The Limits of Acyclic Social Choice
and
Nash Implementability
PRELIMINARY AND INCOMPLETE

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Abstract
I prove that there is no systematic rule for aggregating individual preferences that satisfies the standard axioms of Independence, Pareto, and Acyclicity, that avoids making any individual a weak dictator, and that is sensitive to substantial changes in individual preferences. When there are three or more alternatives, the latter axiom requires that a preference reversal in the same direction by one fourth of all individuals is sufficient to break social indifference; when there are four or more alternatives, a higher threshold of one third of all individuals can be used. These results substantially strengthen the acyclicity theorem of Mas-Colell and Sonnenschein (1972). Corresponding to any social choice rule is a ‘‘revealed social preference’’ rule that satisfies Independence and Acyclicity; when we impose monotonicity as well, further axioms on the social choice rule translate to properties of revealed social preferences, and we can apply the acyclicity theorems. We conclude, broadly speaking, that equilibrium outcome correspondences must either concentrate power in small groups or be insensitive to substantial changes in individual preferences.
1 Introduction

Consider a set of agents with possibly heterogenous preferences who must make a collective choice from a given set of alternatives. The problem is to systematically construct a nonempty choice set — a subset of alternatives that may represent normatively appealing choices or, from the analyst's viewpoint, plausible predictions — based on binary comparisons of alternatives. The key condition needed to construct nonempty choice sets is that the binary comparisons, or "social preferences," be acyclic: there should not be a chain of social preferences beginning with one alternative and leading back to it. I provide two results describing the limitations of acyclic social choice. One result maintains the classical assumption that there are three or more alternatives and weakens the standard positive responsiveness axiom of Mas-Colell and Sonnenschein (1972), and the other weakens positive responsiveness even further when there are four or more alternatives. Intuitively, when there are more alternatives, cycles are easier to construct, and the requirement of acyclicity has greater bite.

These results contribute to the social choice theory literature in the tradition of Arrow (1963), but they have implications for the properties of Nash equilibrium outcome correspondences. Specifically, given any social choice rule, we can define a "revealed social preference" rule that satisfies Arrow's independence axiom and generates acyclic social preferences. It is difficult to place further structure on revealed social preferences in general, but Maskin (1977, 1999) has shown that Nash equilibrium correspondences satisfy a condition known as "monotonicity": if an alternative is chosen in one state of the world but not in another, then that alternative must have moved down relative to some other alternative in some agent's ranking. And once monotonicity is imposed, additional axioms on social choices are easily translated to revealed social preferences. This allows us to apply the acyclicity theorems and yields immediate implications for monotonic social choice rules and, as an immediate consequence, for equilibrium correspondences.
To give an overview of the results, I first describe the positive responsiveness axiom of Mas-Colell and Sonnenschein (1972).\footnote{This axiom is also used by May (1954) and Blair et al. (1976).} We assume each agent’s preferences are represented by a weak order of the set of alternatives, and we initially take a profile of weak orders as given. Suppose that two alternatives, say x and y, are socially indifferent. Now change individual preferences so that (i) everyone who preferred x to y still prefers x, (ii) anyone who was indifferent now weakly prefers x, and (iii) there is at least one agent for whom x strictly improved relative to y. By the latter, it is meant that either y was initially strictly preferred and x is now weakly preferred, or the agent was indifferent and now strictly prefers x. Then positive responsiveness demands that as a result of this improvement in the position of x relative to y, x is strictly socially preferred to y. This axiom is restrictive: it requires social preferences to be sensitive to a change in a single agent’s ordering, and it is enough that a strict preference for the agent turns into an indifference (or an indifference turns into a strict preference).\footnote{The condition used by Mas-Colell and Sonnenschein (1972) is stronger than this, because their condition holds whenever x is initially socially at least as good as y (not just when the alternatives are socially indifferent), and it actually implies monotonicity.}

I follow Deb (1981) and Schwartz (1986) in weakening the responsiveness requirement so that a group of agents of some minimum size is required to break social indifference. Obviously, the larger is this minimum size, the weaker is the axiom. Like Schwartz (but unlike Deb), I do not impose neutrality or monotonicity as axioms. Also like Schwartz (but unlike Deb), I focus on the restricted domain in which individual preferences are ‘‘linear,’’ i.e., individual preferences between distinct alternatives are strict. This dulls the axiom, as any preference reversal must be a strict one, and has the effect of strengthening the results.\footnote{The results of the paper hold when all weak orderings are possible: given a social preference rule defined on the unrestricted domain, we can apply our results the smaller linear domain, and then we can argue that they extend when indifferences are allowed.} Letting n denote the number of agents, Schwartz’s (1986) axiom requires
that a preference reversal in a group consisting of at least \( \frac{5}{9} \) members break social indifference, but his result is not tight; I weaken the axiom further so that a preference reversal by one fourth (modulo integer issues) of all agents breaks indifference. In other words, given a case of social indifference, I require that a `vote swing' of \( \frac{5}{9} \) agents is enough to break indifference --- a condition that becomes vastly weaker than positive responsiveness when the number of agents is large. In general, I refer to this condition as `r-Tie Break,' where r is the sensitivity threshold imposed by the axiom.

I show that if there are at least three alternatives and at least five agents, if a social preference rule satisfies the standard Independence and Pareto axioms, Acyclicity, and \( \frac{5}{9} \)-Tie Break, then some agent is a weak dictator: if that agent prefers one alternative to another, then there cannot be a strict social preference in the opposite direction. And with the agreement of \( \frac{5}{9} \) other agents, that weak dictator can actually impose a strict social preference. I paraphrase Theorem 1 next.

Theorem 1 Assume there are at least three alternatives, there are at least five agents, and the possible preferences of each agent are the linear orderings of alternatives. If a social preference rule satisfies Independence, Pareto, Acyclicity, and \( \frac{5}{9} \)-Tie Break, then there is a weak dictator.

Note that the theorem assumes at least five agents out of necessity: it is well-known that majority rule is acyclic (and satisfies the other axioms) when there are three alternatives and four agents. The statement of Theorem 2 is identical, but I assume at least four alternatives and at least four agents; this allows me to weaken the tie-break condition even further to \( \frac{4}{3} \)-Tie Break, so that the axiom now requires a `vote swing' of two thirds of the agents to break social indifference.

I then examine the restrictiveness of monotonicity in light of these acyclicity theorems. First, I reformulate the r-Tie Break
axiom in terms of social choices: if two alternatives, say x and y, are ranked above all others by all agents, if the social choice set contains both, and if we change individual preferences so that x does not move down relative to any alternative in any agent’s preferences and so that r agents reverse their preferences to favor x, then y does not belong to the new choice set. The first theorem on social choice rules also assumes Monotonicity, a necessary condition for Nash implementability (Maskin (1999)), and ‘‘Pareto Consistency,’’ a weakening of the standard Pareto optimality axiom. The conclusion of Theorem 3 is not that a particular agent can unilaterally determine the social choice set, but that there is one agent such that if she prefers any x to any y, and if \( \frac{n}{4} \) or more agents share that preference, then y does not belong to the social choice set.

Theorem 3 Assume there are at least three alternatives, there are at least five agents, and the possible preferences of each agent are the linear orderings of alternatives. If a social choice rule satisfies Monotonicity, Pareto Consistency, and \( \frac{n}{4} \)-Tie Break, then there is an agent who, with the agreement of \( \frac{n}{4} \) others, can reject any alternative.

Theorem 4 is identical, except that it assumes at least four alternatives and at least four agents, it uses the weaker \( \frac{n}{3} \)-Tie Break axiom, and the agreement of \( \frac{n}{3} \) other agents is needed to reject an alternative. Thus, under the mild Pareto Consistency axiom, we conclude that Nash equilibrium outcome correspondences either concentrate power in small groups or are insensitive to substantial changes in individual preferences.

Background: Acyclic Social Choice

The canonical problem of preference aggregation framed by Arrow (1963) is to define a social ranking of alternatives in all possible states of the world and to do so in a way that satisfies appealing or otherwise interesting axioms. The requirement that
social preferences, so-defined, form an ordering of the alternatives is demanding but desirable if the objective is to construct a preference relation that is somehow representative of individual preferences. However, in combination with the Pareto axiom and Arrow’s independence axiom, and assuming there are three or more alternatives and the domain of possible individual preferences is large, the requirement of a social ordering necessitates the existence of a dictator. A less demanding requirement, investigated by Gibbard (1969), is that social preferences form a partial ordering of the set of alternatives. Under this condition it is possible to construct a nonempty choice set given any finite set of alternatives, but the requirement of a social partial ordering still necessitates the existence of an ‘‘oligarchy,’’ a decisive group all members of which can block the choice of one alternative over another. If the oligarchy is small, then power is concentrated in a small group, a small step from dictatorship; the oligarchy could be large (the extreme example being the Pareto extension rule), but then the dispersion of veto power impinges on the ability of the group to discriminate between alternatives --- social choice sets are too big.

More to come.

Background: Nash Implementation

More to come.

2 Preliminaries

Let \( N \) denote a finite set of \( n \) agents, denoted \( i, j, \) etc., considering a set \( X \) of alternatives, denoted \( a, x, \) etc. Let \( \Theta \) denote the set of states of the world, denoted \( \theta, \) which contain information about the agents’ preferences over alternatives. Let \( P_i(\theta) \) denote agent \( i \)’s strict preference relation on \( X \) in state \( \theta, \) and let \( R_i(\theta) \) denote \( i \)'s weak preference relation. Assume that \( P_i(\theta) \) is
asymmetric and negatively transitive, that $R_i(\theta)$ is complete and transitive, and that these relations are dual: for all $x, y \in X$, $x P_i(\theta) y$ if and only if not $y R_i(\theta) x$.\(^4\) Let

\[
P(x, y|\theta) = \{i \in N \mid x P_i(\theta) y\}
\]
\[
R(x, y|\theta) = \{i \in N \mid x R_i(\theta) y\}
\]

denote the set of agents who strictly and weakly, respectively, prefer $x$ to $y$.

Letting $P(\theta) = (P_1(\theta), \ldots, P_n(\theta))$ denote the profile of strict preference relations in state $\theta$, we say **Unrestricted Domain** holds if

\[
P(\Theta) = \left\{ (P_1, \ldots, P_n) \mid \text{for all } i, P_i(\theta) \text{ is an asymmetric and negatively transitive relation on } X \right\},
\]

i.e., all profiles of weak orders are possible. We say a strict preference relation $P_i$ is **total** if for all distinct $x$ and $y$, either $x P_i y$ or $y P_i x$; this precludes indifference between two alternatives. We then say **Linear Domain** holds if

\[
P(\Theta) = \left\{ (P_1, \ldots, P_n) \mid \text{for all } i, P_i(\theta) \text{ is an asymmetric, total, and negatively transitive relation on } X \right\},
\]

i.e., all profile of linear orders are possible. In this paper, I focus mainly on Linear Domain, as results proved on the smaller domain immediately extend to the unrestricted framework.

### 3 Acyclic Social Choice

A **social preference rule**, denoted $F$, is a mapping $\theta \mapsto (P_F(\theta), R_F(\theta))$ defined on the set $\Theta$ of states, where $P_F(\theta)$ is an asymmetric strict social preference relation, $R_F(\theta)$ is a complete weak social preference relation, and these relations are dual: for all $x, y \in X$,

\(^4\)Equivalently, $y R_i(\theta) x$ if and only if not $x P_i(\theta) y$. Obviously, the properties of each version of preference can be derived from the other.
A special case of interest is that of a *quota rule*, where social preferences are completely specified by a single parameter $q > \frac{n}{2}$ as follows: $x P_F(\theta) y$ if and only if $|P(x,y|\theta)| \geq q$. We gain *simple majority rule* as a further special case in which $q = \lceil \frac{n+1}{2} \rceil$. It is instructive to contrast the latter rule with another version of majority rule, *relative majority rule*, in which case

$$x P_F(\theta) y \text{ if and only if } \frac{|P(x,y|\theta)|}{|P(x,y|\theta)| + |P(y,x|\theta)|} > \frac{1}{2},$$

where here we adopt the convention that $\frac{0}{0} = \frac{1}{2}$. The two versions of majority rule are equivalent when Linear Domain holds, but under Unrestricted Domain, the latter is strictly more permissive.

This paper investigates the consistency of a number of axioms on social preference rules. The first class of axioms impose minimal levels of sensitivity with respect to changes in the preferences of individual agents. A classical axiom, called *Positive Responsiveness* by May (1954) and Mas-Colell and Sonnenschein (1972), is that the following holds for all states $\theta$ and $\theta'$ and all alternatives $x$ and $y$:

$$\begin{align*}
P(x,y|\theta) &\subseteq P(x,y|\theta'), \\
R(x,y|\theta) &\subseteq R(x,y|\theta'), \\
\text{at least one inclusion strict,} \\
\text{and } x I_F(\theta) y
\end{align*} \Rightarrow x P_F(\theta') y.$$

Intuitively, the condition says that if two alternatives are socially indifferent, and an agent changes her preferences in favor of one alternative, then this change should break the social indifference in favor of the alternative with strengthened support. The condition is quite strong in two respects. First, a change in the preference of only one agent is sufficient to break the tie. Second, even if the agent initially was indifferent and changes to a strict preference (or initially has a strict preference and changes to indifference), this change is sufficient to break a tie; in other words, a strict preference reversal is not required to fulfill the antecedent condition of the axiom. It is well-known
from May's (1954) theorem that among anonymous, neutral, and monotonic social preference rules, the Positive Responsiveness axiom is uniquely satisfied on the Unrestricted Domain by relative majority rule and, therefore, on the Linear Domain by (what is the same) simple majority rule.

Given an integer r satisfying $0 < r < n$, we say $F$ satisfies $r$-Tie Break if the following holds for all states $\theta$ and $\theta'$ and all alternatives $x$ and $y$:

\[
\begin{align*}
P(x, y|\theta) &\subseteq P(x, y|\theta'), \\
R(x, y|\theta) &\subseteq R(x, y|\theta'), \\
|P(y, x|\theta) \cap P(x, y|\theta')| &\geq r, \\
\text{and } x I_F(\theta) y
\end{align*}
\]

In words, if $x$ and $y$ are socially indifferent in one state, and we consider another state in which no agents have changed their preferences to favor $y$ and in which $r$ or more agents have reversed a strict preference for $y$ to a strict preference for $x$, then $x$ is strictly socially preferred to $y$ at the new state. Obviously, the Tie Break axiom is less restrictive when the sensitivity threshold $r$ is larger, the most restrictive case of the condition being 1-Tie Break. The latter condition is clearly related to Positive Responsiveness: under Unrestricted Domain, 1-Tie Break is strictly weaker than Positive Responsiveness, while under Linear Domain, the two conditions are equivalent. Note also that under Linear Domain, the antecedent assumption $R(x, y|\theta) \subseteq R(x, y|\theta')$ in the above tie break conditions is redundant, so the conditions are equivalently stated without it.

In this paper, I consider tie-break thresholds $r = \lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{3} \rfloor$. Obviously, when the number $n$ of agents is large, and when Positive Responsiveness becomes most restrictive, these thresholds become relatively lax. Most closely related to the sensitivity conditions I propose is Schwartz's (1986) 'weak non-blocker,' which is essentially $\lfloor \frac{n}{5} \rfloor$-Tie Break. As a yardstick to compare these axioms, suppose Linear Domain holds, and note (ignoring integer issues) that the 'worst case' for a quota rule with quota $q$ is

\footnote{See Schwartz's (1986) 'Really General Impossibility Theorem.'}
a state in which \( n - q + 1 \) agents prefer \( x \) and \( q - 1 \) agents prefer \( y \), and \( r \) members of the latter group reverse their preferences; this reversal breaks the tie as long as \( n - q + r + 1 \geq q \). Thus, the quota rule satisfies \( r \)-Tie Break if and only if \( r \geq 2q - n \), or equivalently \( q \leq \frac{n + r + 1}{2} \). So 1-Tie Break (equivalently, Positive Responsiveness) is satisfied only by simple majority rule. Schwartz’s \( \left\lfloor \frac{n}{5} \right\rfloor \)-Tie Break is satisfied for any quota up to \( q = \frac{5}{9}n \), the weaker \( \left\lfloor \frac{n}{4} \right\rfloor \)-Tie Break is satisfied for quotas up to \( q = \frac{5}{8}n \), and the weakest \( \left\lfloor \frac{n}{3} \right\rfloor \)-Tie Break is satisfied for quotas up to two thirds, \( q = \frac{2}{3}n \).

As is standard, an agent \( i \) is a dictator if for all \( \theta \) and all \( x \) and \( y \), \( x P_i(\theta) y \) implies \( x P_F(\theta) y \). We say \( i \) is a weak dictator if for all \( \theta \) and all \( x \) and \( y \), \( x P_i(\theta) y \) implies \( x R_F(\theta) y \). Thus, a weak dictator’s authority is limited in the sense that a strict preference on her part precludes the opposite strict social preference, without necessitating a strict social preference in the same direction. Furthermore, \( i \) is an \( r \)-dictator if she is a weak dictator and for all groups \( G \subseteq N \setminus \{i\} \) with \( |G| \geq r \), all \( \theta \), and all \( x \) and \( y \), \( \{i\} \cup G \subseteq \{x,y|\theta\} \) implies \( x P_F(\theta) y \). Thus, an \( r \)-dictator can not only block a strict social preference herself, but can impose a strict preference with the agreement of \( r \) other agents. A dictator is then the same thing as a 0-dictator. In obvious fashion, we say a social preference rule satisfies the axiom of No \( r \)-Dictator if no agent is an \( r \)-dictator.

The other axioms used in the sequel are standard. We say a social preference rule \( F \) satisfies...

- **Independence** if for all \( \theta \) and \( \theta' \) and all \( x \) and \( y \),
  \[
  \begin{align*}
  P(x,y|\theta) &= P(x,y|\theta'), \\
  P(y,x|\theta) &= P(y,x|\theta'), \\
  \text{and } x P_F(\theta) y
  \end{align*}
  \]
  \[\Rightarrow\] \( x P_F(\theta') y \).

- **Pareto** if for all \( \theta \) and all \( x \) and \( y \), \( P(x,y|\theta) = N \) implies \( x P_F(\theta) y \).

- **Acyclicity** if for all \( \theta \), all natural numbers \( k \), and all se-
lections $x_1, \ldots, x_k$ of $k$ alternatives, it is not the case that
\[ x_1 P_F(\theta) x_2 P_F(\theta) \cdots x_k P_F(\theta) x_1. \]

In words, respectively, the social preference between two alternatives depends only on the agents' preferences over those two alternatives; a common strict preference of the agents is inherited by the social preference relation; and the social preference rule does not admit cycles. It is well-known that the latter condition of Acyclicity is necessary and sufficient for the existence of maximal elements in all finite sets of alternatives at every state.

We can now state the main result on the limitations of acyclic social choice. In keeping with the Arrovian tradition, we allow for any number of three or more alternatives, and (as majority rule is acyclic when $n = 4$) we accordingly assume at least five agents.

Theorem 1 Assume $|X| \geq 3$, $n \geq 5$, and Linear Domain. There does not exist a social preference rule satisfying Independence, Pareto, Acyclicity, No $\lfloor \frac{n}{4} \rfloor$-Dictator, and $\lfloor \frac{n}{4} \rfloor$-Tie Break.

The question arises: can the $\lfloor \frac{n}{4} \rfloor$-Tie Break axiom be further weakened? As the next result will show, this axiom can indeed be weakened if we know there are at least four alternatives. Thus, an ideal response to this question would establish consistency of the axioms for three alternatives, any number of agents, and any $r \leq \lfloor \frac{n}{3} \rfloor - 1$. That is, I would construct, for three alternatives and any number of agents, a social preference rule that satisfies Independence, Pareto, Acyclicity, No $(\lfloor \frac{n}{3} \rfloor - 1)$-Dictator, and $(\lfloor \frac{n}{3} \rfloor - 1)$-Tie Break. What I provide is short of that: I show that the level of responsiveness cannot be weakened to $\lfloor \frac{n}{3} \rfloor$. Indeed, I will assume the set of alternatives is ordered by $\succeq$, and I assume for simplicity that $n$ is divisible by three. The rule is defined as follows. Given $x$ and $y$, if $x \succeq y$, then we specify that $x P_F(\theta) y$ when

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6When $X$ the cardinality of $X$ is less than or equal to the cardinality of the continuum, this assumption is unrestrictive; more generally, the existence of such an ordering (in fact, a well-ordering) follows from the axiom of choice.
This clearly satisfies Independence, Pareto, Acyclicity, and No \( \lfloor \frac{n}{3} \rfloor \)-Dictator. A moment’s consideration shows that it satisfies \( \lfloor \frac{n}{3} \rfloor \)-Tie Break as well. Indeed, assume \( x \mathrel{I}_F \theta y \), the difficult cases being \( |P(x,y|\theta)| = \frac{n}{3} - 1 \) and \( |P(y,x|\theta)| = \frac{n}{3} \). Consider the former, as the argument is similar in each case. If \( \frac{n}{3} \) agents reverse their preference for \( x \) to now favor \( y \) in \( \theta' \), then we have \( |P(y,x|\theta)| = \frac{n}{3} + 1 \), so \( y \mathrel{P}_F (\theta') x \), as required.

If we now assume there are at least four alternatives, then, intuitively, cycles become easier to construct, and the axiom of Acyclicity has greater bite. This allows us to weaken other assumptions; in particular, we can now allow for any number of four or more agents (majority rule is acyclic if \( n = 3 \)), and we can weaken the tie-break condition to \( \lfloor \frac{n}{3} \rfloor \)-Tie Break.

**Theorem 2** Assume \( |X| \geq 4 \), \( n \geq 4 \), and Linear Domain. There does not exist a social preference rule satisfying Independence, Pareto, Acyclicity, No \( \lfloor \frac{n}{3} \rfloor \)-Dictator, and \( \lfloor \frac{n}{3} \rfloor \)-Tie Break.

As mentioned above, the results extend when we allow arbitrary individual indifferences.

**Corollary 1** The results of Theorems 1 and 2 hold under Unrestricted Domain.

**Proof** Consider a social preference rule \( F \) defined on the domain of all profiles of weak orders and satisfying Independence, Pareto, Acyclicity, and \( r \)-Tie Break for \( r \in \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{2} \rfloor \). Let \( F' \) denote the restriction of \( F \) to the Linear Domain, and note that \( F' \) satisfies the same four axioms. Now further suppose that \( F' \) violates No \( r \)-Dictator, so that some agent \( i \) is an \( r \)-dictator. I claim that \( F \) violates No \( r \)-Dictator, which proves the corollary. We first prove that \( i \) is a weak dictator for \( F \). Consider any \( \theta \) and any \( x \) and \( y \) such that \( x \mathrel{P}_i(\theta) y \), and suppose \( y \mathrel{P}_F(\theta) x \). Now let \( z \) be distinct from \( x \) and \( y \), and consider \( \theta' \) with individual preferences over these three
alternatives as follows, where we partition $N \setminus \{i\}$ into groups $G$ and $H$ such that $G, H \geq r$.

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Here, "$x, y?$" indicates that individual preferences between $x$ and $y$ are as in $\theta$. Of course, there exists $\theta''$ such that $P_j(\theta'')$ is a linear order for all $j$ and $P(x, z|\theta'') = P(x, z|\theta')$ and $P(z, y|\theta'') = P(z, y|\theta')$. Since $i$ is an $r$-dictator for $F'$, we then have $x P_{F'}(\theta'') z$ and $z P_{F'}(\theta'') y$. Since $F'$ is the restriction of $F$, this implies $x P_{F}(\theta') z$ and $z P_{F}(\theta') y$. By Independence, we then have $x P_{F}(\theta') z P_{F}(\theta') y P_{F}(\theta') x$, contradicting Acyclicity. We conclude that $i$ is a weak dictator for $F$.

Now consider $\theta$, $x$, and $y$ such that $x P_i(\theta) y$ and, letting $G = P(x, y|\theta) \setminus \{i\}$, $|G| \geq r$. Suppose that $y R_F(\theta) x$. I claim that $y I_F(\theta) x$, for suppose $y P_F(\theta) x$. Let $z$ be distinct from $x$ and $y$, and consider $\theta'$ with individual preferences over these three alternatives as follows, where $H = N \setminus (G \cup \{i\})$.

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Again, there exists $\theta''$ such that $P_j(\theta'')$ is a linear order for all $j$ and $P(x, z|\theta'') = P(x, z|\theta')$ and $P(z, y|\theta'') = P(z, y|\theta')$. Since $i$ is an $r$-dictator for $F'$, we then have $x P_{F'}(\theta'') z$ and $z P_{F'}(\theta'') y$, and this leads to a violation of Acyclicity of $F$, as above. We conclude that $y I_F(\theta) x$. Now consider $\theta$ such that $x P_i(\theta) y$ and $P(y, x|\theta) = N \setminus \{i\}$. Note that the antecedent of $r$-Tie Break is satisfied, and in particular that $|P(y, x|\theta) \cap P(x, y|\theta)| \geq |G| \geq r$. Therefore, we have $y P_F(\theta) x$, contradicting the fact that $i$ is a weak dictator. Q.E.D.

We conclude that in the binary framework, in which we model social choices via a social preference rule, the problem of collective choice is deep: either social preferences must be insensitive
to substantial changes in individual preferences, or a significant amount of authority must be afforded a single agent. In the next section, we will see that the analysis of acyclic social choice has restrictive implications well beyond the binary framework.

4 Nash Implementability

A social choice rule, denoted $f$, is a mapping $\theta \mapsto f(\theta)$, where $f(\theta)$ is a non-empty subset of $X$. A common interpretation is that $f(\theta)$ is the set of collective choices that satisfy some normative criterion, but an alternative interpretation is that $f(\theta)$ is the set of alternatives that are plausible predictions under some positive criterion, such as Nash equilibrium. Following the above, an agent $i$ is a dictator if for all $\theta$, all $x \in f(\theta)$, and all $y \in X$, we have $x R_i(\theta) y$. We say $f$ satisfies...

- **Resoluteness** if for all $\theta$, $|f(\theta)| = 1$.
- **Full Range** if $\bigcup_{\theta \in \Theta} f(\theta) = X$.
- **No Dictator** if no $i$ is a dictator.

We define a game form as a pair $(S, g)$, where $S = S_1 \times \cdots \times S_n$ is an $n$-fold product of sets, each factor $S_i$ is a set of strategies available to agent $i$, and $g: S \rightarrow X$ is an outcome function mapping strategy profiles to alternatives. As is standard, we write $s = (s_1, \ldots, s_n)$ for a strategy profile and $s_{-i}$ for a profile of strategies for all agents but $i$. Then $s$ is a (pure strategy) Nash equilibrium in state $\theta$ if for all $i \in N$ and all $s_i' \in S_i$, we have $g(s) R_i(\theta) g(s_i', s_{-i})$; and we write $N(S, g)(\theta)$ for the set of Nash equilibria in $\theta$. Finally, the game form $(S, g)$ Nash implements $f$ if for all $\theta \in \Theta$, we have $f(\theta) = g(N(S, g)(\theta))$, i.e., if the social choice set matches the set of Nash equilibrium outcomes of the game form $(S, g)$ in every state, and $f$ is Nash implementable if there is some game form that Nash implements it. Put succinctly, a social choice rule is Nash implementable if it is the equilibrium outcome correspondence for some game form.
By studying implementable social choice rules, we may therefore derive restrictions on equilibrium behavior independent of the particular game form under consideration. Maskin (1977, 1999) has shown, for example, that every Nash implementable social choice rule satisfies Monotonicity: for all \( \theta, \theta' \in \Theta \) and all \( x \in f(\theta) \setminus f(\theta') \), there exists \( i \in N \) and \( y \in X \) such that \( x R_i(\theta)y \) and \( y P_i(\theta')x \). In words, if \( x \) is a viable social choice in state \( \theta \) but not in \( \theta' \), then there must be a preference reversal involving \( x \) and another alternative, where \( x \) is weakly preferred in \( \theta \) but not in \( \theta' \). This opens an avenue for the general analysis of equilibrium behavior by deducing implications of Monotonicity, and in fact, we know via this logic that when the domain of preferences is large, a game form admits a unique equilibrium in each state only under very restrictive conditions. Indeed, assuming at least three alternatives and Linear Domain, Muller and Satterthwaite (1977) show that there is no social choice rule satisfying Resoluteness, Monotonicity, Full Range, and No Dictator; when the preference domain is expanded to Unrestricted Domain, Saijo (1987) has established the even starker implication that if a social choice rule is Resolute and Monotonic, then it is constant.

The analysis of acyclic social preferences in the preceding section has immediate implications for implementable social choice rules beyond those satisfying the strong Resoluteness condition. To draw these implications, we define conditions on social choice rules paralleling No \( r \)-Dictator and \( r \)-Tie Break, as well as a weakening of the Full Range condition. Given a set \( Y \) of alternatives, let \( \Theta^Y \) denote the set of states in which all agents prefer every alternative in \( Y \) to every alternative outside \( Y \):

\[
\Theta^Y = \{ \theta \mid \forall i \in N : \forall y \in Y : \forall z \in X \setminus Y : y P_i(\theta) z \}.
\]

We say \( f \) satisfies \( r \)-Tie Break if the following holds for all \( x \) and \( y \) and all \( \theta, \theta' \in \Theta^{(x,y)} \):

\[
\begin{align*}
\{ &P(x,y|\theta) \subseteq P(x,y|\theta'), \\
&\{x,y\} \subseteq f(\theta) \setminus f(\theta'), \\
&|P(y,x|\theta) \cap P(x,y|\theta')| \geq r, \\
&\Rightarrow y \notin f(\theta').
\end{align*}
\]
In words, if \( x \) and \( y \) are ranked above all other alternatives by all agents, if the social choice set contains both, and if we consider another state in which \( x \) and \( y \) are still ranked above all other alternatives by all agents, in which no agents have changed their preferences to prefer \( x \) over \( y \), and in which \( r \) or more agents have reversed their preferences to favor \( x \), then \( y \) is no longer socially viable. Obviously, under Linear Domain the condition can be simplified by omitting the antecedent requirement that \( R(x, y|\theta) \subseteq R(x, y|\theta') \). An agent \( i \) is an \( r \)-dictator if for all \( \theta \), all \( x \) and \( y \), and all groups \( G \subseteq N \setminus \{i\} \) with \( |G| \geq r \), \( \{i\} \cup G \subseteq P(x, y|\theta) \) implies \( y \notin f(\theta) \); thus, an \( r \)-dictator can reject any alternative with the support of \( r \) or more other agents. We say \( \text{No } r \text{-Dictator} \) holds if no agent is an \( r \)-dictator. A social choice rule \( f \) satisfies \text{Pareto Consistency} if (i) for all finite \( Y \subseteq X \) and all \( \theta \in \Theta^Y \), we have \( f(\theta) \cap Y \neq \emptyset \), and (ii) for all \( x \) and all \( \theta \in \Theta^x \), we have \( f(\theta) = \{x\} \). This is obviously weaker than the usual Pareto optimality condition, which would require that whenever some alternative is strictly preferred to \( x \) by all agents, the social choice set does not contain \( x \).

The analysis of social choices is connected to the analysis of social preference by the following version of revealed preference: given a state \( \theta \), we define \( x \preceq_f(\theta) y \) if and only if for all \( \theta' \),

\[
\begin{align*}
P(x, y|\theta) &= P(x, y|\theta') \\
R(y, x|\theta) &= R(y, x|\theta') \quad \Rightarrow \quad y \notin f(\theta').
\end{align*}
\]

As long as \( f \) is Pareto consistent, the revealed preference relation \( P_f(\theta) \) is asymmetric: given \( x \) and \( y \) and state \( \theta \), we may choose a state \( \theta' \in \Theta^{(x,y)} \) such that individual preferences between \( x \) and \( y \) are unchanged; then part (i) of Pareto Consistency requires either \( x \in f(\theta') \), in which case not \( y \in f(\theta) \), or \( y \in f(\theta') \), in which case not \( x \in f(\theta') \). Defining \( x \preceq_f(\theta) y \) to hold if there exists \( \theta' \) such that \( P(x, y|\theta) = P(x, y|\theta') \), \( R(x, y|\theta) = R(x, y|\theta') \), and \( x \in f(\theta') \), we have defined a social preference rule \( \theta \mapsto (P_f(\theta), R_f(\theta)) \). As usual, we define \( x \succeq_f(\theta) y \) if and only if \( x \preceq_f(\theta) y \) and \( y \preceq_f(\theta) x \). This social preference rule satisfies IIA by construction. Furthermore, it satisfies Acyclicity: given any finite set \( Y \) of alternatives, choose \( \theta' \in \Theta^Y \) and apply part (i) of Pareto Consistency to conclude
that some alternative is maximal in \( Y \), i.e., there is no \( P_2(\theta) \)-
cycle through \( Y \). Note that these properties hold for an arbitrary
social choice rule, regardless of whether it is Monotonic.

The goal is to show that Pareto Consistency and Monotonicity
are inconsistent with our Tie Break and No \( r \)-Dictator conditions.
Of course, the strategy of proof is to show that when \( f \) is Mono-
tonic, the revealed preference rule inherits key properties from
\( f \), allowing us to apply Theorems 1 and 2 on acyclic social prefer-
ences.

**Lemma 1** Assume Linear Domain or Unrestricted Domain. If a social
choice rule \( f \) satisfies Monotonicity, Pareto Consistency, No \( r \)-
Dictator, and \( r \)-Tie Break, then the revealed preference rule \( \theta \mapsto
(P_f(\theta), R_f(\theta)) \) satisfies Pareto, No \( r \)-Dictator, and \( r \)-Tie Break.

**Proof** Assume \( f \) satisfies Monotonicity, Pareto Consistency, No
\( r \)-Dictator, and \( r \)-Tie Break. To establish that the revealed pref-
erence rule satisfies Pareto, consider \( \theta, x, \) and \( y \) such that
\( P(x,y|\theta) = N \). If not \( x P_f(\theta) y \), then there exists \( \theta' \) such that
\( P(x,y|\theta') = P(x,y|\theta) \) and \( y \in f(\theta') \). Let \( \theta'' \in \Theta_{\{x,y\}} \) be a state such
that individual preferences between \( x \) and \( y \) are unchanged with
those alternatives above all others. Then Monotonicity implies
that \( y \in f(\theta'') \), but note that \( \theta'' \in \Theta_{\{x\}} \), so that part (ii) of Pareto
Consistency implies \( y \not\in f(\theta'') \), a contradiction. This establishes
Pareto.

To prove No \( r \)-Dictator, suppose that there exists an \( r \)-dictator
\( i \) for the revealed preference rule; in particular, for all groups
\( G \subseteq N \setminus \{i\} \) with \( |G| \geq r \), all \( \theta \), and all \( x \) and \( y \), \( \{i\} \cup G \subseteq P(x,y|\theta) \)
implies \( x P_f(\theta) y \). I claim that \( i \) is an \( r \)-Dictator for \( f \). Indeed,
consider any \( \theta \), any \( x \) and \( y \), and any \( G \subseteq N \setminus \{i\} \) with \( |G| \geq r \) such
that \( \{i\} \cup G \subseteq P(x,y|\theta) \). By supposition, \( x P_f(\theta) y \), which implies
\( y \not\in f(\theta) \), as required.

To prove \( r \)-Tie Break, consider any \( \theta \) and \( \theta' \) and \( x \) and \( y \) such that
\( P(x,y|\theta) \subseteq P(x,y|\theta') \), \( R(x,y|\theta) \subseteq R(x,y|\theta') \), \( |P(x,y|\theta) \cap P(x,y|\theta')| \geq r \), \( r \),
and \( x I_f(\theta) y \). Revealed indifference implies that there exist \( \theta_x \) and
\( \theta_y \) such that \( P(x,y|\theta_x) = P(x,y|\theta_y) = P(x,y|\theta) = R(x,y|\theta_x) =
R(x,y|\theta_y) =

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R(x,y|θ), x ∈ f(θ_x), and y ∈ f(θ_y). Let \( \hat{\theta} \in \Theta^{x,y} \) be a state such that individual preferences between x and y are unchanged from \( \theta \) (and \( \theta_x \) and \( \theta_y \)) with those alternatives above all others; then Monotonicity implies \( \{x,y\} \subseteq f(\hat{\theta}) \). Let \( \theta' \in \Theta^{x,y} \) be a state in which individual preferences between x and y are unchanged from \( \theta' \) with those alternatives above all others, and note that P(x,y|\( \hat{\theta} \)) \( \subseteq P(x,y|\theta') \), R(x,y|\( \hat{\theta} \)) \( \subseteq R(x,y|\theta') \), and \( |P(x,y|\theta) \cap P(x,y|\theta')| \geq r \); then r-Tie Break implies y\( \not\in f(\theta') \). To complete the proof, suppose that y R_x(\( \theta' \)) x, so there exists \( \theta'' \) such that P(x,y|\( \theta'' \)) = P(x,y|\( \theta' \)) = P(x,y|\( \theta'' \)), R(x,y|\( \theta'' \)) = R(x,y|\( \theta' \)) = R(x,y|\( \theta'' \)), and y \( \in f(\theta'') \); but then Monotonicity implies y \( \in f(\theta') \), a contradiction.

Q.E.D.

With the above lemma, and the fact (noted above) that the revealed preference rule satisfies IIA and Acyclicity, the proofs of the next results are immediate. Note that, using Corollary 1, we can state the results for either Linear or Unrestricted Domain. The analysis has direct implications for equilibrium outcome correspondences: if the equilibrium outcomes of a game form satisfy Pareto Consistency, then the game form must either concentrate power in small groups containing a particular agent (i.e., violate No r-Dictator) or be insensitive to significant changes in individual preferences (i.e., violate r-Tie Break). We first allow for any number of three or more alternatives.

**Theorem 3** Assume \( |X| \geq 3, \ n \geq 5 \), and Linear or Unrestricted Domain. There does not exist a social choice rule satisfying Monotonicity, Pareto Consistency, No \( \lfloor \frac{n}{4} \rfloor \)-Dictator, and \( \lfloor \frac{n}{4} \rfloor \)-Tie Break.

As expected, we can weaken the tie break condition if we know there are at least four alternatives.

**Theorem 4** Assume \( |X| \geq 4, \ n \geq 4 \), and Linear or Unrestricted Domain. There does not exist a social choice rule satisfying Monotonicity, Pareto Consistency, No \( \lfloor \frac{n}{3} \rfloor \)-Dictator, and \( \lfloor \frac{n}{3} \rfloor \)-Tie Break.

Duggan and Schwartz (1995) prove a result related to Theorem 3 assuming a stronger tie break condition (which requires singleton
social choice sets in some situations) and imposing a ‘‘minimal resoluteness’’ condition: if two alternatives, say \( x \) and \( y \), are preferred to all others by all agents, and if all or all but one agent prefers \( x \) to \( y \), then the social choice set is a singleton.\(^7\)

With the latter condition, they weaken Pareto Consistency to Full Range, and they weaken No r-Dictator to No Dictator. In contrast, the only trace of resoluteness in the conditions of Theorems 3 and 4 is in part (ii) of Pareto Consistency, which only applies when one alternative is preferred to all others by all agents.

5 Proofs of Acyclicity Theorems

We begin with lemmas that will lay the groundwork for the proofs of all theorems. Throughout this section, assume that \( r \in \{\lfloor \frac{n}{4} \rfloor,\lfloor \frac{n}{3} \rfloor\} \), that Linear Domain holds, and that there exists a social preference rule \( F \) satisfying Independence, Pareto, P-Acyclicity, No r-Dictator, and r-Tie Break. Regarding the number of agents, we assume that \( n \geq 4 \); furthermore, if \( r = \lfloor \frac{n}{4} \rfloor \), then we assume \( n \geq 5 \). Regarding the number of alternatives, we assume that \(|X| \geq 3\); furthermore, if \( r = \lfloor \frac{n}{3} \rfloor \), then we assume \(|X| \geq 4\). Define

\[
\mu(r) = \left\lfloor \frac{n - 2r}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor - r,
\]

so that \( \mu(\lfloor \frac{n}{4} \rfloor) = \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{4} \rfloor \) and \( \mu(\lfloor \frac{n}{3} \rfloor) = \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor \). It follows that

\[
\mu\left(\frac{n}{4}\right) \geq \frac{n}{4} \quad \text{and} \quad \mu\left(\frac{n}{3}\right) \geq \frac{n}{6}.
\]

Define

\[
\mathcal{G}^-(r) = \{ G \subseteq N \mid |G| \leq \mu(r) \}, \quad \mathcal{G}^+(r) = \{ G \subseteq N \mid |G| > r \}
\]

as the collections of groups with no more members than \( \mu(r) \) and no fewer members than \( r \), respectively.

\(^7\)See also Duggan and Schartz (2000) for an application of minimal resoluteness to strategy-proof social choice.
Given a group \( G \) and distinct alternatives \( x \) and \( y \), we say \( G \) is *semi-decisive for \( x \) over \( y \)*, written \( x \overset{G}{\rightarrow} y \), if for all \( \theta \) such that \( P(x,y|\theta) = G \), we have \( x \overset{F}{\rightarrow} y \). We say \( G \) is *semi-decisive* if for all distinct \( x \) and \( y \), \( G \) is semi-decisive for \( x \) over \( y \). Then \( G \) is *decisive* if \( G \) and all supersets of \( G \) are semi-decisive. In parallel fashion, given distinct \( x \) and \( y \), we say \( G \) is *semi-blocking for \( x \) over \( y \)*, written \( x \overset{G}{\leftarrow} y \), if for all \( \theta \) such that \( P(x,y|\theta) = G \), we have \( x \overset{F}{\leftarrow} y \). We say \( G \) is *semi-blocking* if for all distinct \( x \) and \( y \), \( G \) is semi-blocking for \( x \) over \( y \). Then \( G \) is *blocking* if \( G \) and all supersets of \( G \) are semi-blocking.

**Lemma 2** For all distinct \( x \) and \( y \) and all \( \theta \),

(i) \( x \overset{F}{\rightarrow} y \) if and only if \( x \overset{P(x,y|\theta)}{\rightarrow} y \)

(ii) \( x \overset{F}{\leftarrow} y \) if and only if \( x \overset{P(x,y|\theta)}{\leftarrow} y \)

(iii) for all \( G \subseteq N \), either \( x \overset{G}{\rightarrow} y \) or \( y \overset{N \setminus G}{\rightarrow} x \).

**Proof** Parts (i) and (ii) follow immediately from Linear Domain and Independence. For part (iii), take any \( G \subseteq N \). By Linear Domain, there exists \( \theta \in \Theta \) such that \( P(x,y|\theta) = G \), and therefore \( P(y,x|\theta) = N \setminus G \). By duality, either \( x \overset{F}{\rightarrow} y \) or \( y \overset{F}{\leftarrow} x \). In the first case, part (i) implies \( x \overset{G}{\rightarrow} y \), and in the second case, part (ii) implies \( y \overset{N \setminus G}{\rightarrow} x \). Q.E.D.

The next lemma is implied by a result of Ferejohn and Fishburn (1979). I provide a self-contained proof.

**Lemma 3** There do not exist natural number \( k \), alternatives \( x_1, \ldots, x_k \), and groups \( G_1, \ldots, G_k \) such that

(i) \( x_h \overset{G_h}{\rightarrow} x_{h+1} \) for all \( h = 1, \ldots, k - 1 \) and \( x_k \overset{G_k}{\rightarrow} x_1 \),

(ii) \( \bigcup_{h=1}^{k} G_h = N \),

(iii) \( \bigcap_{h=1}^{k} G_h = \emptyset \).
Proof Suppose there exist such \(k\), \(x_1, \ldots, x_k\), and \(G_1, \ldots, G_k\). Letting \(\phi^h_1 = G_h\) and \(\phi^{-1}_h = N \setminus G_h\), partition \(N\) into

\[
\tilde{\mathcal{G}} = \left\{ \bigcap_{h=1}^{k} \phi^h_{\alpha_h} \mid (\alpha_1, \ldots, \alpha_k) \in \{1, -1\}^k \right\}.
\]

Given arbitrary non-empty \(H \in \tilde{\mathcal{G}}\), let \(\alpha(H) \in \{1, -1\}^k\) be the unique vector satisfying

\[
H = \bigcap_{h=1}^{k} \phi^h_{\alpha_h(H)},
\]

where \(\alpha_h(H)\) is the \(h\)th coordinate of \(\alpha(H)\).

Now construct an asymmetric, total, and transitive relation \(P_H\) on the set \(\{x_1, \ldots, x_k\}\) as follows. First, given any \(\alpha \in \{-1, 1\}\) and any \(h \in \{1, \ldots, k\}\), define the singleton relation

\[
Q^\alpha_h = \begin{cases} 
(x_h, x_{h+1}) & \text{if } \alpha = 1 \\
(x_{h+1}, x_h) & \text{if } \alpha = -1,
\end{cases}
\]

where henceforth \(h+1\) will implicitly mean \(h+1\) modulo \(k\). Define the asymmetric relation

\[
S_H = \bigcup_{h=1}^{k} Q^\alpha_{\alpha_h(H)}.
\]

In words, \(x_h S_H x_{h+1}\) if \(H\) is a subset of \(G_h\), and \(x_{h+1} S_H x_h\) if \(H\) is disjoint from \(G_h\). Note that the relation \(S_H\) is asymmetric, i.e., for all \(h \in \{1, \ldots, k\}\), it is not the case that both \(x_h S_H x_{h+1}\) and \(x_{h+1} S_H x_h\); furthermore, \(S_H\) is total, i.e., for all \(h \in \{1, \ldots, k\}\), either \(x_h S_H x_{h+1}\) or \(x_{h+1} S_H x_h\); and finally, \(S_H\) is vacuous on non-adjacent alternatives, i.e., for all \(h \in \{1, \ldots, k\}\) and all \(\ell \notin \{h-1, h+1\}\), it is not the case that \(x_h S_H x_\ell\).

Second, letting \(T_H\) denote the transitive closure of \(S_H\), I claim that \(T_H\) is asymmetric. If not, then \(S_H\) admits a cycle, and there exist a natural number \(\ell\) and an enumeration of \(\ell\) elements of \(\{x_1, \ldots, x_k\}\) such that

\[
y_1 S_H y_2 \cdots y_{\ell-1} S_H y_\ell S_H y_1.
\]
We may assume without loss of generality that \( y_1 = x_1 \), so that either \( y_2 = x_2 \) or \( y_2 = x_k \). In the first case, by construction we then have \( \ell = k \) and \( y_h = x_h \) for all \( h = 1, \ldots, k \). This implies that \( \alpha_h(H) = 1 \) for all \( h = 1, \ldots, k \), i.e., \( H = \bigcap_{h=1}^{k} G_h \), which is empty by (iii), contradicting \( H \neq \emptyset \). In the second case, a similar argument leads to a contradiction using (ii), establishing the claim. Thus, \( T_H \) is asymmetric and transitive. An implication is that \( T_H \) only adds arcs between alternatives that are not adjacent in the enumeration \( x_1, \ldots, x_k, x_1 \), i.e.,

\[
T_H \setminus S_H \subseteq \left\{ (x,y) \mid \text{there does not exist } h = 1, \ldots, k \text{ such that } (x,y) = (x_h, x_{h+1}) \text{ or } (x,y) = (x_h, x_{h-1}) \right\}.
\]

Third, by Spilrajn’s theorem (Spilrajn, 1930), there exists an asymmetric, total, and transitive relation \( U_H \) on \( \{x_1, \ldots, x_k\} \) such that \( T_H \subseteq U_H \). Note that this compatible extension of \( T_H \) also only adds arcs between alternatives that are not adjacent in the enumeration \( x_1, \ldots, x_k, x_1 \), i.e.,

\[
U_H \setminus T_H \subseteq \left\{ (x,y) \mid \text{there does not exist } h = 1, \ldots, k \text{ such that } (x,y) = (x_h, x_{h+1}) \text{ or } (x,y) = (x_h, x_{h-1}) \right\}.
\]

Given any \( h \in \{1, \ldots, k\} \) such that \( x_h U_H x_{h+1} \), we therefore deduce that \( x_h S_H x_{h+1} \).

Fourth, and finally, set \( P_H = U_H \). By Linear Domain, there exists a state \( \theta \) such that for all \( H \in \mathcal{G} \) and all \( i \in H \), \( P_1(\theta) \{x_1, \ldots, x_k\} = P_H \). To complete the proof of the lemma, consider any \( h \in \{1, \ldots, k\} \) and any \( i \in \mathbb{N} \). Let \( i \in H \). I claim that \( x_h P_1(\theta) x_{h+1} \) if and only if \( i \in G_h \). Indeed, \( x_h P_1(\theta) x_{h+1} \) if and only if \( x_h S_H x_{h+1} \), which holds if and only if \( \alpha_h(H) = 1 \). This is equivalent to \( H \subseteq G_h \), which holds if and only if \( i \in G_h \), establishing the claim. Therefore, by (i), we have

\[
x_1 P_F(\theta) x_2 \cdots x_{k-1} P_F(\theta) x_k P_F(\theta) x_1,
\]

contradicting P-Acyclicity. Q.E.D.

Let \( \mathcal{D} \) denote the collection of semi-decisive groups, and define the following variant of the Nakamura number of the social preference rule \( F \) in two possible cases. First, if there exists
a collection \( \mathcal{G} \subseteq \mathcal{D} \) of semi-decisive groups such that \( \bigcup \mathcal{G} = N \) and \( \bigcap \mathcal{G} = \emptyset \), then let

\[
\mathcal{N}^*(F) = \min \left\{ |\mathcal{G}| \mid \mathcal{G} \subseteq \mathcal{D}, \bigcup \mathcal{G} = N, \text{ and } \bigcap \mathcal{G} = \emptyset \right\}.
\]

That is, \( \mathcal{N}^*(F) \) is the size of the smallest collection of semi-decisive groups with union containing all agents and with empty intersection. Second, if there is no such collection, set \( \mathcal{N}^*(F) \) to any cardinality greater than \( |X| \). This index is defined exactly as the conventional Nakamura number but in terms of semi-decisive, rather than decisive, groups.

Lemma 4 \( \mathcal{N}^*(F) > |X| \).

Proof Suppose \( \mathcal{N}^*(F) \leq |X| \). Then, letting \( \mathcal{N}^*(F) = k \), there exist semi-decisive groups \( G_1, \ldots, G_k \) satisfying (ii) and (iii) of Lemma 2. Furthermore, there exist distinct alternatives \( x_1, \ldots, x_k \). By semi-decisiveness, (i) of Lemma 3 is also satisfied, a contradiction. Q.E.D.

An immediate implication of the following lemma is that if an agent \( i \) is a weak dictator, then he is an \( r \)-dictator.

Lemma 5 For all \( G \subseteq N \), all \( H \in \mathcal{G}^+(r) \) with \( G \cap H = \emptyset \), and all \( x \) and \( y \), if \( x \succeq_G y \), then \( x \succeq_{G \cup H} y \).

Proof Consider any \( G \subseteq N \), any \( H \in \mathcal{G}^+(r) \) with \( G \cap H = \emptyset \), any \( x \) and \( y \) such that \( x \succeq_G y \), and any \( \theta \) such that \( G \cup H = P(x,y|\theta) \). By Lemma 2, to establish \( x \succeq_{G \cup H} y \), it suffices to show \( x \succeq_F \theta \) for \( y \). Let \( \theta' \) be such that \( G = P(x,y|\theta') \). Since \( x \succeq_G y \), we have either \( x \succeq_F \theta \) or \( x \succeq_F \theta' \). In the latter case, \( r \)-Tie Break immediately implies \( x \succeq_F \theta \). Consider the former case. If not \( x \succeq_F \theta \) for \( y \), then either \( y \succeq_F \theta \) for \( x \) or \( y \succeq_F \theta \) for \( y \). In the latter case, \( r \)-Tie Break implies \( y \succeq_F \theta' \) for \( x \), a contradiction. Thus, we must address the possibility of a social preference reversal: \( x \succeq_F \theta \) for \( y \) and \( y \succeq_F \theta \) for \( x \). Note, by Lemma 2, that \( x \succeq_G y \) and \( y \succeq_G \theta \). Letting \( z \in X \setminus \{x,y\} \), I claim that \( x \succeq_G z \). Otherwise, by Lemma 2, we have \( z \succeq_G x \). Of
course, y D N z by Pareto. But then we have x D G y D N z D N(G x, we have G \ N \ (N \ G) = N, and we have G \ N \ (N \ G) = \emptyset, contradicting Lemma 3. Thus, x B G z. Consider \( \theta' \) such that G = P(x,z|\( \theta' \)). Since G is semi-blocking for x over z, there are two cases. Case 1: x P F (\( \theta' \)) z. By Lemma 2, x D G z. Then we have x D G \ N \ (G \cup H) x, we have G \cup N \cup (N \setminus (G \cup H)) = N, and we have G \cap N \cap (N \setminus (G \cup H)) = \emptyset, contradicting Lemma 3. Case 2: x I F (\( \theta' \)) z. Letting \( \theta'' \) be such that G = P(x,z|\( \theta'' \)), r-Tie Break implies x P F (\( \theta'' \)) z, and then x D G z by Lemma 2. Then we have x D G \ H z by Lemma 2. Then we have x D G \ H z \ D N y D I (G,H) x, we have (G \cup H) \cup N \cup (N \setminus (G \cup H)) = N, and we have (G \cup H) \cap N \cap (N \setminus (G \cup H)) = N, contradicting Lemma 3. Q.E.D.

We can now show that if one agent is semi-blocking for each alternative over every other, then he is a weak dictator (and therefore an r-dictator).

Lemma 6 If for all distinct x and y, x B(1) y, then i is a weak dictator.

Proof Consider i as in the statement of the lemma, any distinct x and y, and any \( \theta \) such that x P_i(\( \theta \)) y. Let G = P(x,y|\( \theta \)) \{i\}. Suppose not x R_F(\( \theta \)) y, so that y P_F(\( \theta \)) x. By Lemma 2, y D G \ H x. Let {H,I} be a partition of N \{i\} such that H,I \in G^+(r), and let z \notin X \setminus \{x,y\}. Using Lemma 5, we have y D G \ H y D I (G,H) z, we have (N \setminus (G \cup \{i\})) \cup (\{i\} \cup H) \cup (\{i\} \cup I) = N, and we have (N \setminus (G \cup \{i\})) \cap (\{i\} \cup H) \cap (\{i\} \cup I) = \emptyset, contradicting Lemma 3. Q.E.D.

Lemma 7 2\( \mu(r) \) + 2r \leq n.

Proof Note that 

\[
2\mu(r) + 2r = 2 \left\lfloor \frac{n - 2r}{2} \right\rfloor + 2r \\
= 2 \left\lfloor \frac{n - 2r}{2} + r \right\rfloor \\
= 2 \left\lfloor \frac{n}{2} \right\rfloor \\
\leq n,
\]

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as required. Q.E.D.

The next lemma provides further information for the case \( r = \left\lfloor \frac{n}{3} \right\rfloor \).

**Lemma 8** Given groups \( G, H \in S^{-}\left( \left\lfloor \frac{n}{3} \right\rfloor \right) \), there exists a partition \( \{ G, H, I, J, K \} \) of \( N \) such that

\[
|G \cup I| \geq \left\lfloor \frac{n}{3} \right\rfloor, |H \cup J| \geq \left\lfloor \frac{n}{3} \right\rfloor, |I \cup J| \geq \left\lfloor \frac{n}{3} \right\rfloor, |H \cup I| \geq \left\lfloor \frac{n}{3} \right\rfloor, |K| = \left\lfloor \frac{n}{3} \right\rfloor.
\]

**Proof** Since \( G, H \in S^{-}\left( \left\lfloor \frac{n}{3} \right\rfloor \right) \), we have

\[
|G| \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor \quad \text{and} \quad |H| \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor.
\]

By Lemma 7, we can choose disjoint groups \( K, L \subseteq N \setminus (G \cup H) \) such that

\[
|K| = \left\lfloor \frac{n}{3} \right\rfloor \quad \text{and} \quad |L| \geq \left\lfloor \frac{n}{3} \right\rfloor. \quad \text{Thus, we obtain the last equality of the lemma.}
\]

To fulfill the first three inequalities, we must partition \( L \) into groups \( I \) and \( J \) such that

\[
|G| + |I| \geq \left\lfloor \frac{n}{3} \right\rfloor, \quad |H| + |J| \geq \left\lfloor \frac{n}{3} \right\rfloor.
\]

Since \( |L| = n - |G| - |H| - |K| + |G \cap H| \), these inequalities are consistent if

\[
\left\lfloor \frac{n}{3} \right\rfloor - |G| + \left\lfloor \frac{n}{3} \right\rfloor - |H| \leq n - |G| - |H| - |K|.
\]

This simplifies to \( 2 \left\lfloor \frac{n}{3} \right\rfloor \leq n - |K| \), and the latter follows from \( |K| = \left\lfloor \frac{n}{3} \right\rfloor \), delivering all but the fourth inequality of the lemma. Note that we can assume, relabeling if necessary, that \( |I| \geq |J| \), which yields \( |H \cup I| \geq |H \cup J| \geq \left\lfloor \frac{n}{3} \right\rfloor \), as required. Q.E.D.

A useful implication of the following step is that for all \( G \in S^{-}(r) \), \( B^G \) is transitive. Note also the further implication that for all disjoint \( G, H \in S^{-}(r) \), there do not exist distinct alternatives \( x, y, \) and \( z \) such that \( x B^G y \) \( B^H z \). Indeed, if there were such alternatives, Lemma 9 would imply \( x B^{G \cap H} z \), i.e., \( x B^0 z \), contradicting Pareto.

**Lemma 9** For all \( G, H \in S^{-}(r) \) and all triples \( x, y, \) and \( z \) of distinct alternatives, if \( x B^G y \) \( B^H z \), then for all groups \( K \) with \( G \cap H \subseteq K \subseteq G \cup H \), we have \( x B^K z \).

**Proof** Consider any distinct \( x, y, \) and \( z \), any \( G, H \in S^{-}(r) \) such that \( x B^G y \) \( B^0 z \), and any \( K \) with \( G \cap H \subseteq K \subseteq G \cup H \). Suppose not \( x B^K z \), which
implies $z \overset{N,K}{\rightarrow} x$ by part 3 of Lemma 2. By Lemma 7, we may partition $N \setminus (G \cup H)$ into $\{I,J\}$ such that $I,J \in S^+(r)$. By Lemma 5, we have $x \overset{G,J}{\rightarrow} y$ and $y \overset{H,J}{\rightarrow} z$. But then we have $x \overset{G,J}{\rightarrow} z \overset{N,K}{\rightarrow} x$, we have $(G \cup I) \cup (H \cup J) \cup (N \setminus K) = N$, and we have $(G \cup I) \cap (H \cup J) \cap (N \setminus K) = \emptyset$, contradicting Lemma 3. Therefore, $x \overset{K}{\rightarrow} z$. Q.E.D.

It is straightforward to extend the arguments of the previous lemma to the case in which two groups are semi-blocking over disjoint pairs. Note the further implication that for all disjoint $G,H \in S^-(r)$, there do not exist disjoint pairs $\{x,y\}$ and $\{a,b\}$ such that $x \overset{G}{\rightarrow} y$ and $a \overset{H}{\rightarrow} b$. Indeed, if there were such pairs, Lemma 10 would imply $x \overset{G \cap H}{\rightarrow} b$, i.e., $x \overset{H}{\rightarrow} b$, contradicting Pareto. Obviously, the lemma is vacuously true if there are just three alternatives.

Lemma 10 For all $G,H \in S^-(r)$ and all disjoint pairs $\{x,y\}, \{a,b\} \subseteq X$, if $x \overset{G}{\rightarrow} y$ and $a \overset{H}{\rightarrow} b$, then for all groups $K$ with $G \cap H \subseteq K \subseteq G \cup H$, we have $x \overset{B^K}{\rightarrow} b$ and $a \overset{B^K}{\rightarrow} y$.

Proof Consider disjoint pairs $\{x,y\}$, groups $G,H \in S^-(r)$ with $x \overset{G}{\rightarrow} y$ and $a \overset{H}{\rightarrow} b$, and any $K$ with $G \cap H \subseteq K \subseteq G \cup H$. Suppose not $x \overset{B^K}{\rightarrow} b$, which implies $b \overset{N-K}{\rightarrow} x$ by Lemma 2. Using Lemma 7, partition $N \setminus (G \cup H)$ into $\{I,J\}$, where $I,J \in S^+(r)$. By Lemma 5, $x \overset{G,I,J}{\rightarrow} y$ and $a \overset{H,I,J}{\rightarrow} b$. By Pareto, $y \overset{D^I}{\rightarrow} a$. Then we have $x \overset{G,I,J}{\rightarrow} y \overset{D^I}{\rightarrow} a \overset{D^H}{\rightarrow} b \overset{D^K}{\rightarrow} x$, we have $(G \cup I) \cup (H \cup J) \cup (N \setminus K) = N$, and we have $(G \cup I) \cap (H \cup J) \cap (N \setminus K) = \emptyset$, contradicting Lemma 3. Therefore, $x \overset{B^K}{\rightarrow} b$. The proof that $a \overset{B^K}{\rightarrow} y$ is analogous. Q.E.D.

The next lemma establishes a strong restriction on the structure of semi-blocking groups.

Lemma 11 For all $G,H \in S^-(r)$, if $B^G \neq \emptyset$ and $B^H \neq \emptyset$, then $B^G = B^H$.

Proof By symmetry of the roles of $G$ and $H$ in the lemma, it suffices to show that a contradiction obtains if there exist $G,H \in S^-(r)$ and alternatives $x, y, a,$ and $b$ such that $x \overset{G}{\rightarrow} y$ and $a \overset{H}{\rightarrow} b$. We first establish that $\{x,y\} \cap \{a,b\} \neq \emptyset$. Suppose instead that the pairs are disjoint. Use Lemma 7 to partition $N \setminus (G \cup H)$ into groups $I,J \in S^+(r)$. By Lemma 10, $x \overset{G \cap H}{\rightarrow} b$ and $a \overset{G \cap H}{\rightarrow} y$. By Lemma 5, $x \overset{(G \cap H) \cup I}{\rightarrow} y$. Q.E.D.
and a $D^{(G\cup H)\cup J} b$. By assumption, not a $B^G b$, which implies b $D^{N\setminus G} a$ by Lemma 2. Of course, y $D^N x$ by Pareto. Then we have $x D^{(G\cup H)\cup J}$ b $D^{N\setminus G}$ a $D^{(G\cup H)\cup J}$ y $D^N x$, we have $(G \cap H) \cup (N \setminus G) \cup ((G \cap H) \cup J) \cup N = N$, and we have $(G \cap H) \cap (N \setminus G) \cap ((G \cap H) \cup J) \cup N = \emptyset$, contradicting Lemma 3. This completes the first part of the proof.

We conclude that \{x,y\} $\cap \{a,b\} \neq \emptyset$. Since we cannot have x = a and y = b, there are now five cases. By assumption, not a $B^G b$, which implies b $D^{N\setminus G} a$ by Lemma 2. The remainder of the proof requires separate arguments for r = $\lfloor \frac{n}{4} \rfloor$ and r = $\lceil \frac{n}{5} \rceil$. We first assume r = $\lfloor \frac{n}{4} \rfloor$. In all cases, we use Lemma 7 to obtain $I \subseteq N \setminus (G \cup H)$ satisfying $|I| = r = \lfloor \frac{n}{4} \rfloor$, so $I \in J^-(r) \cap J^+(r)$. By Lemma 5, x $D^{G\cup J} y$.

Case 1: a = y and b $\neq$ x. By assumption, x $B^G y$ = a $B^H b$, and Lemma 9 implies x $B^G b$. By Lemma 5, x $D^{G\cup J}$ b. I claim that x $B^I y$. Otherwise, Lemma 2 implies y $D^{N\setminus I} x$. Then we have x $D^{G\cup J}$ b $D^{N\setminus I} a = y$ $D^{N\setminus I} x$, we have $(G \cup I) \cup (N \setminus G) \cup (N \setminus I) = N$, and we have $(G \cup I) \cap (N \setminus G) \cap (N \setminus I) = \emptyset$, contradicting Lemma 3. Thus, x $B^I y$. Then we have x $B^I y$ = a $B^G b$ for disjoint H,I $\in J^-(r)$, contradicting Lemma 9.

Case 2: a $\neq$ y and b = x. By assumption, a $B^H b$ = x $B^G y$, and Lemma 9 implies a $B^G y$. By Lemma 5, a $D^{G\cup J} y$. I claim that x $B^I y$. Otherwise, Lemma 2 implies y $D^{N\setminus I} x$. Then we have a $D^{G\cup J}$ y $D^{N\setminus I} x = b$ $D^{N\setminus I} x$, we have $(G \cup I) \cup (N \setminus I) \cup (N \setminus G) = N$, and we have $(G \cup I) \cap (N \setminus I) \cap (N \setminus G) = \emptyset$, contradicting Lemma 3. Thus, x $B^I y$. Then we have a $B^H b$ = x $B^I y$ for disjoint H,I $\in J^-(r)$, contradicting Lemma 9.

Case 3: a = y and b = x. Letting z $\in X \setminus \{x,y\}$, I claim that y $D^{N\setminus I} z$. Otherwise, z $B^I y$ by Lemma 2, but then we have z $B^I y$ = a $B^H b$ for disjoint H,I $\in J^-(r)$, contradicting Lemma 9. Thus, y $D^{N\setminus I} z$. Now I claim that x $B^G z$. Otherwise, z $D^{N\setminus G} x$ by Lemma 2, but then we have x $D^{G\cup J}$ y $D^{N\setminus I} z$ $D^{N\setminus G} x$, we have $(G \cup I) \cup (N \setminus I) \cup (N \setminus G) = N$, and we have $(G \cup I) \cup (N \setminus I) \cup (N \setminus G) = N$, contradicting Lemma 3. Thus, we have a $B^H \setminus B^G b$ = x $B^I z$, which is exactly analogous to Case 2.

Case 4: a = x and b $\neq$ y. I claim that b $B^I y$. Otherwise, Lemma 2 implies y $D^{N\setminus I} b$. Then we have x $D^{G\cup J}$ y $D^{N\setminus I} b$ $D^{N\setminus G} a = x$, we have $(G \cup I) \cup (N \setminus I) \cup (N \setminus G) = N$, and we have $(G \cup I) \cap (N \setminus I) \cap (N \setminus G) = \emptyset$,
contradicting Lemma 3. Thus, \( b \cap I \cap y \). Then we have \( b \cap I \cap y \) for disjoint \( H, I \in \mathcal{S}^-(r) \), contradicting Lemma 9.

Case 5: \( x \neq a \) and \( y = b \). I claim that \( x \cap I \). Otherwise, Lemma 2 implies \( a \cap D^{|I|} x \). Then we have \( x \cap D^{|I, J|} y = b \cap D^{|I|} \) \( a \cap D^{|I|} x \), we have \((G \cap I) \cup (N \cap G) \cup (N \cap I) = N\), and we have \((G \cap I) \cup (N \cap G) \cup (N \cap I) = N\), contradicting Lemma 3. Thus, \( x \cap I \). Then we have \( x \cap I \). We then have \( b \cap I \) \( a \cap H \) \( b \) for disjoint \( H, I \in \mathcal{S}^-(r) \), contradicting Lemma 9.

We now assume \( r = \left\lfloor \frac{n}{2} \right\rfloor \). Since \( |X| \geq 4 \), there exists \( z \in X \setminus \{x, y, a, b\} \). Let groups \( I, J, K \) be as in Lemma 8.

Case 1': \( a = y \) and \( b \neq x \). I claim that \( \cap B^{G \cap H} z \). Otherwise, \( z \cap D^{|G \cap H|} x \) by Lemma 2. By Lemma 5, \( x \cap D^{|G \cap H \cup (I \cup J)|} y \) and \( a \cap D^{|H, I \cup J|} b \). Then we have \( x \cap D^{|G \cap H \cup (I \cup J)|} y = a \cap D^{|H, I \cup J|} \cap D^{|G \cap H|} x \), we have \((G \cap K) \cup (H \cup (I \cup J)) \cup (I \cup J)) \cup (N \setminus (G \cap H)) = N \), and we have \((G \cap K) \cap (H \cup (I \cup J)) \cap (N \setminus (G \cap H)) = \emptyset \), contradicting Lemma 3. Thus, \( x \cap B^{G \cap H} z \). Furthermore, it follows similarly that \( z \cap B^{G \cap H} b \).

By Lemma 5, \( x \cap D^{|H \cap K \cup (I \cup J)|} y \) and \( z \cap D^{|H \cap K \cup (I \cup J)|} b \). But then we have \( x \cap D^{|G \cap H \cup (I \cup J)|} y \) \( z \cap D^{|H \cap K \cup (I \cup J)|} b \) \( \cap D^{|G \cap H|} a = y \cap D^{|H \cap K \cup (I \cup J)|} b \), we have \(((G \cap H) \cup K) \cup ((G \cap H) \cup (I \cup J)) \cup (N \setminus G) \cup N = N \), and \(((G \cap H) \cup K) \cap ((G \cap H) \cup (I \cup J)) \cap (N \setminus G) \cap N = \emptyset \), contradicting Lemma 3.

Case 2': \( a \neq y \) and \( b = x \). The argument in this case is analogous to Case 1', with the sequence \( a \cap H \setminus B^G b = a \cap B^G y \) in place of \( x \cap B^G y = a \cap B^H \setminus B^G b \).

Case 3': \( a = y \) and \( b = x \). Note that there exists \( w \in X \setminus \{x, y, a, b, z\} \). Furthermore, it is not the case that \( w \cap B^G z \), for then we would have disjoint pairs \( \{w, z\} \) and \( \{a, b\} \) such that \( w \cap B^G z \) and \( a \cap B^H \setminus B^G b \), which is precluded in the first part of the proof. By Lemma 2, \( z \cap D^{|H \cap G|} w \). I claim that either \( x \cap B^I \) \( z \) or \( w \cap B^I \). Otherwise, Lemma 2 implies \( z \cap D^{|H \cap J|} x \) and \( y \cap D^{|N \setminus I|} w \). By Lemma 5, \( x \cap D^{|G \cup (I \cup J)|} y \). By Pareto, \( w \cap D^{|N \setminus I|} z \). Then we have \( x \cap D^{|G \cup (I \cup J)|} y \) \( w \cap D^{|N \setminus I|} \cap D^{|N \setminus J|} x \), we have \((G \cup (I \cup J)) \cap (N \setminus I) \cap (N \setminus J) = N \), and we have \((G \cup (I \cup J)) \cap (N \setminus I) \cap (N \setminus J) = \emptyset \), contradicting Lemma 3. Consider two subcases.

Case 3: 1': \( x \cap B^J \). By Lemma 5, \( x \cap D^{|G \cup (I \cup J)|} \) \( z \) and \( a \cap D^{|H \cup K|} b \). By Pareto, \( w \cap D^{|N \setminus I|} a \). Then we have \( x \cap D^{|G \cup (I \cup J)|} \) \( D^{|N \setminus I|} \cap D^{|N \setminus K|} a = x \), we have \((J \cup (G \cup I)) \cup (N \setminus G) \cup (N \setminus K) = N \), and we have \((J \cup (G \cup I)) \cap (N \setminus G) \cap (N \setminus K) = \emptyset \),
Case 3:2': w B^2 y. The argument in this case is analogous to Case 3:1', with w B^1 y replacing x B^1 z.

Case 4': a = x and b ≠ y. I claim that b B^K z. Otherwise, z D^{H|K} b by Lemma 2. By Lemma 5, x D^{G|K} y. By Pareto, y D^H z. Then we have x D^{G|K} y D^H z D^{N|K} b D^{N|G} a = x, we have (G ∪ K) ∪ N ∪ (N \ K) ∪ (N \ G) = N, and we have (G ∪ K) ∩ N ∩ (N \ K) ∩ (N \ G) = ∅, contradicting Lemma 3. Thus, b B^K z.

Next, I claim that y B^K z. Otherwise, z D^{N|K} y by Lemma 2. By Lemma 5, a D^{R|K} b and b D^{R|I\cupJ} z. By Pareto, y D^H x. Then we have a D^{R|K} b D^{R|I\cupJ} z D^N y D^N x = a, we have (H ∪ K) ∪ (K ∪ (I ∪ J)) ∪ (N \ K) ∪ N = N, and we have (G ∪ K) ∩ (K ∪ (I ∪ J)) ∩ (N \ K) ∩ N = ∅, contradicting Lemma 3. Thus, y B^K z.

Finally, I claim that b B^J z. Otherwise, z D^{N\J} b by Lemma 2. By Lemma 5, x D^{G\(I\cupJ\)} y and y D^{G\(H\cupJ\)} z. Then we have x D^{G\(I\cupJ\)} y D^{K\(I\cupJ\)} z D^{N\J} b D^{N\G} a = x, we have (G ∪ (I ∪ J)) ∪ (K ∪ (H ∪ J)) ∪ (N \ J) ∪ (N \ G) = N, and we have (G ∪ (I ∪ J)) ∩ (K ∪ (H ∪ J)) ∩ (N \ J) ∩ (N \ G) = ∅, contradicting Lemma 3. Thus, b B^J z.

By Lemma 5, a D^{R|K} b and b D^{R\(G\cupJ\)} z. Note that z D^{N\G} y, for otherwise, using Lemma 2, we have disjoint pairs such that y B^G z and a B^H \ B^G b. By Pareto, y D^H x. Then we have a D^{R|K} b D^{R\(G\cupJ\)} z D^{N\G} y D^H x = a, we have (H ∪ K) ∪ (J ∪ (G ∪ I)) ∪ (N \ G) ∪ N = N, and we have (H ∪ K) ∩ (J ∪ (G ∪ I)) ∩ (N \ G) ∩ N = ∅, contradicting Lemma 3.

Case 5': x ≠ a and y = b. This in this case is analogous to Case 4'; what is critical is that the arcs (x, y) ∈ B^g and (a, b) ∈ B^H \ B^g point in opposite directions and have one node in common. The proof of Case 4' goes through with the roles of x (= a), y, z, and b in Case 4' occupied by x, y (= b), a, and x, respectively, in Case 5'.

Q.E.D.

In what follows, we let G^* be a group in \(S^-(r)\) that is maximally semi-blocking in \(S^-(r)\), i.e., for all \(G \in S^-(r)\), B^g ⊆ B^G*. By the preceding lemma, if there is a group such that B^G ≠ ∅, it suffices to select G^* from \(\{G \mid B^G ≠ ∅\}\) arbitrarily. If B^G = ∅ for all \(G \in S^-(r)\), then G^* is arbitrary.
Lemma 12. $B^g$ is negatively transitive.

Proof. Suppose $B^g$ is not negatively transitive, so there exist $x$, $y$, and $z$ such that $x \not\in B^g z$, not $x \not\in B^g y$, and not $y \not\in B^g z$. The remainder of the proof requires separate arguments for $r = \lfloor \frac{9}{7} \rfloor$ and $r = \lceil \frac{9}{7} \rceil$. By Lemma 2, $z \not\in B^g y$. Let $H \subseteq N$ satisfy $G^* \cap H = \emptyset$ and $|H| = r = \lfloor \frac{9}{7} \rfloor$, so $H \not\in G^+(r) \cap G^-(r)$. Then Lemma 5 implies $x \not\in D^{G^* \cup H} z$. By choice of $G^*$, we have $B^H \subseteq B^g$, so not $x \in B^H y$, which implies $y \not\in D^{N \setminus H} x$ by Lemma 2. But then we have $x \not\in D^{G^* \cup H} z \cup D^{N \setminus H} y \cup D^{N \setminus H} x$, we have $(G^* \cup H) \cup (N \setminus G^*) \cup (N \setminus H) = N$, and we have $(G^* \cup H) \cap (N \setminus G^*) \cap (N \setminus H) = \emptyset$, contradicting Lemma 3.

Now consider $r = \lfloor \frac{9}{7} \rfloor$. Since $|X| \geq 4$, there exists $w \in X \setminus \{x,y,z\}$. Let $H$ be any group such that $H \not\in G^-(r)$ and $G^* \cap H = \emptyset$, and let groups $I,J,K$ be as in Lemma 8. I claim that either $w \not\in B^{G^* \cup H} z$ or $w \not\in B^g w$. Otherwise, Lemma 2 implies $x \not\in D^{I \cup J} w \cup D^{N \setminus K} y$. Since $H \not\in G^-(r)$ and not $xB^g y$, it follows from the construction of $G^*$ that not $xB^H y$, which implies $y \not\in D^{N \setminus H} x$ by Lemma 2. But then we have $x \not\in D^{K \cup I} w \cup D^{N \setminus K} y \cup D^{N \setminus H} x$, we have $(K \cup H) \cup (N \setminus K) \cup (N \setminus H) = N$, and we have $(K \cup H) \cap (N \setminus K) \cap (N \setminus H) = \emptyset$, contradicting Lemma 3. This establishes the claim.

Suppose $y \not\in B^K w$. Since $G^* \cup I \not\in G^+(r)$, Lemma 5 then implies $y \not\in D^{K \cup G^* \cup I} w$. By Pareto, $w \in D^H x$. Since $x \not\in B^g z$ and $H \cup J \not\in G^+(r)$, Lemma 5 implies $x \not\in D^{G^* \cup H \cup J} z$. And $z \not\in D^{G^* \cup I \cup J} y$ by Lemma 2. But then we have $x \not\in D^{G^* \cup H \cup I} z \cup D^{N \setminus K} y \cup D^{N \setminus H} x$, we have $(G^* \cup H \cup J) \cup (N \setminus G^*) \cup (N \setminus J) \cup (N \setminus H) = N$, and we have $(G^* \cup H \cup J) \cap (N \setminus G^*) \cap (N \setminus J) \cap (N \setminus H) = \emptyset$, contradicting Lemma 3. This establishes the claim. But since $x \not\in B^g z$ and $H \cup I \not\in G^+(r)$, Lemma 5 implies $x \not\in D^{G^* \cup H \cup I} z$. By Pareto, $z \not\in D^H w$. Since $w \not\in B^J y$ and $K \not\in G^+(r)$, Lemma 5 implies $w \not\in D^J y$. And $y \not\in D^{N \setminus H} x$ by Lemma 2. Then we have $x \not\in D^{G^* \cup H \cup J} z \cup D^H w \cup D^{N \setminus H} y \cup D^{N \setminus I} x$, we have $(G^* \cup H \cup I) \cup N \cup (J \cup K) \cup (N \setminus G^*) = N$, and we have $(G^* \cup H \cup I) \cap N \cup (J \cup K) \cap (N \setminus G^*) = \emptyset$, a final contradiction of Lemma 3.

Q.E.D.
The next lemma contains the single application of the No \( r \)-Dictator axiom in the proof. In the sequel, \( \Delta = \{(x,x) \mid x \in X\} \) denotes the diagonal relation on \( X \).

Lemma 13 \( B^g \neq (X \times X) \setminus \Delta \).

Proof Suppose \( B^g = (X \times X) \setminus \Delta \), and let \( G \) be a minimal subset of \( G^* \) such that \( B^g \neq \emptyset \), and note that from Lemma 11, \( B^g = (X \times X) \setminus \Delta \). By No \( r \)-Dictator and Lemma 6, we have \( |G| > 1 \), and we can partition \( G \) into \( \{G_1, G_2\} \), where \( G_1 \) and \( G_2 \) are both non-empty. By Lemma 7, we may partition \( N \setminus G \) into \( \{H_1, H_2\} \) such that \( H_1, H_2 \in \mathcal{S}^+(r) \), with further qualification that \( |H_1| < \left\lceil \frac{n}{2} \right\rceil \) and \( |H_2| < \left\lceil \frac{n}{2} \right\rceil \). Take any distinct \( x \) and \( y \), and note that by Lemma 2, either \( x \in B^g \cup H_x \) or \( y \in B^g \cup H_y \). Without loss of generality, assume the former. Note that \( B^g \neq (X \times X) \setminus \Delta \), and therefore Lemma 11 implies \( B^g = \emptyset \). Let \( H \) be a minimal subset of \( G_1 \cup H_x \) such that \( B^H \neq \emptyset \). Then \( H \setminus G_1 \neq \emptyset \), for otherwise we have \( H \subseteq G_1 \subseteq G \), contradicting minimality of \( G \). Let \( i \in H \setminus G_1 \), and note in particular that \( i \in H_1 \subseteq N \setminus G \). Since \( B^H \neq \emptyset \), there exist \( a, b \in X \) such that a \( B^H \) b, and Lemma 5 implies a \( D_{(H \cup H_2)} b \). Furthermore, there exists \( c \in X \setminus \{a, b\} \). Since \( H \) is minimal, it follows that \( B^H \{i\} = \emptyset \), so not \( c \in B^H \{i\} \), which implies \( b \in D_{(H \cup H_2)} \{a\} \). By construction, \( H_2 \cup \{i\} \) has at most \( \left\lceil \frac{n}{2} \right\rceil \) members, and \( G \subseteq N \setminus (H_2 \cup \{i\}) \). Thus, using \( |H_2 \cup \{i\}| \leq \left\lceil \frac{n}{2} \right\rceil \) and \( |G| \leq \mu(x) \), we have

\[
|\{(X \setminus (H_2 \cup \{i\}))) \setminus \Delta| \geq n - \frac{n}{2} - \left[ \frac{n}{2} - r \right] = r,
\]

so \( \{(X \setminus (H_2 \cup \{i\}))) \setminus \Delta \in \mathcal{S}^+(r) \). Therefore, \( c \in B^g \) and Lemma 5 imply that \( c \in D_{(H \cup H_2)} \{a\} \). But then we have a \( D_{(H \cup H_2)} b \in D_{(H \cup H_2)} \{a\} \), we have \( (H \cup H_2) \cup (N \setminus (H \setminus \{i\})) \cup (N \setminus (H_2 \cup \{i\})) = N \), and we have \( (H \cup H_2) \cap (N \setminus (H \setminus \{i\})) \cap (N \setminus (H_2 \cup \{i\})) = \emptyset \), contradicting Lemma 3. We conclude that \( B^g \neq (X \times X) \setminus \Delta \). Q.E.D.

The next lemma draws further implications when \( G^* \) is semi-blocking over no pairs and the threshold is \( r = \left\lceil \frac{n}{3} \right\rceil \).

Lemma 14 Assume \( r = \left\lceil \frac{n}{3} \right\rceil \) and \( B^g = \emptyset \). For all \( \hat{G} \subseteq N \) with \( |\hat{G}| \leq \mu(x) + 1 \), we have \( B^\hat{G} = \emptyset \).

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Proof. Assume \( r = \lfloor \frac{n}{3} \rfloor \) and \( B^r = \emptyset \), and consider any \( \hat{G} \) such that \( |\hat{G}| \leq \mu(r) + 1 \). It follows from construction of \( G^* \) that \( B^\hat{G} = \emptyset \) if \( \hat{G} \in S^-(r) \), so assume \( |\hat{G}| = \mu(r) + 1 \). Suppose contrary to the lemma that \( B^\hat{G} \neq \emptyset \), so there exist \( x \) and \( y \) such that \( x B^\hat{G} y \). Since \( |X| \geq 4 \), there exist distinct \( z, w \in X \setminus \{x, y\} \). Let \( G \subseteq \hat{G} \) and \( H \subseteq N \setminus \hat{G} \) satisfy \( |G| = |H| = \mu(r) \). Let \( I, J, K \) be as in Lemma 8, and without loss of generality let \( i \in I \) satisfy \( \hat{G} = G \cup \{i\} \). There are two cases.

Case 1: \( z \in D_{G^* \cup I \cup K} \). Since \( x B^\{\{\}\} y \), we have \( x \in D_{G^* \cup K} \cup \{\} \) by \( r \)-Tie Break. Since \( B^\{\{\}\} = \emptyset \) and \( G \in S^-(r) \), we have \( B^i = \emptyset \), so Lemma 2 implies \( y \in D_{N \setminus \{i\}} \). I claim that \( x B^\{\} \), for otherwise Lemma 2 implies \( w \in D_{N \setminus \{\} \setminus \{i\}} \). But then we have \( x \in D_{G^* \cup I \cup K} \cup \emptyset \cup \{\} \cup \emptyset \cup \{\} \), \( y \in D_{N \setminus \{\} \setminus \{i\}} \cup \emptyset \cup \{\} \cap \emptyset \). I claim that \( x B^\{\} \), for otherwise Lemma 2 implies \( z \in D_{N \setminus \{\} \setminus \{i\}} \). But then we have \( x \in D_{G^* \cup K} \cup \emptyset \cup \{\} \cup \emptyset \cup \emptyset \), which is clearly true. Note that \( |H| \leq 2 \mu(r) + 1 \), which holds if \( \lfloor \frac{n}{3} \rfloor \leq 2 \mu(r) + 1 = 2 \lfloor \frac{n}{3} \rfloor + 1 \); indeed, this is equivalent to \( 2 \lfloor \frac{n}{3} \rfloor + 1 \geq 3 \lfloor \frac{n}{3} \rfloor \), which is clearly true. Thus, partition \( K \) into groups \( K_1 \) and \( K_2 \) with \( |K_1|, |K_2| \leq \mu(r) + 1 \). By Lemma 14, we have \( B^K = B^{K_1} = B^{K_2} = \emptyset \). Following the construction in the proof of Lemma 8, we can partition \( N \setminus K \) into \( \{G, H, I, J\} \).
with \(|G| = \mu(H) = \mu(r)\) and \(|G \cup I|, |H \cup J| \geq \left\lceil \frac{n}{e}\right\rceil\); furthermore, since \(\left\lceil \frac{n}{e}\right\rceil \leq 2\mu(r) + 1\), we may specify \(J\) so that \(|J| \leq \mu(r) + 1\) and maintain \(|H \cup J| \geq \left\lceil \frac{n}{e}\right\rceil\). Now, since \(|X| \geq 4\), there exist distinct \(z, w \in X \setminus \{x, y\}\). There are two cases.

Case 1: \(w \in D^H(\{H,i\})\). Since \(x \in B^K \) and \(H \cup J \in \splus\), Lemma 5 implies \(x \in D^{K \cup H \cup J} y\). Since \(|J| \leq \mu(r) + 1\), Lemma 14 implies \(B^J = \emptyset\), so \(y \in B^H\) by Lemma 2. Similarly, since \(B^K = \emptyset\), Lemma 2 implies \(z \in D^H x\). But then we have \(x \in D^{K \cup H \cup J} y \cap D^H(\{H,i\}) \cap D^H x\), we have \((K \cup H \cup J) \cap (N \setminus J) \cap (N \setminus (H \cup K_1)) \cap (N \setminus K_2) = \emptyset\), and we have \((K \cup H \cup J) \cap (N \setminus J) \cap (N \setminus (H \cup K_1)) \cap (N \setminus K_2) = \emptyset\), contradicting Lemma 3.

Case 2: \(y \in D^H(\{H,i\})\). By Lemma 2, \(x \in B^{H,i} \) and, since \(G \cup I \in \splus\), Lemma 5 implies \(x \in D^{H,i \cup G,j} w\). As above, since \(x \in B^K\) and \(H \cup J \in \splus\), Lemma 5 implies \(x \in D^{K \cup H \cup J} y\). Since \(H \in \sminus\) and \(B^H = \emptyset\), we have \(B^K = \emptyset\), which implies \(y \in D^{H \cup H} z\) by Lemma 2. Similarly, since \(B^K = \emptyset\), Lemma 2 implies \(w \in D^{K \cup K_1} x\). But then we have \(x \in D^{K \cup H \cup J} y \cap D^{H \cup H} z \cap D^{K \cup K_1} x\), we have \((K \cup H \cup J) \cap (N \setminus H) \cap (H \cup K_1) \cup (G \cup I) \cap (N \setminus K_1) = \emptyset\), and we have \((K \cup H \cup J) \cap (N \setminus H) \cap (H \cup K_1) \cup (G \cup I) \cap (N \setminus K_1) = \emptyset\), contradicting Lemma 3.

Lemma 16 Assume \(B^* = \emptyset\). For all \(x\) and \(y\) and all \(G \subseteq N\), if \(G\) is semi-decisive for \(x\) over \(y\), then for all \(z \in X \setminus \{x, y\}\), \(G\) is semi-decisive for \(x\) over \(z\) and for \(z\) over \(y\).

Proof Assume \(B^* = \emptyset\), and consider any \(x, y\), and \(G\) such that \(x \in D^G y\). Suppose there exists \(z \in X \setminus \{x, y\}\) such that not \(x \in D^G z\), implying \(z \in B^{G,x}\) by Lemma 2. When \(r = \left\lceil \frac{n}{e}\right\rceil\), \(\mu(r) \geq r\), and therefore \(B^* = \emptyset\) immediately implies \(|G| \geq r\). When \(r = \left\lceil \frac{n}{2}\right\rceil\), we use Lemma 15 to deduce \(|G| > r\). In either case, we choose \(H \subseteq G\) with \(|H| = r\) and conclude \(B^H = \emptyset\). By Lemma 2, \(y \in D^{H^G} z\). By Lemma 5, \(z \in D^{H(G)} z\). But then we have \(x \in D^G y \cap D^{H(G)} z \cap D^{H(G)} z\), we have \(G \cup (N \setminus H) \cup (N \setminus G) \cup H = N\), and we have \(G \cap (N \setminus H) \cap (N \setminus G) \cup H = \emptyset\), contradicting Lemma 3. Now suppose there exists \(z \in X \setminus \{x, y\}\) such that not \(z \in D^G y\), implying \(y \in B^{G,x}\) by Lemma 2. Either by construction or by Lemma 15, we again have \(|G| > r\), and we choose \(H \subseteq G\) with \(|H| = r\) and \(B^H = \emptyset\). By Lemma 2, \(z \in D^{H^G} x\). By Lemma 5, \(y \in D^{H(G)} z\). But then we have \(x \in D^G y \cap D^{H(G)} z \cap D^{H^G} z\), contradicting Lemma 3 as above.

Q.E.D.
The following lemma assumes $B^G = \emptyset$ and establishes a form of neutrality: if a group is semi-decisive for one alternative over another, then it is semi-decisive for every alternative over every other.

**Lemma 17** Assume $B^G = \emptyset$. For all $x$ and $y$ and all $G \subseteq N$, if $G$ is semi-decisive for $x$ over $y$, then $G$ is semi-decisive.

**Proof** Assume $B^G = \emptyset$. Consider any $G \subseteq N$, alternatives $x$ and $y$ such that $x B^G y$, and any distinct alternatives $a, b \in X$. If $a = x$ and $b = y$, clearly $a B^G b$. Suppose $a \neq x$, and take any $z$ distinct from $x$ and $a$. Then $x D^G z$ by Lemma 16, and then a $D^G z$ by another application of Lemma 16. If $z = b$, then we have a $D^G b$; otherwise, $z \neq b$, and another application of Lemma 16 yields a $D^G b$, as required. Now suppose $b \neq y$, and take any $z$ distinct from $a$ and $y$. Then $z D^G y$ by Lemma 16, and then a $D^G y$ by another application of Lemma 16. If $y = b$, then we have a $D^G b$; otherwise, $y \neq b$, and a final application of Lemma 16 yields a $D^G b$. We conclude $G$ is semi-decisive. Q.E.D.

**Proof of Theorem 1**

In this subsection, we assume $r = \lfloor \frac{n}{4} \rfloor$, we allow any number of three or more alternatives, and in line with our maintained assumptions, we assume $n \geq 3$.

**Lemma 18** Let $r = \lfloor \frac{n}{4} \rfloor$. For all distinct $x$ and $y$, there exists $\hat{G} \subseteq N$ such that $\hat{G}$ is semi-decisive for $a$ over $b$, where $\{x, y\} = \{a, b\}$, and either

(i) $|\hat{G}| = \lceil \frac{3n}{8} \rceil + \lfloor \frac{n}{4} \rfloor$ and $|\hat{G} \cap G^*| = \min\{|G^*|, \lceil \frac{n}{8} \rceil \}$, or

(ii) $|\hat{G}| = \lfloor \frac{5n}{8} \rfloor$ and $|\hat{G} \cap G^*| = \max\{0, |G^*| - \lfloor \frac{n}{8} \rfloor \}$.

**Proof** Let $H \subseteq N$ be such that $|H| = \lceil \frac{3n}{8} \rceil$ and $|H \cap G^*| = \min\{|G^*|, \lceil \frac{n}{8} \rceil \}$, and let $I \subseteq N \setminus (H \cup G^*)$ be such that $|I| = \lfloor \frac{n}{4} \rfloor$. By Lemma 2, either
x B^n y or y D^W\setminus x. In the first case, set a = x, b = y, and \( \hat{G} = H \cup I \) to fulfill (i). In the second case, set a = y, b = x, and \( \hat{G} = N \setminus H \), and note that \(|\hat{G}| = \left\lceil \frac{5n}{8} \right\rceil \) and \(|\hat{G} \cap G^*| = \max\{0,|G^*| - \left\lceil \frac{n}{8} \right\rceil\} \), fulfilling (ii).

Q.E.D.

In accordance with Lemma 18, to each pair of distinct alternatives we associate a group \( \hat{G} \) satisfying conditions (i) or (ii) of the lemma.

**Lemma 19** For all distinct \( x \) and \( y \), \(|N \setminus (G^* \cup \hat{G})| \geq \left\lceil \frac{n}{8} \right\rceil\).

**Proof** Note that \(|N \setminus (G^* \cup \hat{G})| = n - |G^*| - |\hat{G}| + |G^* \cap \hat{G}|. Consider (i) \(|\hat{G}| = \left\lceil \frac{3n}{8} \right\rceil + \left\lceil \frac{n}{4} \right\rceil\) and \(|G^* \cap \hat{G}| = \min\{|G^*|,\left\lceil \frac{n}{8} \right\rceil\}. Note that

\[
n - |G^*| - |\hat{G}| + |G^* \cap \hat{G}| \geq n - 2 \left\lceil \frac{n}{4} \right\rceil - \left\lceil \frac{3n}{8} \right\rceil + \left\lceil \frac{n}{8} \right\rceil,
\]

so the desired inequality follows if

\[
n \geq 3 \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{3n}{8} \right\rceil - \left\lceil \frac{n}{8} \right\rceil.
\]

Furthermore, note that

\[
n = -\frac{n}{8} + 3 \cdot \frac{n}{4} + \frac{3n}{8} = -\left(\frac{n}{8} - \alpha\right) + 3 \left(\frac{n}{4} + \beta\right) + \left(\frac{3n}{8} - \gamma\right),
\]

where \( \alpha, \beta, \gamma \in [0,1) \). Since \( n \) is an integer, it follows that \( \alpha + 3\beta - \gamma \in \{-1,0,1\} \), and since \( \alpha, \beta \geq 0 \) and \( \gamma < 1 \), we must have \( \alpha + 3\beta - \gamma \geq 0 \). The desired inequality follows.

Now consider (ii) \(|\hat{G}| = \left\lceil \frac{5n}{8} \right\rceil\) and \(|G^* \cap \hat{G}| = \max\{0,|G^*| - \left\lceil \frac{n}{8} \right\rceil\}. Note that

\[
n - |G^*| - |\hat{G}| + |G^* \cap \hat{G}| \geq n - \left\lceil \frac{5n}{8} \right\rceil - \left\lceil \frac{n}{8} \right\rceil,
\]

so the desired inequality follows if

\[
n \geq \left\lceil \frac{n}{8} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{5n}{8} \right\rceil.
\]
Furthermore, note that

\[ n = \frac{n}{8} + \frac{n}{4} + \frac{5n}{8} = \left(\left\lceil \frac{n}{8} \right\rceil - \alpha \right) + \left(\left\lfloor \frac{n}{4} \right\rfloor + \beta \right) + \left(\left\lfloor \frac{5n}{8} \right\rfloor + \gamma \right), \]

where \( \alpha, \beta, \gamma \in [0, 1) \). Since \( n \) is an integer, it follows that \( \beta + \gamma - \alpha \in \{-1, 0, 1\} \), and since \( \beta, \gamma \geq 0 \) and \( \alpha < 1 \), we must have \( \beta + \gamma - \alpha \geq 0 \). The desired inequality follows. Q.E.D.

Given the neutrality result of Lemma 17 in the previous subsection, the crux of the proof is to show that \( B^G \) is empty. Note that the climactic Case 4.2.2.2.2.2 (with the analogue in Case 5) contains the one critical use of the assumption that \( n \geq 5 \).

Lemma 20 Let \( r = \left\lfloor \frac{n}{4} \right\rfloor \). Then \( B^G = \emptyset \).

Proof Suppose otherwise, so by Lemma 13 we have \( \emptyset \subsetneq B^G \subsetneq (X \times X) \setminus \Delta \). I claim that there exists \( A \subseteq X \) such that \( |A| = 3 \) and \( \emptyset \subsetneq B^G \cap (A \times A) \subsetneq (A \times A) \setminus \Delta \). To see this, let \( x \) and \( y \) be such that \( x B^G y \), and let \( a \) and \( b \) be distinct alternatives such that not \( b B^G a \). If \( |\{x,y\} \cap \{a,b\}| = 2 \), then let \( z \in X \) be any third alternative, and define \( A = \{x,y,z\} \). If \( |\{x,y\} \cap \{a,b\}| = 1 \), then let \( A = \{x,y,a,b\} \).

If \( |\{x,y\} \cap \{a,b\}| = 0 \), then there are two possibilities: if \( x B^G a \), then let \( A = \{x,a,b\} \); and if not \( x B^G a \), then let \( A = \{x,y,a\} \). The claim follows. Let \( B = B^G \cap (A \times A) \) be the restriction of \( B^G \) to \( A \).

By construction, \( B \) must comprise from one to five ordered pairs. By negative transitivity, from Lemma 12, \( B \) cannot consist of a single ordered pair. By transitivity, from Lemma 9, \( B \) cannot consist of five ordered pairs. If \( B \) consists of two ordered pairs, then by negative transitivity we cannot have \( \hat{B} = \{(a,b),(b,a)\} \) for any \( a,b \in A \), and it follows that either \( \hat{B} = \{(a,b),(a,c)\} \) or \( \hat{B} = \{(a,c),(b,c)\} \), where \( A = \{a,b,c\} \). The remaining cases are that \( \hat{B} \) consists of three or four ordered pairs. If \( \hat{B} \) consists of three ordered pairs, then it must take the form \( \hat{B} = \{(a,b),(b,c),(a,c)\} \), where \( A = \{a,b,c\} \), for otherwise transitivity entails the inclusion of further ordered pairs. If \( \hat{B} \) consists of four ordered pairs, then transitivity implies that \( \hat{B} \) is total, and \( \hat{B} \) is an ordering.
of $A$. The two possibilities are $\hat{B}=\{(a,b),(b,a),(a,c),(b,c)\}$ and $\hat{B}=\{(a,b),(b,a),(c,a),(c,b)\}$. We address these cases separately.

Case 1: $\hat{B}=\{(a,b),(b,c),(a,c)\}$. Since not $cB^\ast b$, we have $bD^\ast\setminus G^\ast c$ by Lemma 2. Let $H \subseteq N$ be such that $|H| = \lfloor \frac{n}{4} \rfloor$ and $G^\ast \cap H = \emptyset$, so $\lfloor \frac{n}{4} \rfloor$-Tie Break implies a $D^\ast\cup H b$. There are two subcases.

Case 1.1: $cD^\ast\setminus H a$. Then we have a $D^\ast\cup H b D^\ast\setminus G^\ast c D^\ast\setminus H a$, we have $(G^\ast \cup H) \cup (N \setminus G^\ast) \cup (N \setminus H) = N$, and we have $(G^\ast \cup H) \cap (N \setminus G^\ast) \cap (N \setminus H) = \emptyset$, contradicting Lemma 3.

Case 1.2: not $cD^\ast\setminus H a$. By Lemma 2, a $B^H c$. By Lemma 7, we may choose $I \subseteq N$ be such that $|I| = r$ and $I \cap (G^\ast \cup H) = \emptyset$. By $\lfloor \frac{n}{4} \rfloor$-Tie Break, we have a $D^H \cup I b$. Since $H \in \mathcal{S}^-(\lfloor \frac{n}{4} \rfloor)$ and a $B^G b$, Lemma 9 implies not $bB^H c$. Similarly, Since $I \in \mathcal{S}^-(\lfloor \frac{n}{4} \rfloor)$ and $bB^G c$, Lemma 9 implies not $aB^I b$. Thus, $cD^\ast\setminus H b$ and $bD^\ast\setminus I a$ by Lemma 2. But then we have a $D^H \cup I c D^\ast\setminus H b D^\ast\setminus I a$, we have $(H \cup I) \cup (N \setminus H) \cup (N \setminus I) = N$, and we have $(H \cup I) \cap (N \setminus H) \cap (N \setminus I) = \emptyset$, contradicting Lemma 3.

Case 2: $\hat{B}=\{(a,b),(b,a),(a,c),(b,c)\}$. Let $H \subseteq N$ be such that $|H| = r$ and $G^\ast \cap H = \emptyset$. Then $bB^G a$ and $\lfloor \frac{n}{4} \rfloor$-Tie Break imply $bD^G \cup H a$. Since not $cB^G a$, we have $aD^\ast\setminus G^\ast c$ by Lemma 2. Since $H \in \mathcal{S}^-(\lfloor \frac{n}{4} \rfloor)$ and a $B^G b$, Lemma 9 implies not $bB^H c$, and therefore $cD^\ast\setminus H b$ by Lemma 2. But then we have $bD^G \cup H a D^\ast\setminus G^\ast c D^\ast\setminus H b$, we have $(G^\ast \cup H) \cup (N \setminus G^\ast) \cup (N \setminus H) = N$, and we have $(G^\ast \cup H) \cap (N \setminus G^\ast) \cap (N \setminus H) = \emptyset$, contradicting Lemma 3.

Case 3: $\hat{B}=\{(a,b),(b,a),(c,a),(c,b)\}$. Let $H \subseteq N$ be such that $|H| = r$ and $G^\ast \cap H = \emptyset$. Then $bB^G a$ and $\lfloor \frac{n}{4} \rfloor$-Tie Break imply $bD^G \cup H a$. Since not $bB^G c$, we have $cD^\ast\setminus G^\ast b$ by Lemma 2. Since $H \in \mathcal{S}^-(\lfloor \frac{n}{4} \rfloor)$ and a $B^G b$, Lemma 9 implies not $cB^H a$, and therefore $aD^\ast\setminus H c$ by Lemma 2. But then we have $bD^G \cup H a D^\ast\setminus G^\ast c D^\ast\setminus H b$, we have $(G^\ast \cup H) \cup (N \setminus G^\ast) \cup (N \setminus H) = N$, and we have $(G^\ast \cup H) \cap (N \setminus G^\ast) \cap (N \setminus H) = \emptyset$, contradicting Lemma 3.

Case 4: $\hat{B}=\{(a,b),(a,c)\}$. By Lemma 18 and symmetry with respect to $b$ and $c$, we may assume there exists $\hat{G}$ such that $cD^\hat{G} b$ and either (i) $|\hat{G}| = \lfloor \frac{3n}{8} \rfloor + \lfloor \frac{n}{8} \rfloor$ and $|\hat{G} \cap G^\ast| = \min\{|G^\ast|, \lfloor \frac{n}{8} \rfloor\}$, or (ii) $|\hat{G}| = \lfloor \frac{5n}{8} \rfloor$ and $|\hat{G} \cap G^\ast| = \max\{0, |G^\ast| - \lfloor \frac{n}{8} \rfloor\}$. In particular, $|\hat{G}| \leq \lfloor \frac{5n}{8} \rfloor$ and $|\hat{G} \cap G^\ast| \leq \lfloor \frac{n}{8} \rfloor$. Lemma 19 establishes that $|N \setminus (G^\ast \cup \hat{G})| \geq \lfloor \frac{n}{4} \rfloor$. There are two subcases.

Case 4.1: $bD^\ast\cup (N \setminus (G^\ast \cup \hat{G})) a$. Since $N \setminus (G^\ast \cup \hat{G}) \in \mathcal{S}^+(\lfloor \frac{n}{4} \rfloor)$, $\lfloor \frac{n}{4} \rfloor$-Tie Break implies a $D^G \cup (N \setminus (G^\ast \cup \hat{G})) c$. But then we have a $D^G \cup (N \setminus (G^\ast \cup \hat{G})) c D^\hat{G} b D^\ast\setminus (G^\ast \cup \hat{G})$. 36
we have \((G^* \cup (N \setminus (G^* \cup \hat{G}))) \cup \hat{G} \cup (N \setminus (\hat{G} \cap G^*)) = N\), and we have \((G^* \cup (N \setminus (G^* \cup \hat{G}))) \cap \hat{G} \cap (N \setminus (\hat{G} \cap G^*)) = \emptyset\), contradicting Lemma 3.

Case 4.2: not \(b \in D^{|G \cap a|} \). By Lemma 2, a \(\hat{b} \in G^* \). Let \(H \subseteq N\) satisfy \(|H| = \lceil \frac{n}{4} \rceil\) and \(G^* \setminus \hat{G} \subset H \subseteq N \setminus \hat{G}\), so \(\lceil \frac{n}{4} \rceil\)–Tie Break implies a \(D^{(G^* \cap a)}_{H} \) b. Consider a subset \(H' \subseteq H\) such that \(H' \cap G^* = \emptyset\) and \(|G^* \cup H'| = \lceil \frac{n}{4} \rceil\). Since \(G^* \cup H' \subseteq G^*\) and not \(c \in G^*\), it follows that not \(c \in H^* \cup H'\) b, and we therefore have \(b \in D^{(G^* \cup H')} \) by Lemma 2. Define

\[ I = (N \setminus ((\hat{G} \cap G^*) \cup H)) \cup (G^* \cup H'), \]

and note that \(\hat{G} \subseteq I\). Since \(G^* \cup H' \subseteq (\hat{G} \cap G^*) \cup H\), we have

\[ |I| = |N \setminus ((\hat{G} \cap G^*) \cup H)| + |G^* \cup H'| \geq n - \left\lfloor \frac{n}{8} \right\rfloor.\]

There are two subcases.

Case 4.2.1: \(c \in D^I\) a. Then we have a \(D^{(G^* \cap a)}_{H} \) b \(D^{|G^* \cup H'|} \) c \(D^I\) a, we have \(((\hat{G} \cap G^*) \cup H) \cup (N \setminus (G^* \cup H')) \cup I = N\), and we have \(((\hat{G} \cap G^*) \cup H) \cap (N \setminus (G^* \cup H')) \cap I = \emptyset\), contradicting Lemma 3.

Case 4.2.2: not \(c \in D^I\) a. Letting \(J = N \setminus I\), this implies a \(B^J\) c by Lemma 2. Furthermore, \(\hat{G} \subseteq I\) implies \(G \cap J = \emptyset\), and we have \(|J| = |N \setminus I| \leq \left\lfloor \frac{n}{8} \right\rfloor\). Now define

\[ K = (N \setminus J) \setminus \hat{G} = I \setminus \hat{G}, \]

so that \(|K| \geq n - \left\lfloor \frac{n}{8} \right\rfloor - |\hat{G}| \geq \left\lfloor \frac{n}{8} \right\rfloor - 1\). There are two subcases.

Case 4.2.2.1: \(|K| \geq \left\lfloor \frac{n}{8} \right\rfloor\). Since a \(B^J\) c and \(K \in \mathcal{G}^+([\frac{n}{4}])\), \(\left\lfloor \frac{n}{4} \right\rfloor\)–Tie Break implies a \(D^{J \cup \{i\}}_{K \cup \{i\}}\) c. By Pareto, b \(D^N\) a. But then we have a \(D^{J \cup \{i\}}_{K} \) c \(D^N\) b \(D^N\) a, we have \((J \cup K) \cup \hat{G} \cup N = N\), and we have \((J \cup K) \cap \hat{G} \cap N = \emptyset\), contradicting Lemma 3.

Case 4.2.2.2: \(|K| = \left\lfloor \frac{n}{8} \right\rfloor - 1\). Consider \(i \in \hat{G}\). Since a \(B^J\) c and \(K \cup \{i\} \in \mathcal{G}^+([\frac{n}{4}])\), \(\left\lfloor \frac{n}{4} \right\rfloor\)–Tie Break implies a \(D^{J \cup \{i\}}_{K \cup \{i\}}\) c. There are two subcases.

Case 4.2.2.2.1: b \(D^{N \setminus \{i\}} \) a. But then we have a \(D^{J \cup \{i\}}_{K \cup \{i\}} \) c \(D^N\) b \(D^N\) a, we have \((J \cup K \cup \{i\}) \cup \hat{G} \cup (N \setminus \{i\}) = N\), and we have \((J \cup K \cup \{i\}) \cap \hat{G} \cap (N \setminus \{i\}) = \emptyset\), contradicting Lemma 3.
Case 4.2.2.2.2: not b $D^{\{i\}}$ a. By Lemma 2, a $B^{\{i\}}$ b. Since $G^* \cup H' \in S^+(\lfloor \frac{n}{4} \rfloor)$, $\lfloor \frac{n}{4} \rfloor$-Tie Break implies a $D^{\{i\} \cup G^* \cup H'}$ b. From Case 4.2, we have b $D^{\{i\} \cup G^* \cup H'}$ c. There are two subcases.

Case 4.2.2.2.2.1: c $D^{\{i\}}$ a. But then we have a $B^{\{i\}}$ b $D^{\{i\} \cup G^* \cup H'}$ c $D^{\{i\}}$ a, we have $(\{i\} \cup G^* \cup H') \cup (N \setminus (G^* \cup H')) \cup (N \setminus \{i\}) = N$, and we have $(\{i\} \cup G^* \cup H') \cup (N \setminus (G^* \cup H')) \cup (N \setminus \{i\}) = \emptyset$, contradicting Lemma 3.

Case 4.2.2.2.2.2: not c $D^{\{i\}}$ a. By Lemma 2, a $B^{\{i\}}$ c. Define $K' = (N \setminus \{i\}) \setminus \hat{G}$. Since $\hat{G} \leq \lceil \frac{3n}{8} \rceil + \lfloor \frac{n}{4} \rfloor$, we have $|K'| \geq \lfloor \frac{n}{4} \rfloor$ if $n - 1 \geq 2 \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{3n}{8} \right\rfloor$.

which holds for $n = 5, 6, 7$ by inspection and clearly holds for $n \geq 8$. The argument in Case 4.2.2.1 may now be applied with $J = \{i\}$ and $K = K'$, yielding a final contradiction.

Case 5: $\hat{B} = \{(a,c),(b,c)\}$. This case is similar in structure to Case 4, as any cycle must ‘‘go with’’ one arc in $B^g$, ‘‘go against’’ a second arc in $B^g$, and traverse a ‘‘gap’’ in the relation. The proof is omitted. Q.E.D.

Thus, we can make use of neutrality from Lemma 17. The next lemma extends the implications of $B^g = \emptyset$ and makes use of $n \geq 8$ agents.

Lemma 21 Let $r = \lfloor \frac{n}{4} \rfloor$ and $n \geq 8$. For all $G \subseteq N$, if $|G| \leq \lfloor \frac{n}{4} \rfloor + 2$, then $B^g = \emptyset$.

Proof Assume $n \geq 8$. Let $G \subseteq N$ satisfy $|G| = \lfloor \frac{n}{4} \rfloor + 1$, and suppose that $x B^g y$ for some $x$ and $y$. Let $i \in G$, and let $z \in X \setminus \{x,y\}$. By Lemma 20, we have $B^g \setminus \{i\} = \emptyset$, which implies that $N \setminus (G \setminus \{i\})$ is semi-decisive by Lemma 2. There are two cases.

Case 1: there exists $H \in S^+(\lfloor \frac{n}{4} \rfloor)$ such that $G \cap H = \emptyset$ and $H \cup \{i\}$ is not semi-blocking. By Lemmas 2 and 17, $N \setminus (H \cup \{i\})$ is semi-decisive. By $\lfloor \frac{n}{4} \rfloor$-Tie Break, $x D^{G \setminus H} y$ $D^{G \setminus H} z$ $D^{G \setminus (G \setminus \{i\})} x$, we have $(G \cup H) \cup (N \setminus (H \cup \{i\})) \cup (N \setminus (G \setminus \{i\})) = N$, ...
and we have \((G \cup H) \cap (N \setminus (H \cup \{i\})) \cap (N \setminus (G \setminus \{i\})) = \emptyset\), contradicting Lemma 3.

Case 2: for all \(H \in \mathcal{G}^+(\lfloor \frac{n}{4} \rfloor)\) such that \(G \cap H = \emptyset\), the group \(H \cup \{i\}\) is semi-blocking. We may partition \(N \setminus \{i\}\) into \(H, I, J, K\) such that \(|H|, |I|, |J| \geq \lfloor \frac{n}{4} \rfloor\), \(|K| \geq \lfloor \frac{n}{4} \rfloor - 1\), and \(G \cap (H \cup J) = \emptyset\). Choose any \(j \in I\). Since \(H \in \mathcal{G}^+(\lfloor \frac{n}{4} \rfloor)\) and \(G \cap H = \emptyset\), the supposition of Case 2 and \([\frac{n}{3}]\)-Tie Break imply that \(H \cup \{i\} \cup I\) is semi-decisive; similarly, \(J \cup \{i\} \cup K \cup \{j\}\) is semi-decisive. Note that \(n \geq 8\) implies \(|\{i,j\}| = 2 \leq \lfloor \frac{n}{4} \rfloor\), and then using Lemma 20, we have \(B^{i,j} \subseteq B^G = \emptyset\). Thus, Lemma 2 implies that \(N \setminus \{i,j\}\) is semi-decisive. But then we have \(x \in D^{G \setminus \{i,j\} \cup \{k\}}\) \(y \in D^{G \setminus \{i\} \cup \{j\}}\) \(z \in D^{G \setminus \{i\} \cup \{j\}}\) \(w \in D^{G \setminus \{i\} \cup \{j\}}\) \(x, y\). Since \(|(G \setminus \{i,j\}) \cup \{k\}| = |H \cup \{i,j\}| = \lfloor \frac{n}{2} \rfloor + 1\), we have shown that not \(x \in B^{G \setminus \{i,j\} \cup \{k\}}\) \(z \in B^{G \setminus \{i,j\} \cup \{k\}}\) \(y \in D^{G \setminus \{i,j\} \cup \{k\}}\) \(z \in D^{G \setminus \{i,j\} \cup \{k\}}\) \(x\), which implies \(z \in D^{H \setminus \{i,j\}}\) \(y \in D^{H \setminus \{i,j\}}\) \(z \in D^{H \setminus \{i,j\}}\) \(x\), we have \((G \cup H \cup \{k\}) \cap (N \setminus (H \cup \{i,j\})) \cap (N \setminus ((G \setminus \{i,j\}) \cup \{k\})) = N\), and we have \((G \cup H \cup \{k\}) \cap (N \setminus (H \cup \{i,j\})) \cap (N \setminus ((G \setminus \{i,j\}) \cup \{k\})) = \emptyset\), contradicting Lemma 3. We conclude that for all \(G \subseteq N\) such that \(|G| = \lfloor \frac{n}{4} \rfloor + 1\), we have \(B^G = \emptyset\).

Now let \(G \subseteq N\) satisfy \(|G| = \lfloor \frac{n}{4} \rfloor + 2\), and suppose that \(x \in B^G\) \(y\) for some \(x\) and \(y\). Choose distinct \(i, j \in G\), and let \(z \in X \setminus \{x, y\}\). Let \(H \subseteq N\) satisfy \(|H| = \lfloor \frac{n}{4} \rfloor - 1\) and \(G \cap H = \emptyset\), and choose \(k \in N \setminus (G \cup H)\). By \([\frac{n}{4}]\)-Tie Break, we have \(x \in D^{G \setminus \{i,j\} \cup \{k\}}\) \(y\). Since \(|(G \setminus \{i,j\}) \cup \{k\}| = |H \cup \{i,j\}| = \lfloor \frac{n}{2} \rfloor + 1\), we have shown that not \(x \in B^{G \setminus \{i,j\} \cup \{k\}}\) \(y\) \(z \in B^{G \setminus \{i,j\} \cup \{k\}}\) \(y\) \(z \in B^{G \setminus \{i,j\} \cup \{k\}}\) \(x\), which implies \(z \in D^{H \setminus \{i,j\}}\) \(y \in D^{H \setminus \{i,j\}}\) \(z \in D^{H \setminus \{i,j\}}\) \(x\), we have \((G \cup H \cup \{k\}) \cap (N \setminus (H \cup \{i,j\})) \cap (N \setminus ((G \setminus \{i,j\}) \cup \{k\})) = N\), and we have \((G \cup H \cup \{k\}) \cap (N \setminus (H \cup \{i,j\})) \cap (N \setminus ((G \setminus \{i,j\}) \cup \{k\})) = \emptyset\), contradicting Lemma 3. Q.E.D.

**Lemma 22** Let \(r = \lfloor \frac{n}{4} \rfloor\). If \(n \geq 8\), then \(N^r(F) \leq 3\).

**Proof** Assume \(n \geq 8\). Partition \(N\) into eight groups, \(G_1, \ldots, G_8\), such that for all \(k = 1, \ldots, 8\), \(|G_k| \in \{\lfloor \frac{n}{8} \rfloor, \lfloor \frac{n}{8} \rfloor + 1\} \). For all \(k, \ell = 1, \ldots, 8\), it follows that \(|G_k \cup G_\ell| \leq \lfloor \frac{n}{4} \rfloor + 2\), and because \(n \geq 8\), Lemma 21 implies that \(B^{G_k \cup G_\ell} = \emptyset\). We can furthermore specify that \(|G_k| \geq \lfloor \frac{n}{4} \rfloor - \lfloor \frac{n}{8} \rfloor\) \(k = 1, 3, 5, 7\); indeed, this is possible if \(4[\lfloor \frac{n}{8} \rfloor + \lfloor \frac{n}{4} \rfloor - \lfloor \frac{n}{8} \rfloor] \leq n\), which clearly holds. Thus, for \(k\) odd and every \(\ell\), we have \(|G_k| + |G_\ell| \geq \lfloor \frac{n}{4} \rfloor\). In the remainder of the proof, I use the following shorthand: given a set \(\alpha \subseteq \{1, \ldots, 8\}\) of indexes, \(G_\alpha = \bigcup_{k \in \alpha} G_k\).
I claim that given a collection of five consecutively indexed groups, \( G_{j+1}, G_{j+2}, G_{j+3}, G_{j+4}, G_{j+5} \), where \( j \in \{1, 2, 3\} \), it is not the case that both \( G_{j+1}, j+2, j+3 \) and \( G_{j+3}, j+4, j+5 \) are semi-blocking. Otherwise, given distinct \( x, y, \) and \( z \), we have \( x D_{j+1, j+2, j+3}^{G_{j+1}, j+2, j+3, j+4, j+5} y D_{j+3, j+4, j+5}^{G_{j+3}, j+4, j+5} z \). Index the remaining groups by \( k, \ell, m \not\in \{j+1, j+2, j+3, j+4, j+5\} \), and note that at least one, say \( k \), is odd. Since \( B_{G_{j+1}, j+2}^{G_{j+1}, j+2, j+3, j+4, j+5} = \emptyset \), it follows that \( N \setminus G_{j+3, k} \) is semi-decisive. Since \( G_{k, \ell, G_{k, m}} \in \mathcal{S}(\frac{1}{k}) \), \( \frac{1}{k} \)-Tie Break then yields

\[
x D_{j+1, j+2, j+3, k, \ell}^{G_{j+1}, j+2, j+3, k, \ell} y D_{j+3, j+4, j+5, k, \ell}^{G_{j+3}, j+4, j+5, k, \ell} z D_{N \setminus G_{j+3, k}}^{N \setminus G_{j+3, k}} x,
\]

but

\[
G_{j+1, j+2, j+3, k, \ell} \cup G_{j+3, j+4, j+5, k, \ell} \cup (N \setminus G_{j+3, k}) = N,
\]

\[
G_{j+1, j+2, j+3, k, \ell} \cap G_{j+3, j+4, j+5, k, \ell} \cap (N \setminus G_{j+3, k}) = \emptyset,
\]

contradicting Lemma 3. This establishes the claim. There are two cases.

Case 1: \( G_{1, 2, 3} \) is semi-blocking. Since \( G_{1, 5} \in \mathcal{S}(\frac{1}{k}) \), \( \frac{1}{k} \)-Tie Break implies \( G_{1, 2, 3, 4, 5} \) is semi-decisive. By the above claim \( G_{1, 2, 3, 4, 5} \) is not semi-blocking. By Lemmas 2 and 17, \( G_{1, 2, 6, 7, 8} \) is semi-decisive. Since \( B_{G_{1, 2}}^{G_{1, 2}, 2} = \emptyset \), \( N \setminus G_{1, 2} \) is semi-decisive. But then we have \( x D_{G_{1, 2}, 3, 4, 5}^{G_{1, 2, 3, 4, 5}} y D_{G_{1, 2, 6, 7, 8}}^{G_{1, 2, 6, 7, 8}} z D_{N \setminus G_{1, 2}}^{N \setminus G_{1, 2}} x \), we have \( G_{1, 2, 3, 4, 5} \cup G_{1, 2, 6, 7, 8} \cup (N \setminus G_{1, 2}) = N \), and we have \( G_{1, 2, 3, 4, 5} \cap G_{1, 2, 6, 7, 8} \cap (N \setminus G_{1, 2}) = \emptyset \), contradicting Lemma 3.

Case 2: \( G_{1, 2, 3} \) is not semi-blocking. Then Lemmas 2 and 17 imply that \( G_{1, 5, 6, 7, 8} \) is semi-decisive. By the above claim, there are two subcases.

Case 2.1: \( G_{1, 5, 6} \) is not semi-blocking. By Lemmas 2 and 17, \( G_{1, 2, 3, 7, 8} \) is semi-decisive. Since \( B_{G_{7, 8}}^{G_{7, 8}} = \emptyset \), \( N \setminus G_{7, 8} \) is semi-decisive. But then we have \( x D_{G_{1, 2}, 3, 4, 5}^{G_{1, 2, 3, 4, 5}} y D_{G_{1, 2, 3, 7, 8}}^{G_{1, 2, 3, 7, 8}} z D_{N \setminus G_{7, 8}}^{N \setminus G_{7, 8}} x \), we have \( G_{1, 2, 3, 7, 8} \cup G_{1, 2, 3, 7, 8} \cup (N \setminus G_{7, 8}) = N \), and we have \( G_{1, 2, 3, 7, 8} \cap G_{1, 2, 3, 7, 8} \cap (N \setminus G_{7, 8}) = \emptyset \), contradicting Lemma 3.

Case 2.2: \( G_{6, 7, 8} \) is not semi-blocking. By Lemmas 2 and 17, \( G_{1, 2, 3, 4, 5} \) is semi-decisive. Since \( B_{G_{4, 5}}^{G_{4, 5}} = \emptyset \), \( N \setminus G_{4, 5} \) is semi-decisive. But then we have \( x D_{G_{1, 2}, 3, 4, 5}^{G_{1, 2, 3, 4, 5}} y D_{G_{1, 2, 3, 7, 8}}^{G_{1, 2, 3, 7, 8}} z D_{N \setminus G_{4, 5}}^{N \setminus G_{4, 5}} x \), we have \( G_{1, 2, 3, 4, 5} \cup G_{1, 2, 3, 4, 5} \cup (N \setminus G_{4, 5}) = N \), and we have \( G_{1, 2, 3, 4, 5} \cap G_{1, 2, 3, 4, 5} \cap (N \setminus G_{4, 5}) = \emptyset \), contradicting Lemma 3.
G_{1,2,3,4,5} \cup (N \setminus G_{4,5}) = N, and we have G_{4,5,6,7,8} \cap G_{1,2,3,4,5} \cap (N \setminus G_{4,5}) = \emptyset, contradicting Lemma 3. Q.E.D.

We can prove a version of Lemma 22 for the case of 5 \leq n \leq 7, but for this range of n, we have \left\lfloor \frac{n}{4} \right\rfloor = \left\lfloor \frac{n}{5} \right\rfloor, and Theorem 1 follows from Schwartz’s (1986) Theorem 3.6.1. To finish the proof of Theorem 1, note that when n \geq 8 and |X| \geq 3, Lemma 22 yields N^*(F) \leq |X|, contradicting Lemma 4. We conclude that there is no social preference rule satisfying Independence, Pareto, Acyclicity, No \left\lfloor \frac{n}{4} \right\rfloor-Dictator, and \left\lfloor \frac{n}{4} \right\rfloor-Tie Break.

Proof of Theorem 2

In this subsection, we assume r = \left\lfloor \frac{n}{3} \right\rfloor, we allow any number of four or more agents, and in line with our maintained assumptions, we assume |X| \geq 4.

Lemma 23 Let r = \left\lfloor \frac{n}{3} \right\rfloor. If B^{G^*} \neq \emptyset, then there exist a, b, c \in X such that a B^{G^*} b D^{N \setminus G^*} c or c D^{N \setminus G^*} a B^{G^*} b.

Proof Assume B^{G^*} \neq \emptyset, and suppose there do not exist such a, b, and c. Note that for all x, y \in X such that x B^{G^*} y and all z \in X \setminus \{x, y\}, we have x B^{G^*} z B^{G^*} y. Indeed, otherwise Lemma 2 implies z D^{N \setminus G^*} x B^{G^*} y or x B^{G^*} y D^{N \setminus G^*} z, allowing us to set a = x, b = y, and c = z. Now, B^{G^*} \neq \emptyset implies there exist s, t \in X such that s B^{G^*} t. By the preceding argument, given any u \in X \setminus \{s, t\}, s B^{G^*} t implies s B^{G^*} u B^{G^*} t. Furthermore, s B^{G^*} u implies t B^{G^*} u, giving us u B^{G^*} t B^{G^*} u. Likewise, given any v \in X \setminus \{s, t, u\}, we have v B^{G^*} t B^{G^*} v. By transitivity, from Lemma 9, we then have u B^{G^*} v B^{G^*} u. This establishes that B^{G^*} \cap (X \setminus \{s\}) \times (X \setminus \{s\}) = ((X \setminus \{s\}) \times (X \setminus \{s\})) \setminus \Delta.

I claim that, given u \in X \setminus \{x, y\}, we have s B^{N \setminus G^*} u. Otherwise, we have u B^{G^*} s by Lemma 2, and then Lemma 9 yields B^{G^*} = (X \times X) \setminus \Delta, contradicting Lemma 13. Q.E.D.

Lemma 24 Let r = \left\lfloor \frac{n}{4} \right\rfloor. Then B^{G^*} = \emptyset.
Proof Suppose $B^* \neq \emptyset$. Given $G^*$, let $H \in \mathcal{S}^-(\frac{4}{3})$ satisfy $|H| = \mu(\frac{4}{3})$ and $G^* \cap H = \emptyset$, let $I$, $J$, and $K$ be as in Lemma 8, and let $h \in H$ and $k \in K$. By Lemma 23, there are two cases.

Case 1: there exist $x,y,z \in X$ such that $x \in B^*$ and $y \in D^{|G^*|} z$. Let $w \in X \setminus \{x,y,z\}$. There are two subcases.

Case 1.1: $x \in B^*$ w. There are two further subcases.

Case 1.1.1: $y \in B^*$ z. Since $x \in B^*$ y (by Case 1) and $H \cup J \in \mathcal{S}^+(\frac{4}{3})$, $\lfloor \frac{4}{3} \rfloor$-Tie Break implies $x \in D^{|G^*|,I,H,J} y$, and since $y \in B^*$ z and $H \cup I \in \mathcal{S}^+(\frac{4}{3})$, $\lfloor \frac{4}{3} \rfloor$-Tie Break implies $y \in D^{|G^*|,J} z$. Since $x \in B^*$ y (again Case 1), since $\{x,y\} \cap \{z,w\} = \emptyset$, and since $G^* \cap H \in \mathcal{S}^-(\frac{4}{3})$, Lemma 10 implies not $w \in B^*$ z, which implies $z \in D^{|H|} w$ by Lemma 2. By Pareto, $w \in D^X x$. But then we have $x \in D^{|G^*|,I,H,J} y \in D^{|G^*|,K} z \in D^{|H|} w \in D^X x$, we have $(G^* \cup H \cup J) \cap (K \cup H \cup I) \cup (N \setminus H) \cup N = N$, and we have $(G^* \cup H \cup J) \cap (K \cup H \cup I) \cap (N \setminus H) \cap N = \emptyset$, contradicting Lemma 3.

Case 1.1.2: not $y \in B^*$ z. By Lemma 2, $z \in D^{|N|} I$. There are two further subcases.

Case 1.1.2.1: $w \in D^{|G^*|} z$. Since $x \in B^*$ w (by Case 1) and $K \in \mathcal{S}^+(\frac{4}{3})$, $\lfloor \frac{4}{3} \rfloor$-Tie Break implies $x \in D^{|G^*|,K} w$. Also, $z \in D^{|N|} I$ (by Case 1.1.2). By Pareto, $y \in D^X x$. But then we have $x \in D^{|G^*|,K} w \in D^{|G^*|,K} y \in D^X x$, we have $(G^* \cup K) \cap (N \setminus G^*) \cap (N \setminus K) \cap N = N$, and we have $(G^* \cup K) \cap (N \setminus G^*) \cap (N \setminus K) \cap N = \emptyset$, contradicting Lemma 3.

Case 1.1.2.2: not $w \in D^{|N|} z$. By Lemma 2, $z \in B^*$ w. Since $I \cup J \in \mathcal{S}^+(\frac{4}{3})$, $\lfloor \frac{4}{3} \rfloor$-Tie Break implies $z \in D^{|G^*|,I,H,J} w$. Since $x \in B^*$ y (Case 1) and $K \in \mathcal{S}^+(\frac{4}{3})$, $\lfloor \frac{4}{3} \rfloor$-Tie Break implies $y \in D^{|G^*|,K} z$. Also, $y \in D^{|G^*|,z}$ (Case 1). By Pareto, $w \in D^X x$. But then we have $x \in D^{|G^*|,K} y \in D^{|G^*|} z \in D^{|G^*|,I,H,J} w \in D^X x$, we have $(G^* \cup K) \cap (N \setminus G^*) \cup (G^* \cup I \cup J) \cup N = N$, we have $(G^* \cup K) \cap (N \setminus G^*) \cap (G^* \cup I \cup J) \cap N = \emptyset$, contradicting Lemma 3.

Case 1.2: not $x \in B^*$ w. By Lemma 2, $w \in D^{|N|} x$. There are two subcases.

Case 1.2.1: $z \in D^{|N|} z$. Since $x \in B^*$ y (by Case 1) and $H \cup J \in \mathcal{S}^+(\frac{4}{3})$, $\lfloor \frac{4}{3} \rfloor$-Tie Break implies $x \in D^{|G^*|,H,J} y$. Since $H \in \mathcal{S}^-(\frac{4}{3})$, we have $H \subseteq B^*$. Using Lemma 2 and $y \in D^{|G^*|} z$ (by Case 1), this implies $y \in D^{|N|,H} z$. Also, $z \in D^{|G^*|} x$ (by Case 1.2). But then we have $x \in D^{|G^*|,H,J} y \in D^{|N|,H} z \in D^{|G^*|,I,J} w \in D^{|G^*|} x$, we have $(G^* \cup H \cup J) \cup (N \setminus H) \cup (N \setminus J) \cup (N \setminus G^*) = N$, and we have
Lemma 25 Let \( r = \left\lfloor \frac{n}{6} \right\rfloor \). Then \( N^*(F) \leq 4 \).

Proof Partition \( N \) into six groups, \( G_1, \ldots, G_6 \), such that for all \( k = 1, \ldots, 6 \), \( |G_k| \in \{ \left\lfloor \frac{n}{6} \right\rfloor, \left\lfloor \frac{n}{6} \right\rfloor + 1 \} \). We can furthermore specify that \( |G_k| \geq \left\lfloor \frac{n}{6} \right\rfloor - \left\lfloor \frac{\alpha}{6} \right\rfloor \) for \( k = 1, 3, 5 \); indeed, this is possible if \( 3\left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor - \left\lfloor \frac{\alpha}{6} \right\rfloor \leq n \), which clearly holds. Thus, for \( k \) odd and every \( \ell \), we have \( |G_k| + |G_\ell| \geq \left\lfloor \frac{n}{6} \right\rfloor \). To formulate a final restriction on these groups, write \( \frac{n}{6} = \left\lfloor \frac{n}{6} \right\rfloor + \alpha \), where \( \alpha \in \{ \frac{m}{6} \mid m = 0, 1, 2, 3, 4, 5 \} \), and note that the number of groups \( G_k \) such that \( |G_k| = \left\lfloor \frac{n}{6} \right\rfloor + 1 \) is \( 6\alpha \). In particular, this number is no more than five, so we can require that \( |G_i| = \left\lfloor \frac{n}{6} \right\rfloor \). In the remainder of the proof, I use the shorthand from Lemma 22: given \( \gamma \subseteq \{ 1, \ldots, 6 \} \), \( G_\gamma = \bigcup_{k \in \gamma} G_k \). There are two cases.

Case 1: \( G_{1,2} \) is semi-blocking. Since \( G_{3,4} \in \mathcal{S}^+([\frac{n}{3}]) \), \( \left\lfloor \frac{n}{6} \right\rfloor \)-Tie Break implies that \( G_{1,2,3,4} \) is semi-decisive. I claim we can partition \( G_{1,2,3,4} \) into three groups \( \{ H_1, H_2, H_3 \} \) such that \( |H_k| \leq \left\lfloor \frac{n}{3} \right\rfloor \) for \( k = 1, 2, 3 \). Writing \( \frac{n}{3} = 2\left\lfloor \frac{n}{6} \right\rfloor + 2\alpha \), it is apparent that

\[
\left\lfloor \frac{n}{3} \right\rfloor = \begin{cases} 
2\left\lfloor \frac{n}{6} \right\rfloor & \text{if } 6\alpha = 0, 1, 2 \\
2\left\lfloor \frac{n}{6} \right\rfloor + 1 & \text{if } 6\alpha = 3, 4, 5.
\end{cases}
\]

Consistent with the notational conventions above, index the groups so that \( |G_1| \geq |G_3| \geq |G_2| \geq |G_4| \). When \( 6\alpha = 0, 1, 2 \), define \( H_1 = G_{2,4} \), \( H_2 = G_1 \), and \( H_3 = G_3 \), so that \( |H_1| = 2\left\lfloor \frac{n}{6} \right\rfloor = \left\lfloor \frac{n}{3} \right\rfloor \) and \( |H_2| = |H_3| \leq \left\lfloor \frac{n}{6} \right\rfloor + 1 \leq \left\lfloor \frac{n}{3} \right\rfloor \). When \( 6\alpha = 3, 4, 5 \), define \( H_1 = G_{1,4} \), \( H_2 = G_2 \), and \( H_3 = G_3 \), so that \( |H_2| \leq |H_3| \leq |H_1| = 2\left\lfloor \frac{n}{6} \right\rfloor + 1 = \left\lfloor \frac{n}{3} \right\rfloor \). This establishes

\[(G^* \cup H \cup J) \cap (N \setminus H) \cap (N \setminus J) \cap (N \setminus G^*) = \emptyset, \text{ contradicting Lemma 3.}\]

Case 1.2.2: not \( z \in D_{H,J} \). By Lemma 2, \( w \in B_{I} \). Since \( H \cup I \in \mathcal{S}^+([\frac{n}{3}]) \), \( \left\lfloor \frac{n}{6} \right\rfloor \)-Tie Break implies \( w \in D_{J,U,H,I} \). Since \( x \in B^* y \) (Case 1) and \( K \in \mathcal{S}^+([\frac{n}{3}]) \), \( \left\lfloor \frac{n}{6} \right\rfloor \)-Tie Break implies \( x \in D_{I,U,K} \). By Pareto, \( y \in D^* w \) and \( z \in D \). But then we have \( x \in D_{I,U,K} \) and \( y \in D^* w \) \( D^* \) and \( D^* \) \( x \), we have \( (G^* \cup K) \cup (J \cup H \cup I) \cap N = N \), and we have \( (G^* \cup K) \cap N \cap (J \cup H \cup I) \cap N = \emptyset \), contradicting Lemma 3.

Case 2: there exist \( x, y, z \in X \) such that \( x \in D_{I,U,K} \) \( y \in D^* w \). The arguments for this case are analogous to Case 1. Q.E.D.
the claim. By Lemma 15, we have \( B_{H_1} = B_{H_2} = B_{H_3} = \emptyset \), so Lemma 2 implies \( N \setminus H_k \) is decisive for \( k = 1,2,3 \). But then we have \( x \in D^{G_1,2,3,4} \) and \( y \in D^{N \setminus H_1} \) and \( z \in D^{N \setminus H_2} \) and \( w \in D^{N \setminus H_3} \), we have \( G_1,2,3,4 \cap (N \setminus H_1) \cap (N \setminus H_2) \cap (N \setminus H_3) = \emptyset \), contradicting Lemma 3.

Case 2: \( G_{1,2} \) is not semi-blocking. By Lemmas 2 and 17, \( G_{3,4,5,6} \) is semi-decisive. Relying on the above argument, partition \( G_{3,4,5,6} \) into three groups \( \{H_1, H_2, H_3\} \) such that \( |H_k| \leq \lfloor \frac{n}{3} \rfloor \) for \( k = 1,2,3 \). By Lemma 15, we have \( B_{H_1} = B_{H_2} = B_{H_3} = \emptyset \), so Lemma 2 implies \( N \setminus H_k \) is decisive for \( k = 1,2,3 \). But then we have \( x \in D^{G_{3,4,5,6}} \) and \( y \in D^{N \setminus H_1} \) and \( z \in D^{N \setminus H_2} \) and \( w \in D^{N \setminus H_3} \), we have \( G_{3,4,5,6} \cap (N \setminus H_1) \cap (N \setminus H_2) \cap (N \setminus H_3) = \emptyset \), contradicting Lemma 3. Q.E.D.

To finish the proof of Theorem 2, note that because \( |X| \geq 4 \), Lemma 25 yields \( N^*(F) \leq |X| \), contradicting Lemma 4. We conclude that there is no social preference rule satisfying Independence, Pareto, Acyclicity, No \( [\frac{n}{3}] \)-Dictator, and \( [\frac{n}{3}] \)-Tie Break.

References


