofinitely k, $X \dot{+} e[k] \models T \cup \{\theta_j\}$.

By Corollary (61) of essay #2, let i_0 be least with $\mathcal{S} \models \varphi_{i_0}(\bar{x}_{i_0})[h]$. So, because the θ_i's are mutually exclusive in T, we have:

[18] Strong efficiency was introduced in Definition (78) of essay #2.

(90) $T \models \varphi_{i_0}(\bar{x}_{i_0}) \rightarrow \theta_j$.

By the choice of i_0, $\mathcal{S} \models T \cup \{\psi_{i_0}\}[h]$, and for all $i < i_0$, $\mathcal{S} \models \neg\varphi_i(\bar{x}_i)[h]$. Since the φ_i's are refutable formulas, and since h is onto $|\ \mathcal{S}\ |$, it follows that there is $k_0 \in N$ such that for all $k \geq k_0$:

(91) (a) for all $i < i_0$, $\psi_i \models \neg \bigwedge e[k]$, and
(b) $T \cup \{\psi_i \,|\, i \geq i_0\} \not\models \neg \bigwedge e[k]$.

With Exercise (57), this implies that for all $k \geq k_0$, $X \dotplus e[k] = T \cup \{\psi_i \,|\, i \geq i_0\} \cup \mathit{content}(e[k])$. Hence for all $k \geq k_0$, $T \cup \{\psi_{i_0}\} \cup \mathit{content}(e[k]) \subseteq X \dotplus e[k]$. With (91)a, we infer that for all $k \geq k_0$, $X \dotplus e[k] \models T \cup \{\varphi_{i_0}(\bar{x}_{i_0})\}$. This with (90) yields (89).

We prove part (b). Let T-preserving revision function $\dotplus$ be given. Let $\sigma \in \mathit{SEQ}$ be for $(T, \{\theta_0 \ldots \theta_n\})$. Let $i_0 \in N$ be least such that:

(92) $T \cup \{\psi_{i_0}\} \not\models \neg \bigwedge \sigma$.

By the definition of the φ_i's, there is $j \leq n$ such that:

(93) $T \cup \{\varphi_{i_0}\} \models \theta_j$.

By (92), let structure $\mathcal{S}$ and full assignment h to $\mathcal{S}$ be such that $\mathcal{S} \models \bigwedge \sigma[h]$ and:

(94) $\mathcal{S} \models \psi_{i_0}[h]$.

Let environment e for $\mathcal{S}$ and h extend σ, and let $k \geq \mathit{length}(\sigma)$ be given. The definition of i_0, the fact that e is an environment for $\mathcal{S}$ and h, and (94) then imply that:

(95) (a) for all $i < i_0$, $T \cup \{\psi_i\} \models \neg \bigwedge e[k]$, and
(b) $T \cup \{\psi_i \mid i \geq i_0\} \not\models \neg \bigwedge e[k]$.

Since $\dotplus$ is T-preserving, $T \subseteq X \dotplus e[k]$. Together with (95), this implies that $X \dotplus e[k] = T \cup \{\psi_i \mid i \geq i_0\}$. Hence $\psi_{i_0} \in X \dotplus e[k]$. With the definition of i_0, it follows that $X \dotplus e[k] \models T \cup \{\varphi_{i_0}\}$. With (93), we conclude that $X \dotplus e[k] \models T \cup \{\theta_j\}$. So we have shown that for all $k \geq \mathit{length}(\sigma)$, $\emptyset \neq \mathit{MOD}(X \dotplus e[k]) \subseteq \mathit{MOD}(T \cup \{\theta_j\})$. We conclude with Lemma (108) of essay #2. ■

8.15. Proof of Corollary (29)

Once again, we prove something slightly stronger.

> (29) STRENGTHENED COROLLARY: Suppose that **Obs** is closed under negation. Let $T \subseteq \mathcal{L}_{sen}$ be consistent and finitely axiomatizable. Let solvable problem of form $(T, \{\theta_0 \ldots \theta_n\})$ be given. Then there is consistent extension $X \subseteq \mathcal{L}_{form}$ of T such that for every revision function $\dotplus$, $\lambda\sigma\,.\,X \dotplus \sigma$ strongly efficiently solves $(T, \{\theta_0 \ldots \theta_n\})$.[19]

Proof: Recall the enumeration $\{\psi_i \mid i \in N\}$ defined from $(T, \{\theta_0 \ldots \theta_n\})$ in Theorem (28). Let $t \in \mathcal{L}_{sen}$ axiomatize T. Set $X = T \cup \{t \wedge \psi_i \mid i \in N\}$. Let revision function $\dotplus$ be given. It is easy to verify that for all $\sigma \in \mathit{SEQ}$, if σ is consistent with T then $T \subseteq \mathit{Inn}(X, \sigma)$. An immediate adaptation of the proof of Theorem (28) then shows that $\lambda\sigma\,.\,X \dotplus \sigma$ strongly efficiently solves $(T, \{\theta_0 \ldots \theta_n\})$. ■

[19] Strong efficiency was introduced in Definition (78) of essay #2.

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8.16. Proof of Theorem (30)

(30) THEOREM: Suppose that **Sym** is limited to countably many constants and **Obs** $= \mathcal{L}_{basic}$. Then there is a problem of form $(T, \{\theta, \neg\theta\})$ and a definite maxichoice revision function $\dotplus$ with the following properties.

(a) T is recursive.

(b) $(T, \{\theta, \neg\theta\})$ is solvable.

(c) For all $B \subseteq \mathcal{L}_{form}$, if $\lambda\sigma \,.\, B \dotplus \sigma$ solves $(T, \{\theta, \neg\theta\})$ then $\lambda\sigma \,.\, B \dotplus \sigma$ is dominated on $(T, \{\theta, \neg\theta\})$.

Proof: Let a, b, and $\overline{n}$, $n \in N$, be distinct constants. (In view of (1) of essay #2 we assume that this can be done via a total computable isomorphism between N and the set of constants.) Choose a recursive subset E of N that is not primitive recursive. We take θ to be $a = b$, and T to be the following set of sentences:

$$\begin{cases} \theta \leftrightarrow a = \overline{n}, & \text{for all } n \in E, \\ \theta \leftrightarrow a \neq \overline{n}, & \text{for all } n \notin E. \end{cases}$$

It is immediate that (30)a,b are satisfied. For (30)c, we say that $B \subseteq \mathcal{L}_{form}$ *disagrees* with T if there is $n \in N$ with either ($n \in E$ and $B \models a \neq \overline{n}$) or ($n \in \overline{E}$ and $B \models a = \overline{n}$); otherwise, we say that B *agrees* with T. Fix an enumeration $\{\varphi_i \,|\, i > 0\}$ of $\mathcal{L}_{form}$, and set $S_i = \{\varphi_i\}$ for all $i > 0$. Let $\mathbf{S} = \{S_i \,|\, 0 < i < \omega\}$, and let $\dotplus$ be constructed from $\mathbf{S}$ as in Definition (42). So, $\dotplus$ is definite maxichoice by Proposition (43). To verify (30)c, let $B \subseteq \mathcal{L}_{form}$ be given, and define $\Psi = \lambda\sigma \,.\, B \dotplus \sigma$. We suppose that Ψ solves $(T, \{\theta, \neg\theta\})$ since otherwise there is nothing left to prove. To complete the proof it must be shown that Ψ is dominated on $(T, \{\theta, \neg\theta\})$. Let $P_1 = MOD(T \cup \{\theta\})$, $P_2 = MOD(T \cup \{\neg\theta\})$, and let scientist Θ be defined as follows.

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(96) For all $\sigma \in SEQ$,

$$\Theta(\sigma) = \begin{cases} P_1 & \text{if } T \cup \mathit{content}(\sigma) \models \theta. \\ P_2 & \text{if } T \cup \mathit{content}(\sigma) \models \neg\theta. \\ P_1 & \text{if } T \cup \mathit{content}(\sigma) \not\models \theta,\ T \cup \mathit{content}(\sigma) \not\models \neg\theta, \text{ and neither} \\ & \emptyset \neq \Psi(\sigma) \subseteq P_1, \text{ nor } \emptyset \neq \Psi(\sigma) \subseteq P_2. \\ \Psi(\sigma) & \text{otherwise.} \end{cases}$$

It follows that Θ solves $(T, \{\theta, \neg\theta\})$. Hence, for all environments e for $P \in \{P_1, P_2\}$, $SP(\Theta, e, P)$ and $SP(\Psi, e, P)$ are well defined, and $SP(\Theta, e, P) \leq SP(\Psi, e, P)$, as easily verified. Thus, by Definition (76) of essay #2, it suffices to show that there is an environment e for some $P \in \{P_1, P_2\}$ such that $SP(\Theta, e, P) < SP(\Psi, e, P)$. This is demonstrated via the following, exhaustive cases.

(97) (a) $B \models \theta$ iff $B \models \neg\theta$.
(b) B is consistent, $B \models \theta$, and B disagrees with T.
(c) B is consistent, $B \models \theta$, and B agrees with T.
(d) Same as (b) except that θ is replaced by $\neg\theta$.
(e) Same as (c) except that θ is replaced by $\neg\theta$.

If (97)a, let e be any environment for P_1 such that for some $n \in E$, $e(0) = (a = \overline{n})$. By Definition (3), $B \dotplus e[0] = B \dotplus \emptyset = B$, so by (97)a, neither $\emptyset \neq \Psi(e[0]) \subseteq P_1$ nor $\emptyset \neq \Psi(e[0]) \subseteq P_2$. Since $T = T \cup \mathit{content}(e[0]) \not\models \theta$ and $T \cup \mathit{content}(e[0]) \not\models \neg\theta$, the third clause of (96) implies that $\Theta(e[0]) = \Theta(\emptyset) = P_1$. Moreover, for all $k > 0$, $T \cup \mathit{content}(e[k])$ is consistent and implies θ. Hence, the first clause of (96) implies $SP(\Theta, e, P_1) = 0$. On the other hand, since it is not the case that $\emptyset \neq \Psi(e[0]) \subseteq P_1$, $SP(\Psi, e, P_1) > 0$.

If (97)b, then either there is $n \in E$ with $B \models a \neq \overline{n}$ or $n \in \overline{E}$ with $B \models a = \overline{n}$. We consider these subcases in turn. Suppose n is such that $n \in E$ and $B \models a \neq \overline{n}$. Let e be an environment for P_2 such that $e(0) = (a \neq \overline{n})$. Then by Lemma (4) and the fact that B is consistent, $\Psi(e[1]) = B \dot{+} e[1] \models B$, so $\Psi(e[1]) \models \theta$. It follows that $SP(\Psi, e, P_2) > 1$. In contrast, the definition (96) of Θ implies that $SP(\Theta, e, P_2) = 1$. Now suppose n is such that $n \in \overline{E}$ and $B \models a = \overline{n}$. Let e be an environment for P_2 such that $e(0) = (a = \overline{n})$. Then again by Lemma (4) and the fact that B is consistent, $\Psi(e[1]) = B \dot{+} e[1] \models B$, so $\Psi(e[1]) \models \theta$. It follows that $SP(\Psi, e, P_2) > 1$, whereas (96) implies $SP(\Theta, e, P_2) = 1$.

Suppose that (97)c holds. We rely on the following fact, an immediate corollary to the proof that the monadic predicate calculus is decidable (see [5, Ch. 25]).

(98) FACT: Suppose that **Sym** is limited to constants. Then the function that associates to every $\phi \in \mathcal{L}_{form}$ the value 1 if $\models \phi$, and the value 0 if $\not\models \phi$, is primitive recursive.

Let $i_0 > 0$ be least such that $D = B \cap \{\varphi_i \,|\, 0 < i < i_0\} \models \theta$. Then there exists $n \in E$ such that $D \not\models a = \overline{n}$, or there exists $n \in \overline{E}$ such that $D \not\models a \neq \overline{n}$. Indeed, if this were not the case then (98) would imply that the characteristic function of E is primitive recursive, contradicting the choice of E. We consider the two subcases in turn. Assume n is such that $n \in E$ and $D \not\models a = \overline{n}$. Let e be an environment for P_2 such that $e(0) = (a \neq \overline{n})$. Then by the choice of i_0 and the definition of $\dot{+}$, $D \subseteq B \dot{+} e[1] = \Psi(e[1])$, hence $\Psi(e[1]) \models \theta$. So $SP(\Psi, e, P_2) > 1$. In contrast, $SP(\Theta, e, P_2) = 1$ since $T \cup content(e[1]) \models \neg\theta$. Now assume n is such that $n \in \overline{E}$ and $D \not\models a \neq \overline{n}$. Let e be an environment for P_2 such that $e(0) = (a = \overline{n})$. Then, once again, by the choice of i_0 and the definition of $\dot{+}$, $D \subseteq B \dot{+} e[1] = \Psi(e[1])$, hence $\Psi(e[1]) \models \theta$. So $SP(\Psi, e, P_2) > 1$, whereas $SP(\Theta, e, P_2) = 1$.

The argument for (97)d is parallel to that for (97)b, and the argument for (97)e is parallel to that for (97)c. ■

8.17. Proof of Theorem (35)

(35) THEOREM: Let **Sym** consist of a binary predicate, and suppose that **Obs** $= \mathcal{L}_{basic}$. Then there exists a problem of form $(T, \{\theta, \neg\theta\})$ (with T finite), and maxichoice revision function $\dot{+}$ such that:

(a) $(T, \{\theta, \neg\theta\})$ is solvable;

(b) for all $B \subseteq \mathcal{L}_{form}$, if B is closed under $\psi, \phi \,/\, \psi \rightarrow \phi$, then $\lambda\sigma \,.\, B \dot{+} \sigma$ does not solve $(T, \{\theta, \neg\theta\})$.

Proof of the theorem relies on the following lemma.

(99) LEMMA: Suppose that $Y \subseteq \mathcal{L}_{form}$ is closed under $\psi, \phi \,/\, \psi \rightarrow \phi$. Let $\sigma \in SEQ$ and $\theta \in \mathcal{L}_{sen}$ be given with $Y \cup content(\sigma)$ inconsistent and $content(\sigma) \not\models \theta$. Then there is maximally consistent $Z \subseteq content(\sigma) \cup Y$ such that $content(\sigma) \subseteq Z$ and $Z \not\models \theta$.

Proof: Suppose that $Y \subseteq \mathcal{L}_{form}$ is closed under $\psi, \phi \,/\, \psi \rightarrow \phi$, and let σ, θ be such that:

(100) (a) $Y \cup content(\sigma)$ is inconsistent.

(b) $content(\sigma) \not\models \theta$.

Let $\Omega = \{X \subseteq content(\sigma) \cup Y | content(\sigma) \subseteq X \wedge X \not\models \theta\}$. By (100)(b), Ω is nonempty. It follows immediately from the Compactness Theorem that Ω is closed under unions of chains and hence, by Zorn's Lemma, that Ω contains maximal elements (with respect to $\subseteq$). So let $Z \in \Omega$ be such a maximal element, that is, for every $\varphi \in Y$, if $\varphi \notin Z$, then $Z \cup \{\varphi\} \models \theta$. We must show that Z is a maximal consistent subset of $content(\sigma) \cup Y$ such that $content(\sigma) \subseteq Z$ and $Z \not\models \theta$. By construction, $Z \subseteq content(\sigma) \cup Y$, $content(\sigma) \subseteq Z$, and $Z \not\models \theta$ (thus Z is consistent). To show that Z is maximal with the foregoing properties, let $\varphi \in Y - Z$. We must show that $Z \cup \{\varphi\}$ is inconsistent.

By (100)(a) (and because $content(\sigma) \subseteq Z$), to show that $Z \cup \{\varphi\}$ is inconsistent, it suffices to show for every $\psi \in Y$, $\varphi \rightarrow \psi \in Z$. So fix $\psi \in Y$. By the closure condition on Y, $\varphi \rightarrow \psi \in Y$. Since Z is a maximal element of Ω, to prove that $\varphi \rightarrow \psi \in Z$, it suffices to show that $Z \cup \{\varphi \rightarrow \psi\} \not\models \theta$.

Since $Z \cup \{\varphi\} \models \theta$ and $Z \not\models \theta$, there is a structure $\mathcal{S}$ and assignment h such that $\mathcal{S} \models Z \cup \{\neg\theta\}[h]$ and $\mathcal{S} \not\models \varphi[h]$. So also, $\mathcal{S} \models \varphi \rightarrow \psi[h]$. Hence, $\mathcal{S}, h$ witness $Z \cup \{\varphi \rightarrow \psi\} \not\models \theta$. ■

We now exploit Lemma (99) to prove the theorem.

Proof of Theorem (35): Let R be the binary predicate of **Sym**. Let $T = \{\, \exists x \forall y Rxy \leftrightarrow \forall xy(x = y)\,\}$ and $\theta = \exists x \forall y Rxy$. We claim that $(T, \{\theta, \neg\theta\})$ witnesses the proposition. Clause (35)a is evident. For Clause (35)b, observe:

(101) for every $\sigma \in SEQ$, $content(\sigma) \not\models \neg\theta$.

To finish the proof we define a maxichoice revision function $\dot{+}$ such that for all $B \subseteq \mathcal{L}_{form}$, if B is closed under $\psi,\ \phi\ /\ \psi \rightarrow \phi$, then $\lambda\sigma\,.\,B \dot{+} \sigma$ does not solve $(T, \{\theta, \neg\theta\})$. It follows directly from (101) and Lemma (99) that there exists a maxichoice revision function $\dot{+}$ with the following property.

(102) Suppose that $B \subseteq \mathcal{L}_{form}$ is closed under $\psi,\ \phi\ /\ \psi \rightarrow \phi$. Then for every $\sigma \in SEQ$, if $B \cup content(\sigma)$ is inconsistent, $B \dot{+} \sigma \not\models \neg\theta$.

Let environment e be for $MOD(T \cup \{\theta\})$. By compactness:

(103) $T \cup content(e) \not\models \theta$.

Let $B \subseteq \mathcal{L}_{form}$ be closed under $\psi, \phi \,/\, \psi \to \phi$. If $B \dotplus e[k] \not\models \theta$ for all $k \in N$, then $\lambda\sigma \,.\, B \dotplus \sigma$ does not solve $MOD(T \cup \{\theta\})$, and we are done. So let $\sigma \subset e$ be such that

(104) $B \dotplus \sigma \models \theta$.

By (103) there is a structure $\mathcal{U} \in MOD(T \cup \{\neg\theta\})$ and a full assignment h to $\mathcal{U}$ such that $\mathcal{U} \models content(\sigma)[h]$. So we can choose environment e' for $\mathcal{U}$ via h such that $\sigma \subset e'$. Let $k > length(\sigma)$ be given. It suffices to show that $B \dotplus e'[k] \not\models \neg\theta$. For this purpose we distinguish two cases.

Case 1: $B \cup content(e'[k])$ is consistent. Then by (104), Lemma (4), and the fact that $\sigma \subset e'$, $B \dotplus e'[k] \models \theta$. Hence, by Lemma (4), $B \dotplus e'[k] \not\models \neg\theta$.

Case 2: $B \cup content(e'[k])$ is inconsistent. Then (102) implies $B \dotplus e'[k] \not\models \neg\theta$. ■

8.18. Proof of Proposition (37)

(37) PROPOSITION: Suppose that **Obs** is closed under negation. There exists $X \subseteq \mathcal{L}_{form}$ and definite maxichoice revision function $\dotplus$ such that for all consistent $T \subseteq \mathcal{L}_{sen}$, the following holds.

(a) $T \cup X$ is consistent.

(b) $\lambda\sigma \,.\, (T \cup X) \dotplus_{ie} \sigma$ solves every solvable problem of the form $(T, \{\theta_0 \dots \theta_n\})$.

Proof: Let X be defined as in the proof of Theorem (21). Fix an enumeration $\{\chi_i \,|\, i > 0\}$ of $\mathcal{L}_{basic} \cup \mathcal{L}_{sen}$, an enumeration $\{\varphi_i \,|\, i \in N\}$ of X, and an enumeration $\{\psi_i \,|\, i \in N\}$ of $\mathcal{L}_{form}$. For all $i > 0$, set $S_i = \{\chi_i\}$. For all $i \in N$, set $S_{\omega+i} = \{\varphi_i\}$, and $S_{(\omega\times 2)+i} = \{\psi_i\}$. Let $\mathbf{S} = \{S_\alpha \,|\, 0 < \alpha < \omega \times 3\}$, and let $\dotplus$ be constructed from $\mathbf{S}$ as in Definition (42). So $\dotplus$ is definite maxichoice by Proposition (43). Then it is straightforward to adapt the proof of Theorem (21) for the present proposition. ■

8.19. Proof of Proposition (38)

(38) PROPOSITION Suppose that **Obs** is closed under negation. For every solvable problem **P** of the form $(T, \{P_0, P_1, \ldots\})$ there is a consistent extension $X \subseteq \mathcal{L}_{form}$ of T and a definite maxichoice revision function $\dotplus$ such that $\lambda\sigma \,.\, X \dotplus_{ie} \sigma$ solves **P** efficiently.

Proof: Let solvable problem $\mathbf{P} = (T, \{P_0, P_1, \ldots\})$ be given. Recall from the proof of Theorem (22) the enumeration $\{\pi_i \,|\, i \in N\}$ and the enumeration $\{\varphi_i^n \,|\, n \in N\}$ of π_i. Without loss of generality, we may suppose that $\varphi_0^0 \notin \mathcal{L}_{basic}$. Recall from the proof of Theorem (25) the enumeration $\{S_i \,|\, i > 0\}$ of subsets of $\mathcal{L}_{form}$. Fix an enumeration $\{\beta_i \,|\, i \in N\}$ of $\mathcal{L}_{basic}$ and an enumeration $\{\psi_i \,|\, i \in N\}$ of $\mathcal{L}_{form}$. For all $i > 0$ set $S'_i = \{\beta_i\}$, $S'_{\omega+i} = S_i$. For all $i \in N$ set $S'_{(\omega\times 2)+i} = \{\psi_i\}$. Let $\mathbf{S} = \{S'_\alpha \,|\, 0 < \alpha < \omega \times 3\}$, and let $\dotplus$ be constructed from **S** as in Definition (42). So $\dotplus$ is definite maxichoice by Proposition (43). Then it is straightforward to adapt the proof of Theorem (25) for the present proposition. ■

8.20. Proof of Theorem (39)

(39) THEOREM: Suppose that $\mathbf{Sym} = \emptyset$ and $\mathbf{Obs} = \mathcal{L}_{basic}$. Then there exists a definite maxichoice revision function $\dotplus$ such that for all $B \subseteq \mathcal{L}_{form}$, $\lambda\sigma \,.\, B \dotplus_{ie} \sigma$ does not solve $(\emptyset, \{\mathbf{two}, \neg\mathbf{two}\})$.

Proof of the theorem relies on two lemmas. The first is a general fact about iterative revision, straightforward to verify.

(105) LEMMA: Suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$. Let $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$ be given. Then $B \dotplus_{ie} \sigma \subseteq B \cup content(\sigma)$.

(106) LEMMA: Suppose that $\mathbf{Obs} = \mathcal{L}_{basic}$. Let finite, consistent $D \subseteq \mathcal{L}_{form}$ be such that $D \models \mathbf{two}$. Then $\{\beta \in \mathcal{L}_{basic} \mid D \models \beta \text{ and } \not\models \beta\}$ is a finite set of equalities and inequalities among variables.

Proof of Lemma (106): Suppose for a contradiction that Lemma (106) is false. Then since D is finite there is $\beta \in \{\beta \in \mathcal{L}_{basic} \mid D \models \beta \text{ and } \not\models \beta\}$ that contains a variable x that does not appear in D. Up to logical equivalence, β has either the form $x = y$ or $x \neq y$, with y distinct from x. In the first case $D \models \forall x(x = y)$, which is a contradiction since D is consistent and $D \models \mathbf{two}$. In the second case $D \models \forall x(x \neq y)$, which is also a contradiction since D is consistent. ■

Proof of Theorem (39): Fix an enumeration $\{D_i \,|\, i \in N\}$ of all finite, consistent $D \subseteq \mathcal{L}_{form}$ such that $D \models \mathbf{two}$, an enumeration $\{\beta_i \,|\, i \in N\}$ of $\mathcal{L}_{basic}$, and an enumeration $\{\varphi_i \,|\, i \in N\}$ of $\mathcal{L}_{form}$. For all $j \in N$, set $S_{j+1} = D_0 - \{\beta_j, \beta_{j+1} \ldots\}$. For all $i > 0$ and $j \in N$, set $S_{(\omega \times i)+j} = D_i - \{\beta_j, \beta_{j+1} \ldots\}$. Set $S_{\omega^2+i} = \{\varphi_i\}$ for all $i \in N$. Let $\mathbf{S} = \{S_\alpha \,|\, 0 < \alpha < \omega^2 + \omega\}$, and let $\dot{+}$ be constructed from $\mathbf{S}$ as in Definition (42). So $\dot{+}$ is definite maxichoice by Proposition (43). Let $B \subseteq \mathcal{L}_{form}$ be given. To prove that $\lambda\sigma \,.\, B \dot{+}_{ie} \sigma$ does not solve $(\emptyset, \{\mathbf{two}, \neg\mathbf{two}\})$, we distinguish two cases.

Case 1: For all $i \in N$, $D_i - \mathcal{L}_{basic} \not\subseteq B$. Then by Lemma (105) and compactness, $\lambda\sigma \,.\, B \dot{+}_{ie} \sigma$ solves no environment for $MOD(\mathbf{two})$.

Case 2: For some $i \in N$, $D_i - \mathcal{L}_{basic} \subseteq B$. Let $i_0 \in N$ be least such that $D_{i_0} - \mathcal{L}_{basic} \subseteq B$. Let $n \in N$ and integers $j_0 < \ldots < j_{n-1}$ be such that $D_{i_0} = (D_{i_0} - \mathcal{L}_{basic}) \cup \{\beta_{j_0} \ldots \beta_{j_{n-1}}\}$. With the definition of $\dot{+}$ and the choice of i_0, it is easy to verify by induction on $m \leq n$ that $(D_{i_0} - \mathcal{L}_{basic}) \cup \{\beta_{j_0} \ldots \beta_{j_{m-1}}\} \subseteq B \dot{+}_{ie} (\beta_{j_0} \ldots \beta_{j_{m-1}})$. In particular:

(107) $D_{i_0} \subseteq B \dot{+}_{ie} (\beta_{j_0} \ldots \beta_{j_{n-1}})$.

By Lemma (106) there exists structure $\mathcal{S}$ and full assignment h to $\mathcal{S}$ such that $\mathcal{S} \not\models$ **two** and $\mathcal{S} \models \{\beta \in \mathcal{L}_{basic} \mid D_{i_0} \models \beta\}[h]$. We may choose environment e for $\mathcal{S}$ and h that begins with $(\beta_{j_0} \ldots \beta_{j_{n-1}})$. Since $D_{i_0} \cup content(e)$ is consistent, it follows from the definition of $\dot{+}$, the choice of i_0, and (107) that for all $k \geq n$, $D_{i_0} \subseteq B \dot{+}_{ie} e[k]$. Hence $B \dot{+}_{ie} e[k] \models$ **two** for all $k \geq n$. We conclude that $\lambda\sigma \,.\, B \dot{+}_{ie} \sigma$ does not solve $MOD(\neg$**two**$)$ in e. ■

8.21. Proof of Proposition (41)

(41) Proposition: Suppose that **Sym** $= \emptyset$ and **Obs** $= \mathcal{L}_{basic}$. Then there exists a definite maxichoice revision function $\dot{+}$ such that for all $B \subseteq \mathcal{L}_{form}$, $\lambda\sigma \,.\, B \dot{+}^{Cn}_{ie} \sigma$ does not solve $(\emptyset, \{$**two**$, \neg$**two**$\})$.

Proof: Fix an enumeration $\{\varphi_i \mid i > 0\}$ of $\mathcal{L}_{form}$ with $\varphi_1 =$ **two**, and set $S_i = \{\varphi_i\}$ for all $i > 0$. Let $\mathbf{S} = \{S_i \mid i < \omega\}$, and let $\dot{+}$ be constructed from $\mathbf{S}$ as in Definition (42). So Proposition (43) implies that $\dot{+}$ is definite maxichoice. Let $B \subseteq \mathcal{L}_{form}$ be given, and let e_0 be an environment for $MOD($**two**$)$. Suppose that $\lambda\sigma \,.\, B \dot{+}^{Cn}_{ie} \sigma$ solves $MOD($**two**$)$ in e_0 (otherwise we are done). Hence we may choose $k_0 \in N$ such that $B \dot{+}^{Cn}_{ie} e_0[k_0] \models$ **two**. This implies:

(108) **two** $\in Cn(B \dot{+}^{Cn}_{ie} e_0[k_0])$.

It easy to verify the existence of an environment e such that:

(109) (a) $e_0[k_0] \subset e$.

(b) e is for the structure $\{0, 1, 2\}$.

Now for all $\beta \in \mathcal{L}_{basic}$, $\{\mathbf{two}, \beta\}$ is consistent (recall that $\mathbf{Sym} = \emptyset$). So Definition (40), (108), (109)a, and the definition of $\dot{+}$ imply that for all $k > k_0$, $\mathbf{two} \in B \dot{+}^{Cn}_{ie} e[k]$. Thus, (109)b implies that $\lambda\sigma \,.\, B \dot{+}^{Cn}_{ie} \sigma$ does not solve $MOD(\neg\mathbf{two})$ in e. ■

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