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# Theoretical Note

## New Axioms for the Contrast Model of Similarity

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An additive version of Tversky's (1977, *Psychological Review* **84**, No. 4, 327–352) contrast model of similarity is discussed, and an axiomatization proposed for it. A representation theorem for the new system is derived on the basis of results drawn from Krantz *et al.* (1971, *Foundations of Measurement*, Vol. 1, New York: Academic Press). 

### 1. INTRODUCTION

Tversky (1977) conceives the judged similarity of an object $p$ to an object $q$ as a "linear contrast" of three, weighted feature sets: the features common to $p$ and $q$, the features distinctive to $p$ compared with $q$, and the features distinctive to $q$ compared with $p$. This conception is known as the "contrast model" of similarity. Letting $P$, $Q$,... be the feature sets associated with objects $p$, $q$,..., respectively, the contrast model's principal claim may be summarized as follows.

> (∗) There are positive real constants $\alpha$, $\beta$, $\gamma$, and a function $\theta$ from feature sets to nonnegative real numbers such that for all objects $p$, $q$, $r$, $s$ (in some suitably chosen domain of discourse) the similarity of $p$ to $q$ is no less than the similarity of $r$ to $s$ if and only if $\alpha\theta(P \cap Q) - \beta\theta(P - Q) - \gamma\theta(Q - P) \geqslant \alpha\theta(R \cap S) - \beta\theta(R - S) - \gamma\theta(S - R)$.

$\theta$ may be construed as a feature-set weighting function. In the Appendix to Tversky's (1977) article axioms on similarity judgment are presented that imply (∗).

The present paper offers an axiomatization of a strengthened version of the contrast model. Although partially overlapping Tversky's axioms it is hoped that the present system is smoother than the original and avoids certain difficulties that beset it. Our exposition is organized as follows. In the next section three properties of Tversky's system are discussed. Section 3 presents background concepts and notation for the alternative system. The alternative axioms are presented in Section 4. In Section 5 these axioms are used to derive a representation theorem parallel to (∗). Concluding remarks occupy Section 6.

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It should be said at the outset that the present results constitute at most a minor improvement on mathematical and psychological insights due entirely to Tversky.

## 2. Three Properties of the Original Contrast Model

*Additivity.* Let $p$ and $q$ be objects with associated feature sets $P$ and $Q$. It is natural to suppose that the psychological "weight" of $P \cap Q$ is related in a simple way to the weights of its subsets. Indeed, letting $\theta$ be as specified in (*), psychologists often assume that if feature sets $X$ and $Y$ are disjoint, then $\theta(X \cup Y) = \theta(X) + \theta(Y)$. In the case of $P$ and $Q$ finite, this latter, additivity assumption allows $\theta(P \cap Q)$ to be represented as the sum of $\{\theta(x) \mid x \in P \cap Q\}$, and similarly for $P - Q$ and $Q - P$. This assumption underlies the quantitative predictions issuing from the theory of prototype combination advanced by Smith and Osherson (1984; see also, Smith *et al.*, to appear). Much of the discussion in Tversky (1977) also relies explicitly on additivity.

On the other hand, experimental results presented in Gati and Tversky (1984) show that in many stimulus domains additivity does not characterize similarity judgment. Additivity must therefore be viewed as a strong assumption that is warranted in some psychological contexts but not others. The contrast model does not embody the additivity principle, since, as Tversky (1977, p. 322) acknowledges, his axioms do not allow additivity to be deduced. Our alternative axioms will imply a suitable additivity property.

*Size of feature sets.* Let $\Delta$ be a collection of objects whose similarities, one to the other, are under investigation. Tversky's (1977, p. 351) axioms entail that "there exist objects in $\Delta$ whose similarity matches any real value that is bounded by two similarities." In the presence of other axioms this implies that either there are uncountably many features among which subjects can distinguish, or that subjects mentally represent infinitely many distinct properties of most objects. Both of these possibilities appear dubious. The axioms proposed in the next section are compatible with limited discrimination among features and finite mental representation of objects.

*Multiple tokens of the same feature.* Since Tversky's system represents objects as feature sets, the same feature cannot be twice associated with a given object (because $\{x, y, y\} = \{x, y\}$). This is awkward inasmuch as multiple copies of the same feature constitute a convenient way to represent intensity or saliency variations along a dimension, e.g., variations in brightness. One is free, of course, to associate the brightness dimension with a sequence $B = b_0, b_1, b_2, b_3 ...$ of features, and to code any given degree of brightness by a finite initial segment of $B$ (cf., Tversky & Gati, 1982). Such a system, however, requires additional axioms to prevent odd combinations of features from appearing in the same representation; on the foregoing scheme, for example, $\{b_7, b_9\}$ has no evident interpretation.

Such complications can be avoided by the expedience of representing objects not as sets of features but as lists. Multiple copies of the same feature may then enter into a single object representation, distinguished by list position. This is the approach we adopt in what follows.

## 3. Concepts and Notation

Let us assume that human observers reliably agree with each other about the relative similarity of pairs of objects. Then, if $\Delta$ is a nonempty set of objects, a 4-place relation $R$ on $\Delta$ may be defined as follows. For $p, q, r, s \in \Delta$, $R(p, q, r, s)$ if and only if a typical observer assesses the similarity of $p$ to $q$ as not less than the similarity of $r$ to $s$. $R$ is the fundamental relation of the original contrast model.

However, inasmuch as the contrast model portrays similarity as a function of shared and distinctive features, we may construe similarity judgments as bearing directly on featural comparisons between objects. Accordingly, the fundamental relation—denoted: $\geqslant$—of the alternative system is characterized as follows. Let $A$ be a nonempty set of features, and consider the set of all finite sequences (or lists) drawn from $A$. A triple of such finite sequences may be conceived as the featural residue of a similarity judgment bearing on objects $p$, $q$. The first sequence in the triple is a list of the features common to $p$ and $q$, the second is a list of the features distinctive to $p$, and the third is a list of the features distinctive to $q$. Given two such triples of sequences over $A$—$a$, $b$, $c$ and $d$, $e$, $f$—we write $a, b, c \geqslant d, e, f$ if and only if for any four objects $p$, $q$, $r$, $s$ such that

$a$ is a list of the features common to $p$ and $q$.
$b$ is a list of the features distinctive to $p$ compared to $q$.
$c$ is a list of the features distinctive to $q$ compared to $p$,
$d$ is a list of the features common to $r$ and $s$,
$e$ is a list of the features distinctive to $r$ compared to $s$, and
$f$ is a list of the features distinctive to $s$ compared to $r$.

A typical observer assesses the similarity of $p$ to $q$ as not less than the similarity of $r$ to $s$. Our axioms are an attempt to characterize this relation $\geqslant$.

Notice that if $a = f_1, f_2, ..., f_n$ is a list of features common to objects $p$ and $q$, then so is any permutation of $a$. This kind of nonuniqueness is of no consequence in the sequel since our axioms will imply that permutations of the same sequence have identical impact on similarity judgments. It must be pointed out as well that some triples $a$, $b$, $c$ of sequences are anomalous in the sense that the sets of features listed in $b$ and $c$ are not disjoint. It is of course impossible for a feature to be distinctive of an object $p$ compared with $q$ and also distinctive of $q$ compared with $p$. No harm will come, however, from leaving such anomalous triples in the domain of the theory. For, there may turn out to be unusual objects or viewing conditions that lead subjects to anomalous mental representation of featural contrasts, and our model should apply equally well to this strange case. It is well to bear in mind that $\geqslant$ is a relation over mental representations, not over the objects so represented.

We now introduce notation needed for the developments to follow. Let $X$ be a set. $\mathrm{SEQ}_X$ is the set of all finite sequences on $X$. The length of $b \in \mathrm{SEQ}_X$ is denoted: $lh(b)$. The $i$th member of $b$ (counting from 1) is denoted: $b_i$. The length-zero sequence is denoted: $\varepsilon$. We let "*" denote concatenation over finite sequences. The set of natural numbers is denoted: $N$. The set of (positive) real numbers is denoted: $\mathscr{R}$ ($\mathscr{R}^+$, respectively). For $n \in N$ and $b \in \mathrm{SEQ}_X$ the result of concatenating $b$ onto itself $n$ times is denoted: $nb$. Given a binary relation $\geqslant$ on $X$, we define $\simeq$ and $>$ as follows. For all $x, y \in X$, $x \simeq y$ iff $x \geqslant y$ and $y \geqslant x$; $x > y$ iff $x \geqslant y$ and not $y \geqslant x$.

## 4. Similarity Structures

Our alternative version of the contrast model takes the form of a set-theoretical predicate that applies to systems called "similarity structures." In detail: A *similarity structure* is a pair $\langle A, \geqslant \rangle$, where $A$ is a nonempty set and $\geqslant$ is a binary relation on $\mathrm{SEQ}_A^3$, which meets the eight axioms that follow.

Axiom 1 (weak order). *$\geqslant$ is a weak ordering, i.e., connected, transitive, and reflexive.*

Definition 1. Let $a, b \in \mathrm{SEQ}_A$ be given.

(i) The binary relation $\geqslant_{1,a,b}$ on $\mathrm{SEQ}_A$ is defined as follows. For all $x, y \in \mathrm{SEQ}_A$, $x \geqslant_{1,a,b} y$ iff $x, a, b \geqslant y, a, b$. Similar definitions hold for $\geqslant_{2,a,b}$ and $\geqslant_{3,a,b}$.

(ii) The binary relation $\geqslant_{1,2,a}$ on $\mathrm{SEQ}_A^2$ is defined as follows. For all $w, x, y, z \in \mathrm{SEQ}_A$, $w, x \geqslant_{1,2,a} y, z$ iff $w, x, a \geqslant y, z, a$. Similar definitions hold for $\geqslant_{1,3,a}$ and $\geqslant_{2,3,a}$.

Axiom 2 (independence). *For all $a, b, c, d \in \mathrm{SEQ}_A$,*

(i) $\geqslant_{1,a,b} = \geqslant_{1,c,d}$,
$\geqslant_{2,a,b,} = \geqslant_{2,c,d}$,
$\geqslant_{3,a,b} = \geqslant_{3,c,d}$;

(ii) $\geqslant_{1,2,a} = \geqslant_{1,2,b}$,
$\geqslant_{1,3,a} = \leqslant_{1,3,b}$,
$\geqslant_{2,3,a} = \geqslant_{2,3,b}$.

Definition 2. The binary relation $\geqslant_1$ on $\mathrm{SEQ}_A$ is defined as follows. For all $x, y \in \mathrm{SEQ}_A$, $x \geqslant_1 y$ iff for all $a, b \in \mathrm{SEQ}_A$, $x, a, b \geqslant y, a, b$. The binary relations $\geqslant_2$, $\geqslant_3$, $\geqslant_{1,2}$, etc. are defined similarly.

Axiom 3 (congruence). *For all $a, b \in \mathrm{SEQ}_A$, $a \simeq_1 b$ iff $a \simeq_2 b$ iff $a \simeq_3 b$.*

AXIOM 4 (monotonicity). *For all* $a, b, c, d, e, f, g \in \mathrm{SEQ}_A$, $a, b, c \geqslant d, e, f$ *iff* $a^*g, b, c \geqslant d^*g, e, f$ *iff* $a, b^*g, c \geqslant d, e^*g, f$ *iff* $a, b, c^*g \geqslant d, e, f^*g$.

AXIOM 5 (Archimedean principle). *For all* $a, b, c, d, e, f \in \mathrm{SEQ}_A$, *if* $a \neq \varepsilon$ *then*:

(i) *there is* $n \in N$ *such that* $na, b, c \geqslant d, e, f$;
(ii) *there is* $n \in N$ *such that* $e, b, c \geqslant d, na, f$;
(iii) *there is* $n \in N$ *such that* $f, b, c \geqslant d, e, na$.

AXIOM 6 (positivity). *For all* $a, b, c, d \in \mathrm{SEQ}_A$, *if* $d \neq \varepsilon$ *then*:

(i) $a^*d, b, c > a, b, c$;
(ii) $a, b, c > a, b^*d, c$;
(iii) $a, b, c > a, b, c^*d$.

AXIOM 7 (solvability). *For all* $a, b, c, d, e, f, a', b, c \in \mathrm{SEQ}_A$;

(i) *if* $a', b, c \geqslant d, e, f \geqslant a, b, c$, *then there is* $g \in \mathrm{SEQ}_A$ *such that* $a^*g, b, c \simeq d, e, f$.

(ii) *if* $b, a, c \geqslant d, e, f \geqslant b, a', c$, *then there is* $g \in \mathrm{SEQ}_A$ *such that* $b, a^*g, c \simeq d, e, f$.

(iii) *if* $b, c, a \geqslant d, e, f \geqslant b, c, a'$ *then there is* $g \in \mathrm{SEQ}_A$ *such that* $b, c, a^*g \simeq d, e, f$.

AXIOM 8 (commutativity). *For all* $a, b \in \mathrm{SEQ}_A$,

(i) $a^*b \simeq_1 b^*a$,
(ii) $a^*b \simeq_2 b^*a$,
(iii) $a^*b \simeq_3 b^*a$.

## 5. REPRESENTATION THEOREM FOR SIMILARITY STRUCTURES

REPRESENTATION THEOREM. *Let* $\langle A, \geqslant \rangle$ *be a similarity structure. Then there exists* $\theta: A \to \mathscr{R}^+$ *and* $\alpha, \beta, \gamma \in \mathscr{R}^+$ *such that for all* $a, b, c, d, e, f \in \mathrm{SEQ}_A$: $a, b, c \geqslant d, e, f$ *iff* $\alpha \sum_{1 \leqslant i \leqslant lh(a)} \theta(a_i) - \beta \sum_{1 \leqslant i \leqslant lh(b)} \theta(b_i) - \gamma \sum_{1 \leqslant i \leqslant lh(c)} \theta(c_i) \geqslant \alpha \sum_{1 \leqslant i \leqslant lh(d)} \theta(d_i) - \beta \sum_{1 \leqslant i \leqslant lh(e)} \theta(e_i) - \gamma \sum_{1 \leqslant i \leqslant lh(f)} \theta(f_i)$.

Proof of the theorem turns on two fundamental results demonstrated in Krantz *et al.* (1971). The following parade of lemmas simply permits their application to the present setting. Lemmas 1–6 establish the existence of a certain closed, extensive structure. Lemmas 8–14 do the same for a 3-component, additive conjoint structure. In the ensuing discussion let $\langle A, \geqslant \rangle$ be a given similarity structure.

*Extensive Measurement Lemmas*

LEMMA 1 (weak order). *For all* $i \in \{1, 2, 3\}$, $\geqslant_i$ *is a weak order on* $\mathrm{SEQ}_A$.

*Proof.* This follows from the independence axiom. ∎

LEMMA 2 (solvability). *Let* $a, b \in \mathrm{SEQ}_A$ *be given.*

(i) *If* $b >_1 a$ *then there is* $c \neq \varepsilon$ *such that* $a*c \simeq_1 b$.

(ii) *If* $b <_2 a$ *then there is* $c \neq \varepsilon$ *such that* $a*c \simeq_2 b$.

(iii) *If* $b <_3 a$ *then there is* $c \neq \varepsilon$ *such that* $a*c \simeq_3 b$.

*Proof.* From Axiom 7 via Definition 2. ∎

LEMMA 3 (strong Archimedean principle). *For all* $a, b, c, d \in \mathrm{SEQ}_A$ *and* $i \in \{1, 2, 3\}$, *if* $a >_i b$ *then there is* $n \in N$ *such that* $na*c \geqslant_i nb*d$.

*Proof.* We let $i = 1$. For a contradiction suppose that $a >_1 b$ but

(A) for all $n$, $nb*d >_1 na*c$.

By Lemma 2, let $e \neq \varepsilon$ be such that $a \simeq_1 b*e$. Then (A), monotonicity, and the transitivity of $>_1$ yield

(B) for all $n$, $nb*d >_1 n(b*e)*c$.

By Axiom 8 and the associativity of $*$, (B) implies

(C) for all $n$, $nb*d >_1 (ne*c)*nb$.

Hence, by monotonicity,

(D) for all $n$, $d >_1 ne*c$.

Since by positivity $d*c \geqslant_1 d$, (D) yields

(E) for all $n$, $d*c >_1 ne*c$.

Monotonicity applied to (E) implies that for all $n$, $d >_1 ne$, contradicting Axiom 5. ∎

LEMMA 4 (strong congruence). $\geqslant_1 = \leqslant_2 = \leqslant_3$.

*Proof.* Via Axiom 3. ∎

LEMMA 5 (positivity). *For all* $a, b \in \mathrm{SEQ}_A$, *if* $b \neq \varepsilon$ *then* $a*b >_1 a$, $a*b <_2 a$, *and* $a*b <_3 a$.

*Proof.* By Axiom 6. ∎

LEMMA 6. Each of $\langle \mathrm{SEQ}_A - \{\varepsilon\}, \geqslant_1, *\rangle$, $\langle \mathrm{SEQ}_A - \{\varepsilon\}, \leqslant_2, *\rangle$, $\langle \mathrm{SEQ}_A - \{\varepsilon\}, \leqslant_3, *\rangle$, are positive, closed extensive structures in the sense of Krantz *et al.* (1971, Sect. 3.2.1).

*Proof.* By the previous lemmas plus the associativity of the concatenation operator $*$. ▮

LEMMA 7. *There exists a function $\varphi: A \to \mathscr{R}^+$ such that for all $a, b \in \mathrm{SEQ}_A$:*

(i) $a \geqslant_1 b$ *iff* $\sum_{1 \leqslant i \leqslant lh(a)} \varphi(a_i) \geqslant \sum_{1 \leqslant i \leqslant lh(b)} \varphi(b_i)$.

(ii) $a \geqslant_2 b$ *iff* $\sum_{1 \leqslant i \leqslant lh(a)} \varphi(a_i) \leqslant \sum_{1 \leqslant i \leqslant lh(b)} \varphi(b_i)$.

(iii) $a \geqslant_3 b$ *iff* $\sum_{1 \leqslant i \leqslant lh(a)} \varphi(a_i) \leqslant \sum_{1 \leqslant i \leqslant lh(b)} \varphi(b_i)$.

Moreover, another function $\varphi'$ satisfies (i)–(iii) iff there exists $\alpha \in \mathscr{R}^+$ such that $\varphi' = \alpha\varphi$.

*Proof.* This follows from Lemma 6 and Theorem 1 of Krantz *et al.* (1971, Sect. 3.2.1). ▮

Note that a sum with no terms equals 0. Thus $\varphi$ assigns 0 to $\varepsilon$.

*Conjoint Measurement Lemmas*

In the ensuing discussion let $\varphi: A \to \mathscr{R}^+$ be as specified in Lemma 7.

LEMMA 8 (substitution). *For all $a, b, c, a', b', c' \in \mathrm{SEQ}_A$, if $a \simeq_1 a'$, $b \simeq_1 b'$, and $c \simeq_1 c'$, then $a, b, c \simeq a', b', c'$.*

*Proof.* By Definition 2 and the transitivity of $\simeq$. ▮

DEFINITION 3. (i) Extend $\varphi$ to $\varphi': \mathrm{SEQ}_A \to \mathscr{R}^+ \cup \{0\}$ by the stipulation that for all $a \in \mathrm{SEQ}_A$, $\varphi'(a) = \sum_{1 \leqslant i \leqslant lh(a)} \varphi(a_i)$. "$\varphi'$" is abbreviated to $\varphi$.

(ii) Let $\mathrm{SEQ}_{\varphi(A)}$ = the image of $\mathrm{SEQ}_A$ under $\varphi$.

(iii) Define $\geqslant *$ on $\mathrm{SEQ}_{\varphi(A)}{}^3$ as follows. For all $u, v, w, x, y, z \in \mathrm{SEQ}_{\varphi(A)}$, $u, v, w \geqslant * x, y, z$ iff for all $a \in \varphi^{-1}(u)$, $b \in \varphi^{-1}(v)$, $c \in \varphi^{-1}(w)$, $d \in \varphi^{-1}(x)$, $e \in \varphi^{-1}(y)$, $f \in \varphi^{-1}(z)$, $a, b, c \geqslant d, e, f$.

Evidently, $\{\varphi^{-1}(x) \mid x \in \mathrm{SEQ}_{\varphi(A)}\}$ is the partition of $\mathrm{SEQ}_A$ induced by $\simeq_1$ ($= \simeq_2 = \simeq_3$ by Axiom 3). Correspondingly, $x \in \mathrm{SEQ}_{\varphi(A)}$ may be conceived as the equivalence class $\varphi^{-1}(x)$.

LEMMA 9. *$\geqslant *$ is a weak order on $\mathrm{SEQ}_{\varphi(A)}{}^3$.*

*Proof.* Transitivity of $\geqslant *$ follows straightforwardly from Axiom 1 and Definition 3(iii). Connectivity follows similarly from the connectivity of $\geqslant$. ▮

DEFINITION 4. For $x, y \in \mathrm{SEQ}_{\varphi(A)}$. Let the binary relations $\geqslant *_{1,x,y}$, $\geqslant *_{2,3,x}$, etc. be defined in parallel fashion to Definition 1.

LEMMA 10 (independence). *For all $w, x, y, z \in \mathrm{SEQ}_{\varphi(A)}$,*

(i) $\geqslant *_{1,w,x} = \geqslant *_{1,y,z}$,
$\geqslant *_{2,w,x} = \geqslant *_{2,y,z}$,
$\geqslant *_{3,w,x} = \geqslant *_{3,y,z}$,

(ii) $\geqslant *_{1,2,x} = \geqslant *_{1,2,y}$,
$\geqslant *_{1,3,x} = \geqslant *_{1,3,y}$,
$\geqslant *_{2,3,x} = \geqslant *_{2,3,y}$.

*Proof.* Directly from Axiom 2 and Lemma 7 via Lemma 8. ▮

DEFINITION 5. The binary relation $\geqslant *_1$ on $\mathrm{SEQ}_{\varphi(A)}$ is defined as follows. For all $x, y \in \mathrm{SEQ}_{\varphi(A)}$, $x \geqslant *_1 y$ iff for all $r, s \in \mathrm{SEQ}_{\varphi(A)}$, $x, r, s \geqslant * y, r, s$. The binary relations $\geqslant *_2$, $\geqslant *_3$, $\geqslant *_{1,2}$, etc. are defined similarly.

LEMMA 11 (restricted solvability). *For all $u, v, w, x, y, z, u', v, w \in \mathrm{SEQ}_{\varphi(A)}$,*

(i) *if $u', v, w \geqslant * x, y, z \geqslant * u, v, w$, then there is $g \in \mathrm{SEQ}_{\varphi(A)}$ such that $u+g, v, w \simeq * x, y, z$;*

(ii) *if $v, u, w \geqslant * x, y, z \geqslant * v, u', w'$, then there is $g \in \mathrm{SEQ}_{\varphi(A)}$ such that $v, u+g, w \simeq * x, y, z$;*

(iii) *if $v, w, u \geqslant * x, y, z \geqslant * v, w, u'$ then there is $g \in \mathrm{SEQ}_{\varphi(A)}$ such that $v, w, u+g \simeq * x, y, z$.*

*Proof.* Directly from Axiom 7 and Lemma 7 via Lemma 8. ▮

LEMMA 12 (every strictly bounded standard sequence is finite). *Let $x_1, x_2, \ldots$ be a denumerably infinite sequence drawn from $\mathrm{SEQ}_{\varphi(A)}$, and suppose that $s, t, y, z \in \mathrm{SEQ}_{\varphi(A)}$ are such that $x_i, s, t \simeq * x_{i+1}, y, z$ for all $i \in N$.*

(i) *If $s, t >*_{2,3} y, z$ then for every $r \in \mathrm{SEQ}_{\varphi(A)}$ there is $n \in N$ such that $x_n \geqslant *_1 r$.*

(ii) *If $y, z >*_{2,3} s, t$ then for every $r \in \mathrm{SEQ}_{\varphi(A)}$ there is $n \in N$ such that $r \geqslant *_1 x_n$.*

*Similar conditionals hold with respect to $\geqslant *_2$ and $\geqslant *_3$.*

*Proof.* (i) For a contradiction, let $r \in \mathrm{SEQ}_{\varphi(A)}$ be such that $r >*_1 x_i$ for all $i \in N$. Then, for some $b \in \varphi^{-1}(r)$, $a_i \in \varphi^{-1}(x_i)$, we have

(A) $b >_1 a_i$ for all $i \in N$.

Choose $c \in \varphi^{-1}(s)$, $d \in \varphi^{-1}(t)$, $e \in \varphi^{-1}(y)$, and $f \in \varphi^{-1}(z)$ so that $c, d >_{2,3} e, f$ and $a_i, c, d \simeq a_{i+1}, e, f$ for all $i \in N$.

CLAIM. *There is $g \neq \varepsilon$ such that for all $i \in N$, $a_i^* g \simeq_1 a_{i+1}$.*

*Proof of Claim.* Since $c, d >_{2,3} e, f$, $a_{i+1}, c, d > a_{i+1}, e, f \simeq a_i, c, d$, so $a_{i+1} >_1 a_i$ for all $i \in N$. Given $i < j$, Lemma 2 implies the existence of $h, g, g' \in \mathrm{SEQ}_A - \{\varepsilon\}$ such that $a_i^* g \simeq_1 a_{i+1}$, $a_j^* g' \simeq_1 a_{j+1}$ and $a_i^* h \simeq_1 a_j$. Then, $a_i, c, d \simeq a_i^* g, e, f$ and $a_i^* h, c, d \simeq a_j, c, d \simeq a_j^*\, g', e, f \simeq a_i^* h^* g', e, f$. By Axioms 4 and 8 the latter equation yields $a_i, c, d \simeq a_i^* g', e, f$, which by transitivity yields $a_i^* g', e, f \simeq a_i^* g, e, f$. Monotonicity then gives $g \simeq_1 g'$. This proves the claim.

From (A), the claim, Axiom 8, and the associativity of $*$ we have, for some $g \neq \varepsilon$,

(B) $b >_1 a_1^* ng$, for all $n \in N$.

By positivity and transitivity,

(C) $a_1^* b >_1 a_1^* ng$, for all $n \in N$.

Monotonicity applied to (C) yields

(D) $b >_1 ng$, for all $n \in N$,

contradicting Axiom 5. ∎

LEMMA 13 (essentialness of components). (i) *There are* $w, x, y, z \in \mathrm{SEQ}_{\varphi(A)}$ *such that not* $w, y, z \simeq^* x, y, z$.

(ii) *There are* $w, x, y, z \in \mathrm{SEQ}_{\varphi(A)}$ *such that not* $y, w, z \simeq^* y, x, z$.

(iii) *There are* $w, x, y, z \in \mathrm{SEQ}_{\varphi(A)}$ *such that not* $y, z, w \simeq^* y, z, x$.

*Proof.* By positivity. ∎

LEMMA 14 (invariance). *Let* $x, y, z \in \mathrm{SEQ}_{\varphi(A)}$ *be such that for some* $p, q, r, s \in \mathrm{SEQ}_{\varphi(A)}$ $p, q \geqslant^*_{2,3} r, s$, $x, p, q \simeq^* y, r, s$, *and* $y, p, q \simeq^* z, r, s$. *Let* $l, m, n, o \in \mathrm{SEQ}_{\varphi(A)}$ *be such that* $l, x, m \simeq^* n, y, o$. *Then* $l, y, m \simeq^* n, z, o$. *A similar fact holds for any permutation of the first, second, and third components of* $\mathrm{SEQ}_{\varphi(A)}$.

*Proof.* Let $a \in \varphi^{-1}(x)$, $b \in \varphi^{-1}(y)$, $c \in \varphi^{-1}(z)$ be chosen. By the argument of the claim of Lemma 12, there is $h \in \mathrm{SEQ}_A$ such that $b \simeq_1 a^* h$ and $c \simeq_1 a^* h^* h$. Hence, by Axiom 3, $b \simeq_2 a^* h$ and $c \simeq_2 a^* h^* h$. Now let $d \in \varphi^{-1}(1)$, $e \in \varphi^{-1}(m)$, $f \in \varphi^{-1}(n)$, $g \in \varphi^{-1}(o)$ be chosen and suppose that $d, a, e \simeq f, b, g$. Then by monotonicity, $d, a^* h, e \simeq f, b^* h, e$, that is, $d, b, e \simeq f, c, e$. The lemma follows. ∎

LEMMA 15. *There exists a function* $\psi\colon \mathrm{SEQ}_{\varphi(A)} \to \mathscr{R}^+ \cup \{0\}$ *and* $\alpha, \beta, \gamma \in \mathscr{R}^+$ *such that* $\psi(0) = 0$ *and for all* $u, v, w, x, y, z \in \mathrm{SEQ}_{\varphi(A)}$:

$$u, v, w \geqslant^* x, y, z \quad \textit{iff} \quad \alpha\psi(u) - \beta\psi(v) - \gamma\psi(w) \geqslant \alpha\psi(x) - \beta\psi(y) - \gamma\psi(z).$$

*Moreover, another function* $\psi'$ *and constants* $\alpha', \beta', \gamma'$ *meet this condition iff there are* $k_1, k_2, k_3 \in \mathscr{R}$ *such that* $\psi' = k_1\psi + k_2$ *and* $\alpha' = k_3\alpha$, $\beta' = k_3\beta$ *and* $\gamma' = k_3\gamma$.

*Proof.* The preceding lemmas show that $\langle \mathrm{SEQ}_{\varphi(A)}{}^3, \geqslant^* \rangle$ is an invariant, 3-component, additive, conjoint structure in the sense of Krantz *et al.* (1971, Sect. 6.11.2). The present lemma thus follows from Theorem 15 of Krantz *et al.* (1971, Sect. 6.11.2) along with the observations that (a) Lemma 7 constrains the second and third factors to contribute negatively to the similarity associated with a triple in $\mathrm{SEQ}_{\varphi(A)}$, and (b) positivity requires that $\psi(0) \leqslant \psi(x)$ for all $x \in \mathrm{SEQ}_{\varphi(A)}$. ▮

Note that the function $\psi$ of Lemma 15 depends on prior choice of $\varphi$.

We now combine the extensive and conjoint measurement results to complete our proof. In the ensuing discussion let $\psi: \mathrm{SEQ}_{\varphi(A)} \to \mathscr{R}^+ \cup \{0\}$ and $\alpha, \beta, \gamma \in \mathscr{R}^+$ be as specified in Lemma 15.

LEMMA 16. *For all* $a, b, c, d, e, f \in \mathrm{SEQ}_A$,

$$a, b, c \geqslant d, e, f \quad \textit{iff}$$
$$\alpha\psi(\varphi(a)) - \beta\psi(\varphi(b)) - \gamma\psi(\varphi(c)) \geqslant \alpha\psi(\varphi(d)) - \beta\psi(\varphi(e)) - \gamma\psi(\varphi(f)).$$

*Proof.* Using Lemma 8 and Definition 3(iii), the lemma follows immediately from Lemmas 7 and 15. ▮

LEMMA 17 (additivity). *For all* $a, b \in \mathrm{SEQ}_A$, $\psi(\varphi(a{*}b)) = \psi(\varphi(a)) + \psi(\varphi(b))$.

*Proof.* Let $d \neq \varepsilon$ be given. By positivity, $a{*}b, \varepsilon, d \geqslant a, \varepsilon, d$. By Axiom 5 let $n \in N$ be such that $a, \varepsilon, d \geqslant a{*}b, \varepsilon, nd$. By Axiom 7 there is $f \in \mathrm{SEQ}_A$ such that $a{*}b, \varepsilon, d{*}f \simeq a, \varepsilon, d$. By monotonicity, this implies $b, \varepsilon, d{*}f \simeq \varepsilon, \varepsilon, d$. By Lemma 15, these two equations respectively imply

$$\text{(A)} \quad \alpha\psi(\varphi(a{*}b)) - \gamma\psi(\varphi(d{*}f)) = \alpha\psi(\varphi(a)) - \gamma\psi(\varphi(d))$$

and

$$\text{(B)} \quad \alpha\psi(\varphi(b)) - \gamma\psi(\varphi(d{*}f)) = \gamma\psi(\varphi(d)).$$

Subtracting (B) from (A) yields

$$\text{(C)} \quad \alpha\psi(\varphi(a{*}b)) - \alpha\psi(\varphi(b)) = \alpha\psi(\varphi(a)).$$

Canceling the constant $\alpha$ yields the desired result. ▮

The representation theorem now follows immediately from Lemmas 16 and 17 by letting $\theta = \psi \circ \varphi$.

## 6. CONCLUDING REMARKS

It can be shown that the eight axioms on similarity structures are independent. All the axioms except solvability are necessary conditions on the representation theorem.

The axioms provide uniqueness information about the function $\theta$ specified in the representation theorem, namely, that $\theta$ can be analyzed as the composition of functions $\varphi$, $\psi$ enjoying the properties described in Lemmas 7 and 15.

The representation theorem shows that our axioms imply feature additivity in the sense discussed in Section 2. On the other hand, the axioms commit us neither to the uncountability of the feature set $A$ nor to the representation of objects as infinite feature sets. Indeed, there are natural models of the axioms in which $A$ is finite.

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