

Behavioral Competitive Equilibrium and Extreme Prices*

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Abstract

A *behavioral competitive equilibrium* restricts households ability to tailor their consumption to the state of the economy. Compared to standard competitive equilibrium, a behavioral competitive equilibrium yields more consumption risk and extreme price volatility when the realized output is near its maximum or minimum.

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1 Introduction

Cognitive limitations, such as the inability to formulate complex plans, difficulties in information processing or limited memory, hinder individuals' ability to react to changes in the economic environment. Simple plans cannot respond to every variation in economic fundamentals. Limits on information processing and limited memory render plans that incorporate all available information infeasible and, as a result, individuals cannot tailor their actions as precisely to the economic environment as would be desirable.

Models of behavioral optimization modify the description of the standard economic agent, for example, by adding constraints that capture the consequences of cognitive limitations. In this paper, we analyze a competitive exchange economy with behavioral optimizers. Households choose consumption as a function of the realized state of the economy but consumption plans must be simple; they can have at most k distinct consumption realizations in an economy with $n > k$ states. We refer to plans that satisfy this constraint as *crude consumption plans*. They implement the idea that cognitive limitations lead to an imperfect adjustment of consumption to economic circumstances.

To motivate crude consumption plans, consider households who find it difficult to precisely identify the state of the economy. Such households may only distinguish broad categories of economic events and adapt their plans to these broad categories. In the most extreme case, all the household can do is partition circumstances into “good times” and “bad times” and choose one consumption level for each event. Another possibility is that households cannot formulate consumption plans that are too complex, as measured by the number of realized values in a plan. Alternatively, households may incur a fixed cost for each consumption level, either because planning consumption requires cognitive effort or because there are adjustment costs as in Grossman and Laroque (1990). An appropriately chosen fixed cost would lead households to choose crude consumption plans.¹

Except for the restriction to crude consumption plans, our economy is a standard exchange economy with complete markets. A *behavioral competitive equilibrium* (BCE) is a competitive

¹There is a duality between fixed costs and the crudeness constraint k : for a given optimal crude consumption plan we can choose a fixed cost so that the same plan is optimal in the costly adjustment model.

equilibrium in which agents choose optimal crude consumption plans. We study BCEs in both a two period model and in a dynamic Lucas tree economy. In both cases, there is a continuum of identical agents, each endowed with a unit of the productive asset and identical CRRA utility functions.

For ease of exposition our model has a single physical good in every state of nature; households have identical endowments and identical utility functions. In this context, the restriction that consumption cannot perfectly track the state of the economy implies that households cannot consume their endowment. Thus, we require that households sell their endowment on the market and use the receipts to buy a crude consumption plan. A more realistic model would have multiple physical goods, production and households whose endowment consists mainly of time. In such a model, consuming the endowment is not a feasible consumption plan for the typical household and hence our assumption would be satisfied. We analyze the single-good environment to simplify our analysis. The restriction that households cannot consume their endowment should be interpreted as an implication of a more realistic setting.

The two period model has a planning period and a consumption period. Households learn the value of the endowment after the planning period and before the consumption period. In the planning period, each household chooses a crude (state-contingent) consumption plan. We show existence and Pareto optimality (given the restriction to crude consumption plans) of a BCE for this economy. Notice that a single crude consumption plan cannot distinguish between all the states of the economy. Therefore, if all consumers were to choose the same crude consumption plan then markets cannot clear. It follows that in a BCE ex ante identical consumers choose distinct plans. For example, if households are constrained to choose two consumption levels (“good times” and “bad times”) then, in equilibrium, households must differ in how they classify the states of the economy into these two categories. A consequence of this differentiation is that consumption is more risky in a BCE than in a standard competitive economy.

Our main focus is the analysis of equilibrium price and price volatility when the random endowment approaches a continuous distribution. We fix the households’ utility function and the range of possible endowment realizations and consider a sequence of economies that converges to an economy with a non-atomic distribution of endowments. We show that, in the limit,

prices in states near the lowest or the highest possible endowment realization are *extreme*; the price of consumption converges to infinity when the endowment is at or near the lower bound of the distribution; the price converges to zero when the endowment is near or at the upper bound of the distribution. Thus, when the state of the economy is unusually good (high endowment) or unusually bad (low endowment) prices are very volatile in the sense that small changes in the state of the economy lead to large changes in prices.

Next, we consider a dynamic economy. In that case, a finite state Markov chain describes the transitions of the endowment realization. We show existence of a stationary BCE. The equilibrium of the dynamic economy can be mapped to an equilibrium of an appropriately defined two-period model. This allows us to describe equilibrium allocations and prices in the dynamic economy in terms of the prices and allocations of a two-period model.

As in the two-period case, we fix the utility function and the range of endowment realizations and consider a sequence of economies converging to a limit economy with a non-atomic invariant distribution of endowments. The extreme consumption prices of the two-period model imply extreme asset prices in the dynamic economy. Specifically, consider an asset that pays off a share of the endowment in all future periods. The price of this asset is a random variable that depends only on the current state of the economy (the current endowment). When the realized endowment is near or at the lower bound of possible endowments then the price of the asset is essentially zero. Conversely, as we approach the upper bound of possible endowment realizations then the price of this asset converges to infinity. Thus, the asset price is very volatile near unusually high or low realizations of the endowment.

Our results show how behavioral constraints can create extreme price volatility in a standard competitive model. Since households cannot adjust their consumption to every change in the economic environment they must choose which events to react to. Extreme price volatility occurs at events where households have little incentive to adjust their consumption. We show that this is the case near unusually low or unusually high endowment realizations. In those events market clearing requires large price swings.

1.1 Relation to Literature

An empirical literature in financial economics suggests that the high volatility of stock prices is at odds with the relatively low volatility of economic fundamentals, see, e.g., [Shiller \(1981\)](#); [LeRoy and Porter \(1981\)](#), and [Campbell \(2003\)](#) for a literature review. One approach to this equity volatility puzzle is to introduce a combination of nonstandard preferences ([Epstein and Zin \(1989, 1991\)](#); [Weil \(1989\)](#)) and novel assumptions about the growth rate dynamics ([Bansal and Yaron \(2004\)](#); [Bansal, Kiku, and Yaron \(2010\)](#)).² A second approach, more closely related to our work, modifies the household's problem by introducing additional constraints, such as adjustment costs or adjustment lags.

[Grossman and Laroque \(1990\)](#) distinguish liquid and illiquid consumption and assume a transaction cost must be paid when an illiquid good is sold. [Gabaix and Laibson \(2002\)](#) study a model in which only a fraction of agents can make adjustments at a given time. Both papers relate the consumption rigidities to price movements. [Chetty and Szeidl \(2007\)](#) study the effect of consumption rigidities on risk preferences while [Chetty and Szeidl \(2010\)](#) focuses on the extent to which consumption rigidities reduces stock market participation.³ The common feature between our model and these papers is the connection between consumption rigidities and price volatility.

[Sims \(2003\)](#) introduces a model of rational inattention by imposing a bound on the information processing capacity of agents. [Mankiw and Reis \(2002\)](#) study a model where in each period, only a fraction of agents obtains new information about the state of the economy. [Hong, Stein, and Yu \(2007\)](#) study a model where agents are restricted to a small class of forecasting models that does not include the true model of the world.

The game theory literature has developed related ideas in the context of bounded rationality. [Rubinstein \(1986\)](#) limits players to finite automata, [Abreu and Rubinstein \(1988\)](#) impose a utility cost on the complexity of strategies; [Dow \(1991\)](#) analyzes a search model with limited memory; [Piccione and Rubinstein \(1997\)](#) examine the relation between bounded memory and

²See also [Beeler and Campbell \(2009\)](#) who point to some empirical difficulties with the implied high elasticity of intertemporal substitution.

³See also [Güvenen \(2009\)](#), who studies limited stock market participation in a model with heterogeneity in the elasticity of intertemporal substitution in consumption.

time consistency; Wilson (2002) examines the effect of bounded memory on long run inference; Jehiel (2005) and Jehiel and Samet (2007) constrain players to respond the same way in similar situations. Despite differences in motivation (limited complexity, limited memory, limited attention), these papers impose additional constraints on decision makers that serve a role analogous to the role played by our crude consumption plans: they restrict agents' ability to tailor their behavior to their environment. We have chosen a particularly simple form of this restrictions that is tractable in a competitive setting.

2 BCE in a Static Economy

The economy consists of a continuum of identical households, each endowed with a unit of the productive asset. The asset yields a stochastic return (endowment) in each state $i \in N = \{1, \dots, n\}$. Let π_i be the probability of the uncertain state i , s_i be the endowment in state i and $K(s) \subset [a, b]$ be the finite support of s .

Let $\mathcal{C} = \mathbb{R}_+^n$ be the set of all consumption plans. For any set X , let $|X|$ denote the cardinality of X . In a BCE, households optimize over *crude consumption plans*, which restrict the number of distinct consumption levels that a household can choose to k . To avoid trivial cases, we assume throughout that $1 < k < |K(s)|$.

Definition 1. The consumption plan $c \in \mathcal{C}$ is *crude* if $|\{c_i \mid i \in N\}| \leq k$.

Let \mathcal{C}_k be the set of all crude consumption plans. Clearly, $\mathcal{C}_k \subset \mathcal{C}_{k+1}$ for all k . When $k = 2$, a crude consumption plan partitions the endowment realization into good and bad times. In good times consumption is high and in bad times consumption is low. Note that \mathcal{C}_k is not a convex set since a convex combination of $c, c' \in \mathcal{C}_k$ would typically require more than k consumption levels. This non-convexity implies that, in equilibrium, identical households will choose different partitions of the state space. For example, when $k = 2$, some households will consider a particular endowment realizations a good outcome and choose their high consumption while others will consider it bad outcome and choose their low consumption.

A household's utility of the consumption plan c is

$$U(c) = \sum_{i \in N} u(c_i) \pi_i$$

where u is any strictly concave CRRA utility index. That is,

$$u(c_i) = \begin{cases} c_i^{1-\rho}/(1-\rho) & \text{if } \rho \neq 1 \\ \ln c_i & \text{if } \rho = 1 \end{cases}$$

Note that u is strictly concave if $\rho > 0$ and bounded above if $\rho > 1$.

The quartuple $E = (u, k, \pi, s)$ is a static economy. An element p in the $n - 1$ -dimensional simplex $\Delta(N)$ is a (normalized) *price*. At price p , the household's wealth in E is

$$w(p) = \sum_{i \in N} s_i \cdot p_i.$$

The household's budget set, $B_k(p)$, is

$$B_k(p) = \left\{ c \in \mathcal{C}_k : \sum_{i \in N} p_i \cdot c_i \leq w(p) \right\}$$

The crude consumption plan $c \in B_k(p)$ is *optimal at prices* p if $U(c) \geq U(c')$ for all $c' \in B_k(p)$.

For any set X , let $\Delta(X)$ denote the set of simple probabilities on X ; that is $\Delta(X)$ is the set of all functions in $\mu \in [0, 1]^X$ such that $K(\mu) = \{x : \mu(x) > 0\}$ is finite and $\sum_{x \in X} \mu(x) = 1$. For any $Y \subset X$, we let $\mu(Y)$ denote $\sum_{x \in Y} \mu(x)$. We call $K(\mu)$ the support of μ . When X is finite, we identify $\Delta(X)$ with the $|X| - 1$ dimensional simplex.

An *allocation* μ is a probability on consumption plans. Thus, $\Delta(\mathcal{C})$ is the set of allocations. An allocation is crude if $\mu(\mathcal{C}_k) = 1$. The allocation μ is *feasible* in E if

$$\sigma_i(\mu) := \sum_{c \in K(\mu)} c_i \cdot \mu(c) \leq s_i$$

for all $i \in N$. For any $\mathcal{C}' \subset \mathcal{C}$, let $M(\mathcal{C}')$ be the set of all feasible allocations such that $\mu(\mathcal{C}') = 1$.

Hence, $M(\mathcal{C}_k)$ is the set of feasible allocations for our behavioral economy. Henceforth, μ is feasible means $\mu \in M(\mathcal{C}_k)$.

Definition 2. The price-allocation pair (p, μ) is a *BCE* of E if μ is feasible for E and if $\mu(c) > 0$, then c is optimal at prices p .

We call $p(\mu)$ a BCE price (allocation) if (p, μ) is a BCE for some $\mu(p)$. Two consumption plans c, c' are *conformable* if $c_i = c_j$ if and only if $c'_i = c'_j$; i.e., two consumption plans are conformable if they induce the same partition of N . We write $c \sim c'$ if c and c' are conformable.

Definition 3. An allocation is *simple* if $\mu(c) \cdot \mu(c') > 0$ and $c \sim c'$ implies $c = c'$. An allocation is *fair* if $\mu(c) \cdot \mu(c') > 0$ implies $U(c) = U(c')$.

For every partition of N a simple allocation has at most one consumption plan in its support that is measurable with respect to that partition. Therefore, if μ is simple then the cardinality of $K(\mu)$ is finite and at most equal to the number of partitions of N with k or fewer elements. A fair allocation yields the same utility for each consumption plan in its support.

Definition 4. The plan c is *monotone* if (i) $c_i \geq c_j$ whenever $s_i > s_j$ and (ii) $c_i = c_j$ whenever $s_i = s_j$. The allocation μ is *monotone* if all $c \in K(\mu)$ are monotone.

Part (i) of the definition requires that consumption is weakly increasing in the aggregate endowment. Part (ii) says that the allocation remains feasible if states with identical endowments are combined into a single state.

The mean utility, $W(\mu)$, of allocation μ is

$$W(\mu) = \sum_c U(c) \cdot \mu(c)$$

We can also view $W(\mu)$ as the ex ante utility of the representative household before the random consumption plan μ is implemented. We say that $\mu \in M(\mathcal{C}_k)$ solves the planner's problem if $W(\mu) \geq W(\mu')$ for all $\mu' \in M(\mathcal{C}_k)$. In the standard economy the unique solution to the planner's

problem is for each household to consume the endowment. Note that this allocation is simple, monotone and fair. Lemma 1 below shows the solution to the planner's problem retains these properties when the planner is restricted to crude consumption plans.

Lemma 1. *(i) There is a solution to the planner's problem (ii) Every solution to the planner's problem is simple, fair and monotone.*

As long as the utility function is strictly concave, a solution to the planner's problem must be simple. By contrast, fairness and monotonicity rely on the assumption of CRRA. The following example illustrates how fairness or monotonicity may fail if u is not CRRA.

Example 1: Let u be defined as

$$u(x) = \begin{cases} 2x & \text{if } x \in [0, 1) \\ x + 1 & \text{if } x \in [1, 2] \\ x/2 + 2 & \text{if } x > 2 \end{cases}$$

Let $N = \{1, 2, 3, 4\}$ with $s_1 = 1, s_2 = 4/3, s_3 = 5/3, s_4 = 2$ and assume all states are equally likely. For this economy, there is no fair *and* monotone solution to the planner's problem. The following allocation is a *fair but not monotone* solution to the planner's problem: a $2/3$ -fraction of households choose the consumption plan $(c_1, c_2, c_3, c_4) = (1, 1, 2, 2)$ and a $1/3$ -fraction of households choose $(\hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4) = (1, 2, 1, 2)$. It is easy to verify that the planner's problem also has solutions that are *monotone but not fair*. The utility function in this example is concave but not strictly concave. However, it is easy to show that a strictly concave approximation of the above defined piecewise linear utility function would lead to the same conclusion: the solutions to the planner's problem cannot be simultaneously fair and monotone. Hence, Lemma 1 is not true for all strictly concave utility functions.

Households in our model satisfy the standard assumption of local non-satiation and, therefore, the first welfare theorem implies that BCE allocations are Pareto efficient. (Of course, Pareto efficiency is defined with respect to allocations in $M(\mathcal{C}_k)$.) By Lemma 1, the planner's

problem has a fair solution and, since all households have the same endowment, BCE allocations must be fair. Thus, if a BCE allocation did not solve the planner's problem there would exist a fair allocation that yields higher household utility, contradicting the Pareto efficiency of BCEs. Hence, every BCE allocation must solve the planner's problem. Theorem 1, below, establishes the converse.

Theorem 1. *An allocation solves the planner's problem if and only if it is a BCE allocation.*

Theorem 1 and Lemma 1 together establish the existence and Pareto-efficiency of BCE. They also show that every BCE allocation is simple, fair and monotone. In a *pure endowment economy* the realized endowment resolves all uncertainty, that is, s is one-to-one. In that case we can strengthen Theorem 1 and show that equilibrium prices must be *monotone*, that is, $s_i \geq s_j$ implies $p_j/\pi_j \geq p_i/\pi_i$.

Definition 5. The price p is monotone if $s_i \geq s_j$ implies $p_j/\pi_j \geq p_i/\pi_i$

Theorem 2. *If E is a pure endowment economy and $(\mu, p), (\mu, p')$ are BCE for E then $p = p'$ and p is monotone.*

Theorem 2 shows that for a pure endowment economy there is a unique price for every BCE allocation and, moreover, this price is non-increasing in the aggregate endowment. When an economy is not a pure endowment economy there may be multiple prices supporting a particular solution of the planner's problem. However, the proof of Theorem 2 implies that even in this case there exists a monotone price.

3 Consumption Risk and Price Variation in a BCE

In this section, we compare the BCE for economy E to the standard equilibrium for E . In particular, we show that consumption is riskier and prices are more extreme in a BCE than in a standard competitive equilibrium (SCE).

We first establish that BCE yields greater consumption risk. Given π and $z \in \mathbb{R}^N$, we let F_z denote the cumulative distribution of the random variable z ; that is, $F_z(x) = \sum_{i: z_i \leq x} \pi_i$. Hence, F_s is the cdf of the endowment and F_c is the cdf of consumption associated with the plan c . Then, G_μ , the cdf of consumption given the allocation μ is

$$G_\mu(x) = \sum_c F_c(x) \cdot \mu(c).$$

We can think of the (random) consumption as the result of a two stage lottery; the first stage reveals the state $i \in N$ and the second stage reveals the consumption in state i . Since $k < |K(s)|$ consuming the endowment in every state is not feasible. Therefore, there are $c \in K(\mu)$ with $c_i \neq s_i$ for some $i \in N$. Moreover, since μ is feasible it follows that $s_i \geq \sigma_i(\mu)$ for all i and hence the expected value of consumption in state i is less than or equal to s_i , the SCE consumption in state i . This implies that SCE consumption second order stochastically dominates BCE consumption and hence welfare (i.e., mean utility) in a BCE is strictly less than welfare in a SCE. Put differently, the fact that s is not in \mathcal{C}_k , by itself, ensures that households bear greater consumption risk in a BCE than in a SCE.

In the unique SCE the equilibrium price $p^* \in \Delta(N)$ is

$$p_i^* = \frac{\pi_i u'(s_i)}{\sum_{i=1}^n \pi_i u'(s_i)}$$

Since $s_i \in [a, b]$ it follows that for all $i \in N$

$$\frac{p_i^*}{\pi_i} \in \left[\frac{u'(b)}{u'(a)}, \frac{u'(a)}{u'(b)} \right]$$

Thus, the SCE price (normalized by the probability of the state) is bounded away from zero and infinity. Moreover, if $b - a$ is small then $u'(b)/u'(a)$ is close to 1 and therefore the maximal difference in price is small.

By contrast, for any fixed range of the endowment realization $[a, b]$, the equilibrium price $p \in \Delta(N)$ in a BCE (normalized by the probability of the state) can be arbitrarily large or arbitrarily close to zero. Thus, BCE exhibits *extreme prices*. Specifically, extreme prices emerge

when the endowment has many possible realizations and F_s approaches a nonatomic, continuous distribution.

Let E^n be a pure endowment economy with n states and order states so that $s_i < s_j$ if $i < j$. Next, we define a sequence of economies converging to a limit economy with a continuous endowment distribution.

Definition 6. A sequence of economies $E^n = (u, \pi^n, s^n)$ is *almost continuous* if $F_{s^n} \rightarrow F$ converges in distribution to a random variable with a continuous strictly positive density f on $[a, b]$.

To characterize equilibrium prices with many possible endowment realizations it is convenient to represent the equilibrium price p^n as a *cumulative price* $P^n : [0, 1] \rightarrow [0, 1]$. Define $F_N(0) = 0$ and for $i \in N$ define $F_N(i) = \sum_{j \leq i} \pi_j^n$ to be the probability that a state in $\{1, \dots, i\}$ occurs. Define $P^n(0) = 0$ and

$$P^n(F_N(i)) := \sum_{j \leq i} p_j^n \quad (1)$$

and for $x \in [F_N(i-1), F_N(i)]$ define

$$P^n(x) = P^n(F_N(i-1)) + (x - F_N(i-1)) \frac{p_i^n}{\pi_i^n} \quad (2)$$

Note that $P^n(F_N(i))$ is the cost of one unit of consumption in the event $\{1, \dots, i\}$. The function P^n is an extension of this function assuming that agents can purchase consumption in “fractions” of a state. Since the price p_i^n is decreasing in the state i the function $P^n(x)$ can be interpreted as the *maximal cost* of one unit of consumption in events with probability x and $1 - P^n(1 - x)$ is the *minimal cost* of 1 unit of consumption in events with probability x .

For any equilibrium price p^n , the cumulative price P^n is *increasing* with $P^n(1) = 1$. Theorem 2 implies that P^n is *concave*. If P^n is differentiable, that is, if $x \in (F_N(i-1), F_N(i))$ for some $i \in N$, and $[P^n]'$ is the derivative of P^n then

$$[P^n]'(x) = \frac{p_i}{\pi_i}$$

Thus, the derivative of P^n at x is the price per unit of probability if the state is at the x -percentile.

Lemma 11 shows that there exists P such that (for an appropriate subsequence) $P^n(x) \rightarrow P(x)$ for all $x > 0$. We refer to P as the limit cumulative price. Moreover, Lemma 11 shows that the limit cumulative P has the following properties:

- (i) P is concave and non-decreasing with $P(1) = 1$
- (ii) $P(x) = P(0) + \int_0^x \hat{p}(r)dr$ for some $\hat{p} : (0, 1] \rightarrow [0, \infty)$.

Thus, the limit cumulative price may have a mass point at zero ($P(0) > 0$) but has no other mass points. In the following we refer to the limit cumulative price simply as the *limit price* P .

As a benchmark, we first consider the limit price of a sequence of standard economies. In that case, it is straightforward to show that there is a unique equilibrium limit price P^* . Moreover, it satisfies

$$P^*(x) = \int_0^x \hat{p}^*(r)dr$$

with

$$\hat{p}^*(x) = \alpha u'(F^{-1}(x))$$

where α is some strictly positive constant and F is the cdf of the limit endowment. Hence, $P^*(0) = 0$ and $\hat{p}^*(x)$ is uniformly bounded away from zero and infinity. If the variation in limit endowment, $b - a$, is small then \hat{p}^* is nearly constant.

Next, we define two, slightly distinct, notions of extreme prices. First, consider a sequence of states i^n such that the endowment in i^n converges to the lower bound a as $n \rightarrow \infty$. We say that the BCE price has *extreme highs* if the price in state i^n (normalized by the probability of state i^n) converges to infinity. Similarly, consider a sequence of states i^n such that the endowment in i^n converges to the upper bound b . The BCE price has *extreme lows* if the (normalized) price in state i^n converges to zero. The definition below uses the limit price P to capture this notion of extreme prices.

A second, and stronger, notion of extreme prices focuses on the limit value of tail events. Specifically, consider an asset that pays off 1 unit of consumption in the ϵ -fraction of states

with the lowest endowment. This asset has the limit price $P(\epsilon)$. If $P(0) > 0$ then the limit price of this asset remains positive even when expected return is zero. We refer to this phenomenon as *heavy high tails*. Next, consider an asset that pays off 1 unit of consumption in the x -fraction of states with the highest endowment. This asset has an expected return x . The limit price of this asset is $1 - P(1 - x)$. If $1 - P(1 - x) = 0$ then the limit price of this asset is zero even though its expected return is positive. We refer to this as *heavy low tails*. This motivates the following definitions:

Definition 7. Let $P = P(0) + \int_0^x \hat{p}(r) dr$ be a limit BCE price of an almost continuous sequence.

- (i) P has *heavy high tails* if $P(0) > 0$ and *heavy low tails* if $P(x) = 1$ for some $x < 1$.
- (ii) P has *extreme highs* if $\lim_{x \rightarrow 0} \hat{p}(x) = \infty$ and *extreme lows* if $\lim_{x \rightarrow 1} \hat{p}(x) = 0$.

The following theorem characterizes limit BCE prices of an almost continuous sequence.

Theorem 3. Let P be a limit price of an almost continuous sequence $E^n = (u, k, s^n, \pi^n)$.

- (1) If $\rho < 1$ then P has *heavy high tails* and *extreme lows*.
- (2) If $\rho = 1$ then P has *heavy high tails*, *extreme lows* and *extreme highs*.
- (3) If $\rho > 1$ then P has *heavy high tails*, *heavy low tails*, *extreme highs* and *extreme lows*.

Theorem 3 shows that the limit price in all cases has heavy high tails, i.e., $P(0) > 0$. The proof of theorem reveals that the quantity $P(0)$ can be interpreted as the shadow price of the crudeness constraint in the following sense. Let p^n be the equilibrium price of the almost continuous sequence $E^n = (u, k, \pi, s)$. Consider a household with wealth $w(p^n) + P(0)$ and assume that this household must choose a consumption plan in $B_{k-1}(p^n)$. Thus, this household has greater wealth than the representative household in E^n but has a tighter crudeness constraint. The maximal utility of this household converges to the same value as the equilibrium utility of the household in E^n . Thus, increasing the wealth of the household by $P(0)$ exactly compensates for tightening the crudeness constraint from k to $k - 1$.

This observation can be used to establish that $P(0)$ converges to zero as $k \rightarrow \infty$. Thus, when k is large the heavy high tails are small. To see this, note that the value of an additional partition element must converge to zero in k . This can easily be shown using the equivalence between equilibrium allocations and solutions to the planner’s problem established in Theorem 1. Since $P(0)$ is the shadow price of additional partition elements this in turn implies that $P(0)$ converges to zero as k goes to infinity.

In the following subsection, we provide an example that illustrates Theorem 3 and, in the context of this example, provide some intuition for the theorem.

3.1 An Example

Let $E^n = (u, k, \pi^n, s^n)$ be an almost continuous sequence of economies with the following properties: $k = 2$, $u(x) = \ln x$ and the limit endowment is uniformly distributed on the interval $[1, 2]$.

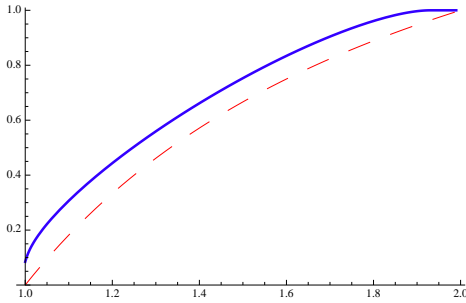


Figure 1: P and P^*

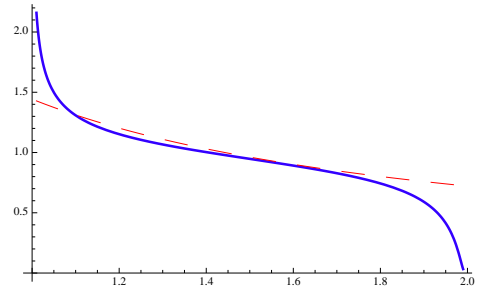


Figure 2: \hat{p} and \hat{p}^*

Figure 1 depicts the BCE limit price and the standard limit price as a function of the realized endowment. The dashed line shows the standard limit price P^* whereas the solid line shows the BCE limit price P . The intercept of P at zero illustrates the heavy high tails for this example. Figure 2 shows the derivatives of the limit BCE price (\hat{p} – solid line) and the standard economy (\hat{p}^* – dashed line). As Figure 2 illustrates, \hat{p} converges to infinity as the endowment converges to 1 and to 0 as the endowment converges to 2.

Next, we describe the key steps in the proof in the context of this example. Let (p^n, μ^n) be a BCE of E^n . First note that the price p^n must be strictly greater than zero in every state

since the utility function is unbounded. This, in turn, implies that the feasibility constraint holds with equality, i.e., $\sigma_i(\mu^n) = s_i$ in every state.

By Theorem 1, every consumption plans in the support of μ^n is monotone. In the $k = 2$ case this implies that there is a cutoff state j such that the consumer chooses low consumption in states $i \leq j$ and high consumption in states $i > j$. Next, we claim that for every $i \in N$ there is a consumption plan in the support of $K(\mu^n)$ with cutoff i . To see this, note that otherwise aggregate consumption would be the same in two consecutive states. Since aggregate endowment is strictly increasing, feasibility would not be satisfied with equality. We conclude that there is a consumption plan in $K(\mu^n)$ that singles out the highest endowment state as the only state with high consumption and chooses the same low consumption in all other states. Similarly, there is a consumption plan in $K(\mu^n)$ that singles out the lowest endowment state as the only state with low consumption and chooses the same (high) consumption in all other states.

Next, consider the utility of a household who chooses the same consumption in all states. This plan is crude and therefore feasible but worse than any of the plans chosen in equilibrium. In fact, the utility of this plan must stay uniformly bounded away from the equilibrium utility along the sequence (p^n, μ^n) . The reason is straightforward: if, in the limit, the constant plan were optimal then prices would have to be constant. But then, by concavity, *every* optimal plan would be constant and feasibility would not hold with equality in every state.

Consider a household who chooses a cutoff at the lowest state. Along the sequence of economies the probability of the lowest state converges to zero. Nonetheless, this household must get a utility that is bounded away from the utility of a constant consumption plan. The only way this can happen is if prices in the lowest state converge to infinity and do so fast enough that the value of one unit of consumption in the lowest state stays bounded away from zero. Thus, the heavy high tail $P(0)$ is needed to compensate this household for singling out the lowest endowment state. When relative risk aversion is greater than 1, i.e., when the utility function is log or more risk averse, then utility is unbounded below. In Lemma 23 we show that extreme highs are a consequence of this feature.

Next, consider a household who chooses a cutoff at the second highest state. The probability

of the highest state converges to zero and, therefore, choosing this cutoff can only be optimal for this household if consumption in the highest state is extraordinarily cheap. In fact, it must be so cheap that the utility gain from consumption in the lowest state stays bounded away from zero along the sequence. Hence, the limit price must exhibit extreme lows. When relative risk aversion is strictly above 1 then utility is bounded above. In that case, the household cannot be compensated for singling out the highest endowment state at any price. As a result, there is $\epsilon > 0$ such that the limit price is zero whenever the endowment is within ϵ of the upper bound. Hence, we must have heavy low tails when risk aversion is strictly above 1.

4 BCE in a Dynamic Economy

In this section, we extend our analysis to a dynamic (Lucas-tree) economy. We show that there is a one-to-one correspondence between the stationary BCE of our dynamic economy and the BCE of a corresponding static economy. This correspondence enables us to relate the extreme consumption prices analyzed in Theorem 3 to extreme asset prices.

As in the static economy, $N = \{1, \dots, n\}$ is a finite set of states and each state $i \in N$ implies a dividend (endowment) realization $s_i \in [a, b]$. A t -period history h is an vector $(i_1, \dots, i_t) \in N^t$; we write hi for the history (h, i) . We call H^t the set of all t -period histories and $H = \bigcup_{t \geq 1} N^t$ the set of (all) histories. Given any t -period history $h = (i_1, \dots, i_t)$, we let $\iota(h) = i_t$ be the state in period t and let $H_i^t = \{h \in N^t : \iota(h) = i\}$ be the set of all t -period histories that end in i .

A matrix of transition probabilities, ϕ , describes the evolution of the state; ϕ_{ij} is the probability that the state at date $t + 1$ is j given that it is i on date t . We assume that ϕ has a stationary distribution π ; that is,

$$\pi = \pi \cdot \Phi \tag{3}$$

The initial state (the period 1 history) is drawn from the stationary distribution π . Therefore, the probability of history $h = (i_1, \dots, i_t) \in H^t$ is

$$\lambda_h = \pi_{i_1} \cdot \phi_{i_1 i_2} \cdots \phi_{i_{t-1} i_t} \tag{4}$$

Households choose a consumption plan prior to the realization of the initial state. In this context, the assumption that the initial state is chosen according to the invariant distribution means that we do not have to consider transitory effects of the initial condition. As we show below, the economy has a stationary equilibrium allocation and stationary equilibrium prices. Moreover, we can map the dynamic economy to a two-period economy as analyzed in the previous section. We show that the equilibrium allocation and prices of the two-period economy together with the transition matrix ϕ can be used to describe the equilibria of the dynamic economy.

A function $d \in \mathbb{R}_+^H$ is a (dynamic) consumption plan and \mathcal{D} is the set of all consumption plans. The definition of crude consumption plans mirrors the corresponding definition for the static economy:

Definition 8. The consumption plan $d \in \mathcal{D}$ is *crude* if $|\{d_h \mid h \in H\}| \leq k$.

Let \mathcal{D}_k be the set of all *crude* consumption plans. As in the static economy, we assume households choose a crude consumption plan *ex ante*, that is, prior to the realization of the initial state. This implies that, prior to the realization of the initial state, the household partitions the set of histories into k categories and chooses the same consumption for each partition element. This partition stays fixed and cannot be revised over time.⁴

The household's utility from the consumption plan d is

$$V(d) = (1 - \beta) \sum_{t \geq 1} \sum_{h \in N^t} u(d_h) \beta^{t-1} \lambda_h \quad (5)$$

where $\beta \in (0, 1)$ is the discount factor. The sextuple $E^* = (u, \beta, k, \pi, s, \phi)$ is a dynamic economy.

An *allocation* is a probability on \mathcal{D} . It is *crude* if its support is contained in the set of crude consumption plans. Thus, the set of dynamic allocations is $\Delta(\mathcal{D})$ and the allocation $\nu \in \Delta(\mathcal{D})$

⁴If the Markov process is not iid then households would benefit from changing the partition as they learn the realized endowment in a given period. In effect, this would relax the crudeness constraint. An alternative would be to assume that crude consumption plans for period t are chosen in period $t - 1$. Such a sequential model would be more complicated to analyze but we conjecture that extreme asset prices would emerge also in this sequential version of the model.

is crude if $K(\nu) \subset \mathcal{D}_k$. The allocation ν is *feasible* in E^* if

$$\sigma_h^*(\nu) := \sum_{d \in K(\nu)} d_h \cdot \nu(d) \leq s_{i(h)}$$

for all $h \in H$. For any $\mathcal{D}' \subset \mathcal{D}$, let $M^*(\mathcal{D}')$ be the set of all feasible allocations such that $\mu(\mathcal{D}') = 1$. Hence, $M^*(\mathcal{D}_k)$ is the set of feasible allocations for E^* . Henceforth, ν is feasible means $\nu \in M^*(\mathcal{D}_k)$.

A function $q \in \mathbb{R}_+^H$ is a (dynamic) *consumption price* if $\sum_H q_h = 1$. The household's budget is

$$B_k^*(q) = \left\{ d \in \mathcal{D}_k : \sum_{h \in H} q_h [d_h - s_{i(h)}] \leq 0 \right\} \quad (6)$$

The crude consumption plan $d \in B_k^*(p)$ is *optimal at prices* q if $V(d) \geq V(d')$ for all $d' \in B_k^*(q)$. As we noted above, the household makes decisions prior to the realization of the initial state i_1 .

Definition 9. The price-allocation pair (q, ν) is a *BCE* of E^* if ν is feasible for E^* and if $\nu(d) > 0$, then d is optimal at prices q .

Fix a dynamic economy $E^* = (u, \beta, k, \pi, s, \phi)$ and consider the static economy $E = (u, \beta, k, \pi, s)$. The two economies share the same utility function and crudeness constraint. In both economies, the initial endowment is chosen according to the distribution π . In the dynamic economy, the endowment evolves according to a Markov process while in the static economy the endowment stays fixed. Since π is the stationary distribution of the Markov process with transition matrix ϕ it follows that

$$\sum_{h \in H_i^t} \lambda_h = \pi_i \quad (7)$$

for all $t \geq 1$. Hence, from an ex ante view, state i is realized with probability π_i in every period of the dynamic economy E^* . Moreover, we have $\sum_t \sum_{h \in H_i^t} \beta^t \lambda_h = \pi_i$, so the dynamic economy can be thought of as a version of the static economy where each state i is split into many identical states corresponding to the branches of the event tree that end with i . We refer to $E = (u, \beta, k, \pi, s)$ as the *static economy for* $E^* = (u, \beta, k, \pi, s, \phi)$. Part (ii) of the definition

of monotonicity of consumption plans in E will manifest itself as stationarity of consumption plans in E^* .

A consumption plan is stationary if consumption depends only on the current state. We can associate stationary consumption plans of the dynamic economy E^* with consumption plans of the corresponding static economy E . The plan d is a *stationary* consumption plan for the dynamic economy $E^* = (u, \beta, k, \pi, s, \phi)$ if there exists a consumption plan, c , for the static economy $E = (u, \beta, k, \pi, s)$ such that $d_h = c_{\iota(h)}$ for all $h \in H$. Let $\bar{\mathcal{D}}$ denote the set of stationary plans and let $T_1 : \mathcal{C} \rightarrow \bar{\mathcal{D}}$ be the one-to-one mapping between stationary dynamic plans and static consumption plans defined above. Thus, $d = T_1(c)$ is the dynamic consumption plan in which a household consumes c_i in every period in which state i occurs. The set of stationary allocations is $\Delta(\bar{\mathcal{D}})$. Let $T_3 : \Delta(\mathcal{C}) \rightarrow \Delta(\bar{\mathcal{D}})$ be the one-to-one mapping between stationary allocations in the dynamic economy and allocations in the static economy defined by $\nu(d) = \mu(T_1^{-1}(d))$ for $d \in \bar{\mathcal{D}}$ and $\nu(d) = 0$ for $d \notin \bar{\mathcal{D}}$.

A consumption price is stationary if the price after history h depends only on the current state $\iota(h)$ and on the discounted probability of history h appropriately normalized. More precisely, a dynamic consumption price, q , is *stationary* if there is a static consumption price p such that for all $t \geq 1$ and all $h \in N^t$

$$q_h = (1 - \beta)\beta^{t-1}\lambda_h \frac{p_{\iota(h)}}{\pi_{\iota(h)}} \quad (8)$$

Equation (7) implies that when q satisfies (8) then

$$\sum_{t=1}^{\infty} \sum_{h \in H_t^t} q_h = p_i$$

and hence $\sum_{h \in H} q_h = \sum_{i=1}^n p_i = 1$.

Let $\bar{Q} \subset \Delta(H)$ be the set of stationary consumption prices. Let $T_2 : \Delta(N) \rightarrow \bar{Q}$ be the one-to-one mapping between stationary prices in the dynamic economy and prices in the static

economy defined above. To summarize:

$$\begin{aligned} T_1 &: \mathcal{C} \xrightarrow{1-1} \bar{\mathcal{D}} \\ T_2 &: \Delta(N) \xrightarrow{1-1} \bar{Q} \\ T_3 &: \Delta(\mathcal{C}) \xrightarrow{1-1} \Delta(\bar{\mathcal{D}}) \end{aligned}$$

The following lemma describes the relation between the static and the dynamic economy.

Lemma 2. *Let $E^* = (u, \beta, k, \pi, s, \phi)$ and let $E = (u, k, \pi, s)$ be the static economy for E^* . Then,*

- (i) $V(T_1(c)) = U(c)$ for all c .
- (ii) $c \in B_k(p)$ if and only if $T_1(c) \in B_k^*(T_2(p)) \cap \bar{\mathcal{D}}$.
- (iii) The allocation $\mu \in \Delta(\mathcal{C})$ is feasible in E if and only if the stationary allocation $T_3(\mu) \in \Delta(\bar{\mathcal{D}})$ is feasible in E^* .

The following theorem shows that we can use the static equilibria to find the BCE allocations and prices of the dynamic economy E^* .

Theorem 4. *Let $E = (u, k, \pi, s)$ and let $E^* = (u, \beta, k, \pi, s, \phi)$.*

- (i) *If (p, μ) is a BCE for E then $(T_2(p), T_3(\mu))$ is a BCE for E^* .*
- (ii) *If ν is a BCE allocation for E^* then $\nu \in \Delta(\bar{\mathcal{D}})$ and $T_3(\nu)$ is a BCE allocation for E .*

Theorem 4 connects BCEs of the dynamic economy and BCEs of the corresponding static economy. To find the equilibrium allocation in the dynamic economy, we must find the equilibrium allocation in the corresponding static economy with the state drawn according to the stationary distribution π . Moreover, there is a simple mapping that associates an equilibrium price of the static economy with an equilibrium price for the dynamic economy. Theorem 4 leaves open the possibility of a non-stationary price BCE price (supporting a stationary BCE

allocation). The possibility of non-stationary prices in the dynamic economy is related to the non-uniqueness of equilibrium prices in the static economy discussed after Theorem 2 above.

Theorem 4 uses the assumption that the initial state is chosen according to the stationary distribution π and that consumers choose their consumption plans prior to the realization of the initial state. Without this assumption, there might still be an analogue of Theorem 4 but the mapping between the dynamic and the static economy would be more complicated and depend on the initial state.

5 Asset Prices

Let $N = \{1, \dots, n\}$ be such that $s_1 < s_2 < \dots < s_n$ and let (p, μ) be a BCE of the static economy $E = (u, k, \pi, s)$. In this section, we use the equilibrium price p of the static economy to analyze asset prices in the dynamic economy $E^* = (u, \beta, k, \pi, s, \phi)$ with equilibrium consumption prices $q = T_2(p)$.

Consider an asset in zero net supply that delivers z_i units of the consumption good in state i in period $t + 1$ and let $z = (z_1, \dots, z_n)$. Recall that $\iota(h) \in N$ is the state after history h and assume $p_{\iota(h)} > 0$. Let $r_h(z)$ be the BCE price of this asset in terms of consumption in the t -period history h . By a standard no-arbitrage argument, it follows that

$$r_h(z) = \frac{\sum_{i \in N} q_{hi} z_i}{q_h} \quad (9)$$

where the numerator is the value of the return z after history h and the denominator is the price of consumption after history h . Using (8) we substitute for q_h and get the following expression for r_h :

$$r_h(z) = \frac{\pi_{\iota(h)}}{p_{\iota(h)}} \beta \sum_{i \in N} \phi_{\iota(h)i} \frac{p_i}{\pi_i} z_i$$

for h such that $p_{\iota(h)} > 0$. The price of the asset is a function of the state $\iota(h)$ only and therefore $r_h(z) = r_{\iota(h)}(z)$. In the following when we refer to an “asset z ” it is understood that z is the return vector of the asset and trade occurs in the period prior to the realization of the return.

Below, we apply Theorems 3 and 4 to show that asset prices in the dynamic economy are

extreme. As in Theorem 3 we consider a sequence of economies that converges to a limit economy with a continuous distribution of endowments. We will show that limit equilibrium asset prices are extremely high if the endowment (dividend) is near its upper bound and extremely low if the endowment is near its lower bound. To simplify the exposition, we restrict to iid transitions. However, this restriction is for expositional ease only. The observations below can be extended to arbitrary Markov transitions if we add the assumption that the ratio of all transition probabilities stays bounded.

We assume that $\phi_{ij} = \pi_j$ for all i, j and refer to a dynamic economy with constant transition probabilities as an *iid-economy*. For an iid economy, the asset price formula (9) for states j such that $p_j > 0$ can be simplified as follows:

$$r_j(z) = \frac{\pi_j}{p_j} \left(\beta \sum_{i \in N} p_i z_i \right) \quad (10)$$

Since p_j may be zero, it is more convenient to analyze the inverse of the asset price $(r_j(z))^{-1}$. The inverse asset price is the amount of asset z that can be purchased with one unit of consumption in state j . Notice that this quantity is always well defined. For $j \in N$

$$\sum_{i=1}^j \frac{1}{r_i(z)} \frac{\pi_i}{F_N(j)}$$

is the expected amount of asset z that the household receives in return for one unit of consumption in states $1, \dots, j$. If we substitute for $r_i(z)$ and define P^n as in section 4, we obtain

$$\sum_{i=1}^j \frac{1}{r_i(z)} \frac{\pi_i}{F_N(j)} = \frac{P^n(F_N(j))}{F_N(j)} \left(\beta \sum_{i \in N} p_i z_i \right)^{-1}$$

This motivates the following definition. For any $z \in \mathbb{R}^n$ and $x \in (0, 1)$ let

$$\bar{Q}^n(z, x) := \frac{P^n(x)}{x} \left(\beta \sum_{i \in N} p_i z_i \right)^{-1} \quad (11)$$

Thus, $\bar{Q}^n(z, x)$ is the expected amount of asset z the household receives in return for one unit

of consumption in the x -fraction of states with the highest consumption prices. Of course, this interpretation only holds if $x = F_N(i)$ for some $i \in N$. The function \bar{Q}^n is an extension to all of $[0, 1]$ using the interpolation used to define P^n in section 4. Conversely, let

$$\underline{Q}^n(z, x) := \frac{1 - P^n(1 - x)}{x} \left(\beta \sum_{i \in N} p_i z_i \right)^{-1} \quad (12)$$

The function $\underline{Q}^n(z, x)$ is the expected number of shares the household receives in return for one unit of consumption in x -fraction of states with the lowest consumption prices. We use the functions \bar{Q}^n and \underline{Q}^n to show that asset prices are extreme.

Definition 10. A sequence of iid-economies $E^{*n} = (u, \beta, k, \pi^n, s^n)$ is *almost continuous* if the corresponding sequence of static economies $E^n = (u, k, \pi^n, s^n)$ is almost continuous.

Corollary 1. *Let E^n be an almost continuous sequence of iid economies and P^n be the corresponding sequence of cumulative BCE prices. Consider a sequence of assets z^n with payoffs in a compact interval of \mathbb{R}_+ bounded away from zero. Let \bar{Q}^n and \underline{Q}^n be the corresponding inverse asset price averages as defined in (11) and (12). Then*

$$\lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} \bar{Q}^n(x, z^n) = \infty$$

and

$$\lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} \underline{Q}^n(x, z^n) = 0.$$

The Corollary implies that when the endowment is near its lower bound then (for large n) asset prices are extremely low in the sense that the expected amount of asset z that the household gets in return for 1 unit of consumption in the lowest x -fraction of states can be made arbitrarily large if x is chosen sufficiently small. Similarly, when the endowment is near its upper bound then (for large n) assets prices are *extremely high* in the sense that the expected amount of asset z that the household gets in return for 1 unit of consumption in the lowest x -fraction of states can be made arbitrarily small if x is chosen sufficiently small. Note that for $\rho > 1$ we have $\lim_{n \rightarrow \infty} \underline{Q}^n(x, z) = 0$ for x small. In that case, giving up one unit of consumption

in the states with the lowest consumption prices buys zero units of the asset. This does not constitute a violation of no-arbitrage because consumption in states close to the upper bound of the endowment distribution has no value when $\rho > 1$.

Next, we compare the price of a (risk-free) bond and a similar asset that is nearly risk-free and pays off one unit of consumption in all but a small set of states. As a corollary of our extreme price results, we demonstrate that the price of the risk-free bond stays bounded away from the price of the risky bond even as the two assets differ on a negligible set of states.

A risk-free bond corresponds to the return vector $e = (1, \dots, 1)$. Assume $p_i > 0$ and substitute e for z in the pricing formula (10) to obtain:

$$r_i(e) = \frac{\pi_i}{p_i} \left(\beta \sum_{i' \in N} p_{i'} \right) = \beta \frac{\pi_i}{p_i}$$

Next, consider the return vector $z = (0, \dots, 0, 1, \dots, 1)$ with $z_{i'} = 0$ for all $i' < j$. We have

$$r_i(z) = \frac{\pi_i}{p_i} \left(\beta \sum_{i'=j}^n p_{i'} \right) = \beta \frac{\pi_i}{p_i} (1 - P^n(F_N(j)))$$

Hence, the price ratio of those two assets is

$$\frac{r_i(z)}{r_i(e)} = 1 - P^n(F_N(j))$$

Consider an almost continuous sequence E^n ; let p^n be an equilibrium price for E^n and let P^n be the cumulative price. The limit price of the sequence is P . Let z^n be a sequence of assets such that $z_i^n = 1$ for all i such that $F_N(i) \geq 1 - \epsilon$ and $z_i^n = 0$ for all i such that $F_N(i) < \epsilon$. Consider a sequence of states i^n such that $p_{i^n}^n > 0$ and hence $r_{i^n}^n(e)$ and $r_{i^n}^n(z^n)$ are well defined. Then,

$$\lim_n \frac{r_{i^n}^n(z^n)}{r_{i^n}^n(e)} = 1 - P(\epsilon)$$

As we let ϵ converge to zero the asset z^n is *almost risk free* in the sense that it differs from asset e on a set of negligible probability. Nevertheless, the price ratio stays bounded away from 1 since $\lim_{\epsilon \rightarrow 0} P(\epsilon) = P(0) > 0$. We can interpret asset e as a *safe-haven asset*. Our extreme

price result implies that the safe haven trades at a premium over assets that are nearly identical and differ only in the lowest endowment states.

A Appendix: Proofs

If $K(\mu) \subset \{c^1, \dots, c^m\}$, we write $\mu = (\mathbf{a}, \mathbf{c})$ where $\mathbf{a} = (\alpha^1, \dots, \alpha^m)$, $\mathbf{c} = (c^1, \dots, c^m)$ and $\mu(c^l) = \alpha^l$ for all l . It will be understood that $\mathbf{a} = (\alpha^1, \dots, \alpha^m)$, $\hat{\mathbf{a}} = (\hat{\alpha}^1, \dots, \hat{\alpha}^m)$, and so forth. We follow the same convention with $\mathbf{c}, \hat{\mathbf{c}}$ etc. If $\{c^1, \dots, c^m\}$ contains exactly one representative from each equivalence class of \sim , we say that $\mu = (\mathbf{a}, \mathbf{c})$ is in simple form. Thus, μ can be expressed in simple form if and only if it is simple.

A.1 Proof of Lemma 1

Lemma 3. *If μ is feasible and not simple, then there is a simple and feasible μ' such that $W(\mu') > W(\mu)$.*

Proof. Let $\mu = (\mathbf{a}, \mathbf{c})$. If μ is not simple, there is $c, c' \in K(\mu)$ such that $c \sim c'$. Let $c^* = \gamma \cdot c + (1 - \gamma)c'$ where $\gamma = \frac{\mu(c)}{\mu(c) + \mu(c')}$ and let μ^* be the allocation derived from μ by replacing c, c' with $(\mu(c) + \mu(c')$ probability of) c^* . Since, c, c' are crude, so is c^* and μ^* . Since u is strictly convex, $W(\mu^*) > W(\mu)$. Note that $|K(\mu^*)| < |K(\mu)|$. Hence, we can repeat this above construction until we get a simple μ^* . \square

Lemma 4. *If μ is feasible but not fair, then there is a feasible and fair μ' such that $W(\mu') > W(\mu)$ and $|K(\mu')| \leq |K(\mu)|$.*

Proof. Let $\mu = (\mathbf{a}, \mathbf{c})$, let x^l be the certainty equivalent of c^l and \bar{x}^l be the corresponding constant consumption plan; that is, $u(x^l) = U(c^l)$ and $\bar{x}_i^l = x^l$ for all i, l . Also, let $x = \sum_{l=1}^m \alpha^l x^l$ and let \bar{x} be the corresponding constant consumption plan. Let $\hat{\mu} = (\hat{\mathbf{a}}, \hat{\mathbf{c}})$ such that $\hat{\alpha}^l = \frac{\alpha^l x^l}{x}$ and $\hat{c}^l = \frac{x c^l}{x^l}$ for all l . Finally, let $\bar{\mu} = (\bar{\mathbf{a}}, \bar{\mathbf{c}})$ such that $\bar{\alpha}^l = \hat{\alpha}^l$ and $\bar{c}^l = \bar{x}^l$ for all l . Since u is strictly concave and μ is not fair, $W(\delta_{\bar{x}}) > W(\bar{\mu})$. Since u is CRRA,

$$u^{-1}(U(\hat{c}^l)) = \frac{x}{x^l} u^{-1}(U(c^l)) = \frac{x}{x^l} x^l = x;$$

hence, $W(\hat{\mu}) = W(\delta_{\bar{x}})$. By definition, $W(\bar{\mu}) = W(\mu)$. Hence, $W(\hat{\mu}) > W(\mu)$. By construction $\hat{\mu}$ is fair. It is easy to verify that $\sum_l \hat{c}_i^l \hat{\alpha}^l = \sum_l c_i^l \alpha^l$ for all $i \in N$ and hence $\hat{\mu}$ is feasible. Clearly, $|\hat{\mu}| \leq |\mu|$. \square

Lemma 5. *A solution to the planner's problem exists and every solution to the planner's problem is simple and fair.*

Proof. As explained in Section 2.1, $W(\mu) < W(\delta_s)$ for every feasible $\mu \in M(\mathcal{C}_k)$. Hence,

$$W_k = \sup_{\mu \in M(\mathcal{C}_k)} W(\mu)$$

is well-defined. By Lemmas 3 and 4, there exists a sequence of feasible, simple, and fair allocations $\mu^t = (\mathbf{a}^t, \mathbf{c}^t)$ such that $W(\mu^t) \geq W_k - 1/t$ and $a^t \in \mathbb{R}_+^m$ for all t , where m is the cardinality of the set of all partitions of N with k or fewer elements.

By passing to a subsequence, $a^t = (\alpha^{1t}, \dots, \alpha^{mt})$ converges to some $a \in \Delta(m)$. If c^{lt} is unbounded for some l , we must have $\alpha^l = 0$. Let $A \subset N$ be the set of l such that $\alpha^l \neq 0$. Then, $A \neq \emptyset$ and c^{lt} is bounded for all $l \in A$. Hence, there exists a subsequence of μ^t along which c^{lt} converges, to some $c^l \in \mathcal{C}_k$, for every $l \in A$.

Let $\mu = (\mathbf{a}, \mathbf{c})$ where $a = (\alpha^l)_{l \in A}$ and $\mathbf{c} = (c^l)_{l \in A}$. Since $\lim W(\mu^t) = W_k$ and each μ^t is fair, $U(c^{lt}) = W(\mu^t)$ so by continuity of u we have that $U(c^l) = W_k$ for all $l \in A$ and therefore $W(\mu) = W_k$. Finally, $\sum_{l \in A} \alpha^{lt} c_i^{lt} \leq \sum_{l \in A} \alpha^{lt} c_i^{lt} \leq s_i$ for all i, l, t and so $\sum_{l \in A} \alpha^l c_i^l \leq \sum_{l \in A} \alpha^l c_i^{lm} \leq s_i$ for all i, l , and hence μ is feasible and therefore μ solves the planner's problem. Then, Lemmas 3 and 4 imply that μ must be simple and fair. \square

Lemma 6. *Let $E = (u, s, \pi)$, $\hat{E} = (u, \hat{s}, \hat{\pi})$ be such that $F_s = F_{\hat{s}}$ and let W_k and \hat{W}_k be the maximal mean utility attainable in E and \hat{E} respectively. Then:*

- (i) *For any feasible allocation μ in E there exists a feasible allocation $\hat{\mu}$ in \hat{E} with $W(\hat{\mu}) = \hat{W}(\mu)$,*
- (ii) *For any feasible allocation $\hat{\mu}$ in \hat{E} there exists a feasible allocation μ in E with $W(\mu) = \hat{W}(\hat{\mu})$,*
- (iii) $W_k = \hat{W}_k$.

Proof. Without loss of generality, we assume that E is a pure endowment economy; that is, $N = \{1, \dots, n\}$, s is one-to-one, $\hat{N} = \{ij : i \in N, j \in N_i\}$ for some collection of N_i ' such that $N_i = \{1, \dots, n_i\}$ for all i , $s_i = s_{ij}$ for all i and $j \in N_i$ and $\sum_{j \in N_j} \hat{\pi}_{ij} = \pi_i$.

(i) For any feasible allocation $\mu = (\mathbf{a}, \mathbf{c})$ in E define the allocation $\hat{\mu} = (\hat{\mathbf{a}}, \mathbf{c})$ for the economy \hat{E} as follows: $\hat{c}_i^l = c_{ij}^l$ for all i and $j \in N_i$. Clearly, $\hat{W}(\hat{\mu}) = W(\mu)$ and $\hat{\mu}$ is feasible (for \hat{E}).

(ii) We will first show that given any consumption \hat{c} for \hat{E} , there is an allocation, $\mu = (\mathbf{a}, \mathbf{c})$, for E such that $W(\mu) = \hat{U}(\hat{c})$ and

$$\sigma_i(\mu) = \frac{1}{\pi_i} \sum_{j \in N_i} \pi_{ij} \hat{c}_{ij}$$

for all $i \in N$. Let $L = \{(j_1, \dots, j_n) : j_i \in N_i \forall i \in N\}$ and let $\gamma^{ij} = \frac{\hat{\pi}^{ij}}{\pi_i}$ for all $i \in N$ and $j \in N_i$. For each $l = (j_1, \dots, j_n) \in L$, let $\alpha^l = \gamma^{1j_1} \cdot \gamma^{2j_2} \dots \gamma^{nj_n}$ and $c_i^l = \hat{c}_{ij_i}$. Since \hat{c} is crude and $\{c_i^l : i \in N\} \subset \{c_{ij} : i \in N, j \in N_i\}$, each c^l is crude. Verifying that $W(\mu) = U(\hat{c})$ and that the equation displayed above holds for all $i \in N$ is straightforward.

To conclude, let $\hat{\mu} = (\hat{\mathbf{a}}, \hat{\mathbf{c}})$ be a feasible allocation in \hat{E} . For each $\hat{c}^l \in K(\hat{\mu})$, choose μ^l as described above. Let $\mu = \sum_l \hat{\alpha}^l \cdot \mu^l$. Then, μ is feasible since $\sigma_i(\mu) \leq \max_{j \in N_i} \sigma_{ij}(\hat{\mu}) = s_i$ for all $i \in N$. Since $W(\mu^l) = \hat{U}(\hat{c}^l)$ for all l , we have $W(\mu) = \hat{W}(\hat{\mu})$ as desired.

(iii) By Lemma 5, there exists a simple solution $\mu = (\mathbf{a}, \mathbf{c})$ to the planner's problem in the economy E . By part (i) there exists a feasible allocation $\hat{\mu}$ in \hat{E} with $W(\mu) = \hat{W}(\hat{\mu})$. Thus, $\hat{W}_k \geq W_k$. Conversely, by Lemma 5, there exists a simple solution $\hat{\mu} = (\hat{\mathbf{a}}, \hat{\mathbf{c}})$ to the planner's problem in the economy \hat{E} . By part (ii) there exists a feasible allocation μ in E with $W(\hat{\mu}) = \hat{W}(\mu)$. \square

To conclude the proof of Lemma 1, we will show that if $\mu = (\mathbf{a}, \mathbf{c})$ is a solution to the planner's problem, then it is also monotone. By Lemma 5 μ is simple and fair. Let E be the original economy and consider the pure endowment economy \bar{E} that has the same endowment distribution as E . Assume c^l fails condition (ii) of monotonicity for some l with $\alpha^l > 0$. Then, the construction in the proof of part (ii) of Lemma 6 (applied to economies \bar{E} and E) reveals that there exists a solution to the planner's problem in \bar{E} that is not fair. This contradicts Lemma 5.

To prove that μ satisfies condition (i) of monotonicity, we note that if $s_i \geq s_j$, then $\sigma_i(\mu) \geq \sigma_j(\mu)$. If not, $\sigma_i(\mu) < s_i$ and there must be some l such that $c_i^l < c_j^l$. Then, let $\hat{c}_t = c_t^l$ for $t \neq i$ and $\hat{c}_i = c_j^l$. Note that \hat{c} is crude and yields a strictly higher utility than c^l . Since

$\sigma_i(\mu) < s_i$, mean utility can be increased by replacing c^l with \hat{c} for a small fraction of households, contradiction the optimality of μ .

So, assume μ fails condition (i) of monotonicity: then, $\hat{c}_i^l > \hat{c}_j^l$ for some i, j such that $\hat{s}_i < \hat{s}_j$. Without loss of generality, we assume $i = 1, j = 2$ and $l = 1$. We showed in the previous paragraph that $\sigma_2(\mu) \geq \sigma_1(\mu)$, so we must have \hat{c}^o such that $\hat{c}_2^o > \hat{c}_1^o$; again without loss of generality, we assume $o = 2$ and that $\hat{\pi}_i \geq \hat{\pi}_j$ (an obvious adjustment is needed if this last inequality is reversed).

Construct a new state space $N = \hat{N} \cup \{n+1\}$ and $s \in [a, b]^{n+1}$ such that $s_t = \hat{s}_t$ for all $t \in N$ and $s_{n+1} = \hat{s}_1$. Also, set $\pi_t = \hat{\pi}_t$ for all $t \neq 1, n+1$, $\pi_1 = \hat{\pi}_2$ and $\pi_{n+1} = \hat{\pi}_1 - \hat{\pi}_2$. If $\pi_{n+1} = 0$, then ignore state $n+1$ in the argument below. Let γ solve $\gamma(c_1^1 - c_2^1) = (1 - \gamma)(c_2^2 - c_1^2)$ and choose $\epsilon > 0$ such that $\epsilon < \min\{\hat{\alpha}^1, \hat{\alpha}^2\}$. Next, we construct, \tilde{c}^{m+1} and \tilde{c}^{m+2} as follows: let $\tilde{c}_t^{m+1} = \hat{c}_t^1$ for all $t \neq 1, 2, n+1$, $\tilde{c}_1^{m+1} = \hat{c}_2$, $\tilde{c}_2^{m+1} = \hat{c}_1$ and $\tilde{c}_{n+1}^{m+1} = \hat{c}_1$; let $\tilde{c}_t^{m+2} = \hat{c}_t^2$ for all $t \neq 1, 2, n+1$, $\tilde{c}_1^{m+2} = \hat{c}_2^2$, $\tilde{c}_2^{m+2} = \hat{c}_1^2$ and $\tilde{c}_{n+1}^{m+2} = \hat{c}_1^2$.

Let $\tilde{\alpha}^r = \hat{\alpha}^r$ for all r such that $2 < r \leq m$, $\tilde{\alpha}^1 = \hat{\alpha}^1 - \epsilon\gamma$, $\tilde{\alpha}^2 = \hat{\alpha}^2 - \epsilon(1 - \gamma)$, $\tilde{\alpha}^{m+1} = \epsilon\gamma$, $\tilde{\alpha}^{m+2} = \epsilon(1 - \gamma)$ and $\tilde{\mu} = (\tilde{\mathbf{a}}, \tilde{\mathbf{c}})$. Our choice of γ ensures that $\tilde{\mu}$ is feasible for E and $W(\tilde{\mu}) = \hat{W}(\hat{\mu})$; therefore, by Lemma 6, $\tilde{\mu}$ solves the planner's problem for E . But $\tilde{\mu}$ is not simple which contradicts Lemma 5. \square

A.2 Proof of Theorem 1

Let $Z = \mathbb{R}_{++}^N$. Then, for all $z \in Z$, let $M^z(\mathcal{C}')$ be the set of all allocations with support contained in \mathcal{C}' that are feasible for the economy $E = (u, \pi, z)$. Let

$$W_k(z) = \max_{\mu \in M^z(\mathcal{C}_k)} W(\mu).$$

Hence, W_k is the planner's value as a function of the endowment. Let $Z^s = \{z \in Z : W_k(z) > W_k(s)\}$.

Clearly, $W_k(z) > W_k(y)$ whenever $z_i > y_i$ for all $i \in N$ since we can take the optimal allocation for y and increase every consumption in every state by a small constant amount. Hence, Z^s is nonempty. For $\epsilon > 0$, choose $\epsilon' > \frac{\epsilon}{a}$. If $|z_i - y_i| < \epsilon$ for all i and if $\mu = (\mathbf{a}, \mathbf{c})$ is

feasible for (u, π, y) , then $(\mathbf{a}, (1 - \epsilon')\mathbf{c})$ is feasible for (u, π, z) . Hence, by CRRA,

$$\frac{1}{1 - \frac{\epsilon}{a}} u^{-1}(W_k(y)) \geq u^{-1}(W_k(z)) \geq \left(1 - \frac{\epsilon}{a}\right) u^{-1}(W_k(y))$$

whenever $\sup_i |z_i - y_i| < \epsilon$, proving that W is continuous at y and hence Z^y is open.

We note that since W is a concave function of μ , W_k is a concave function of z and hence the set Z^s is convex. To see that, fix $z^1, z^2 \in Z^s$ and choose $\mu^i \in M^{z^i}(\mathcal{C}_k)$ such that $W(\mu^i) = W_k(z^i)$ for $i = 1, 2$. By Lemma 1, such μ^i exist. Clearly, $\gamma\mu^1 + (1 - \gamma)\mu^2 \in M^{\hat{z}}(\mathcal{C}_k)$ for $\hat{z} = \gamma z^1 + (1 - \gamma)z^2$ and hence $W_k(\gamma z^1 + (1 - \gamma)z^2) \geq W(\gamma\mu^1 + (1 - \gamma)\mu^2) = \gamma W(\mu^1) + (1 - \gamma)W(\mu^2) = \gamma W_k(z^1) + (1 - \gamma)W_k(z^2)$.

Since Z^s is nonempty, open, and convex, and $s \notin Z^s$, by the separating hyperplane theorem, there exists $p \in \mathbb{R}^n$ such that $p_i \neq 0$ for some i and $\sum_i p_i \cdot z_i > \sum p_i \cdot s_i$ for all $z \in Z^s$. Since there is free disposal, W_k is weakly increasing in each argument and therefore, we must have $p_i \geq 0$ for every $i \in N$ and hence we can normalize p to ensure that $p \in \Delta(N)$.

Let $\mu = (\mathbf{a}, \mathbf{c})$ be a solution to the planner's problem, where $\alpha^l > 0$ for all l . The argument establishing that each c^l must maximize U given budget $B(p)$ is standard and omitted, as is the proof of the following lemma:

Lemma 7. *If (p, μ) is a BCE, then μ is Pareto-efficient.*

Finally, to see that if (p, μ) is a BCE, then μ must be a solution to the planner's problem, note that since every household has the same endowment, μ must be fair. But then, if μ did not solve the planner's problem, the solution to the planner's problem would Pareto-dominate it, contradicting Lemma 7. \square

A.3 Proof of Theorem 2

Lemma 8. *If $(p, \mu), (p', \mu)$ are two BCE for the pure endowment economy E then $p = p'$.*

Proof. First, we show that for all c in the support of μ we have $c_i > 0$. Assume to the contrary that $c_i = 0$ for some i and let $A_i = \{j : c_j = 0\}$. Then, by household optimality it follows that $\sum_{A_i} p_j = 1$ and $\sum_{N \setminus A_i} p_j = 0$, otherwise the consumption can be raised by ϵ on the set A_i

and lowered by $\epsilon \frac{\sum_{A_i} p_j}{\sum_{N \setminus A_i} p_j}$ on the set $N \setminus A_i$ resulting in an overall increase of utility for small ϵ . Since $N \setminus A_i$ must be non-empty it follows that c cannot be an optimal plan. For c with $c_i \neq c_j$ for some i, j define

$$MRS_i = \frac{u'(c_i) \sum_{\{j|c_j=c_i\}} \pi_j}{\sum_{\{j|c_j \neq c_i\}} \pi_j u'(c_j)}$$

For c such that $c_i = c_j$ for all i, j define $MRS_i = 1$ for all i . Household optimality implies that

$$MRS_i = \frac{\sum_{\{j|c_j=c_i\}} p_j}{\sum_{\{j|c_j \neq c_i\}} p_j} \quad (*)$$

First consider states such that $\sigma_i(\mu) < s_i$. Since utility is strictly increasing, it follows that an optimal consumption plan must satisfy the budget constraint with equality. This in turn implies that $p_i = 0$ for all states such that $\sigma_i(\mu) < s_i$.

Let $N^* = \{i^* : \sigma_{i^*}(\mu) = s_{i^*}\}$ and note $p_i = 0$ for $i \notin N^*$. Without loss of generality let $N^* = \{i_1^*, \dots, i_m^*\}$ and assume that $s_{i_1^*} < s_{i_2^*} < \dots < s_{i_m^*}$. There must exist c^1 in the support of μ such that $c_{i_1^*}^1 < c_{i_j^*}^1$ for $j > 1$ and therefore equation (*) implies a unique equilibrium price for state i_1^* (the uniqueness of the price follows from the price normalization). Next, there must exist c^2 in the support of μ such that $c_{i_2^*}^2 < c_{i_j^*}^2$ for $j > 2$. Since $p_{i_1^*}$ is uniquely determined, equation (*) implies a unique price of state i_2^* . We can repeat this argument for i_3^*, \dots, i_{m-1}^* . \square

Lemma 9. *If (p, μ) is a BCE for the pure endowment economy E then p is monotone.*

Proof. Let $E = (u, k, \pi, s)$ be a pure endowment economy with $N = \{1, \dots, n\}$ and $s_i < s_j$ and let μ be a solution to the planner's problem. We will show that $\frac{p_i}{\pi_i} \geq \frac{p_j}{\pi_j}$. Assume that $\pi_i \geq \pi_j$. The proof of for the reverse case is analogous and therefore omitted. Fix $d \in \mathbb{N}$ and choose $m \in \mathbb{N}$ and $\delta \in \mathbb{R}_+$ so that $\pi_i/d = m \cdot \pi_j/d + \delta$. Note that $m \geq 1$ and $\delta \in [0, \pi_j/d]$.

We define the economy $\hat{E} = (u, k, \hat{\pi}, \hat{s})$ as follows: the set of states is $\hat{N} = \hat{N}_i \cup \hat{N}_j \cup (N \setminus \{i, j\})$ where $\hat{N}_i = \{s_{i,1}, \dots, s_{i,dm+1}\}$ and $\hat{N}_j = \{s_{j,1}, \dots, s_{j,d}\}$. The probability $\hat{\pi}$ satisfies $\hat{\pi}_l = \pi_l$ for

$l \neq i, j$ and

$$\begin{aligned}\hat{\pi}_{i,l} &= \pi_j/d \text{ for } l = 1, \dots, dm \\ \hat{\pi}_{i,dm+1} &= \delta d \\ \hat{\pi}_{j,l} &= \pi_j/d \text{ for } l = 1, \dots, d\end{aligned}$$

The endowment satisfies $\hat{s}_l = s_l$ for $l \neq i, j$ and

$$\begin{aligned}\hat{s}_{i,l} &= s_i \text{ for } l = 1, \dots, dm + 1 \\ \hat{s}_{j,l} &= s_j \text{ for } l = 1, \dots, d\end{aligned}$$

Fix an allocation μ in E that solves the planner's problem and let $\hat{\mu}$ be the corresponding allocation in \hat{E} (obtained by Lemma 6). Let $(\hat{\mu}, \hat{p})$ be a BCE of \hat{E} . We now show that $\hat{p}_{i,l} \geq \hat{p}_{j,t}$ for all $l = 1, \dots, md$ and $t = 1, \dots, d$. To prove this, suppose that $\hat{p}_{j,t} > \hat{p}_{i,l}$ for some l, t . It is straightforward to show that for any optimal consumption plan c at price \hat{p} it must be that $c_{i,l} \geq c_{j,t}$ and therefore $\sigma_{i,l}(\hat{\mu}) \geq \sigma_{j,t}(\hat{\mu})$. Since $s_{j,t} > s_{i,l}$ this implies that $\hat{p}_{j,t} = 0$ contradicting $\hat{p}_{j,t} > \hat{p}_{i,l}$.

Let $p_i = \sum_{N_i} \hat{p}_{i,l}$, $p_j = \sum_{N_j} \hat{p}_{j,l}$ and $p_l = \hat{p}_l$ for $l \neq i, j$. Note that the consumption plans in the support of μ satisfy the budget constraint. Each consumption plan in the support of μ must be optimal since the corresponding plan in the support of $\hat{\mu}$ is optimal. Hence, (μ, p) is a BCE of E .

It follows that

$$\frac{p_i}{\pi_i} = \frac{\sum_{l \in N_i} \hat{p}_{i,l}}{\sum_{l \in N_i} \hat{\pi}_{i,l}} \geq \frac{m}{m+1} \frac{\sum_{l \in N_j} \hat{p}_{j,l}}{\sum_{l \in N_j} \hat{\pi}_{j,l}} = \frac{m}{m+1} \frac{p_j}{\pi_j}$$

Since d was arbitrary and $\lim_{d \rightarrow \infty} \frac{m}{m+1} \rightarrow 1$, it follows that $\frac{p_i}{\pi_i} \geq \frac{p_j}{\pi_j}$. □

A.4 Proof of Theorem 3

Let E^n be an almost continuous sequence of economies. Let μ^n be an equilibrium allocation and let $K(\mu^n)$ be the set of consumption plans in its support and let $K_R(\mu^n) = \{c \in K(\mu^n) | c_n \leq R\}$ be the plans with consumption less than $R > 0$. Let p^n be the equilibrium price, let P^n be the price distribution. Let B_k^n be the budget constraint of a consumer at the equilibrium price p^n .

Lemma 10. $\lim_{R \rightarrow \infty} \inf_n \left(\sum_{K_R(\mu^n)} \mu^n(c) \right) = 1$.

Proof. Note that $c_n \geq a > 0$ for all $c \in K(\mu^n)$. Feasibility then implies that for all n $\sum_{K_R(\mu^n)} \mu^n(c) \geq \frac{R-b}{R-a}$. \square

Lemma 11. *There exists a subsequence of n and a continuous increasing function $P : [0, 1] \rightarrow [0, 1]$ such that $P^n(x) \rightarrow P(x)$ for all $x \in (0, 1]$. Moreover, if $P(0) = 0$, then also $P^n(0) \rightarrow P(0)$. Furthermore, $P(x) = P(0) + \int_0^x \hat{p}(y) dy$ where $\hat{p} : (0, 1] \rightarrow [0, \infty)$ is non-increasing, non-negative, and $P(1) = 1$.*

Proof. For every n define $\tilde{P}^n : \mathbb{R} \rightarrow [0, 1]$ as follows

$$\tilde{P}^n(x) := \begin{cases} 0 & \text{if } x < 0 \\ P^n(x) & \text{if } x \in [0, 1] \\ 1 & \text{if } x > 1 \end{cases} \quad (13)$$

and note that P^n is a cdf of some distribution on $[0, 1]$. By the Helly selection theorem there exists $\tilde{P} : \mathbb{R} \rightarrow [0, 1]$ which is a cumulative distribution function of a measure on $[0, 1]$ and, therefore, non-decreasing and right-continuous having left limits. Moreover, $\tilde{P}^n(x) \rightarrow \tilde{P}(x)$ at all continuity points of \tilde{P} .

Define P to be the restriction of \tilde{P} to the interval $[0, 1]$. To show concavity of P let $x, x' \in [0, 1]$, $\lambda \in (0, 1)$ so that $x, y, \lambda x + (1 - \lambda)y$ are continuity points of P . Then, since $P^n(z) \rightarrow P(z)$ for $z \in \{x, y, \lambda x + (1 - \lambda)y\}$, concavity of P^n implies $P(\lambda x + (1 - \lambda)y) \geq \lambda P(x) + (1 - \lambda)P(y)$.

We now show that P is continuous. First, since \tilde{P} is right-continuous, P is continuous at 0. To reach the conclusion, it remains to show that P is left-continuous in the interval $(0, 1]$. To see that let $z \in (0, 1]$. Since P is non-decreasing, it has at most countably many points of discontinuity. This means that there exists an increasing sequence $z_m < z$ such that $z_m \rightarrow z$ and P is continuous at z_m for all m . For any m define λ_m such that $z_m = \lambda_m z_1 + (1 - \lambda_m)z$ and note that $\lambda_m \rightarrow 0$. The preceding paragraph implies that $P(z_m) \geq \lambda_m P(z_1) + (1 - \lambda_m)P(z)$ for all m , which implies that $\lim_{m \rightarrow \infty} P(z_m) \geq P(z)$, which given that the left limits of P exist and P is non-decreasing implies that P is continuous.

Thus, $P^n(x) \rightarrow P(x)$ for all $x \in (0, 1]$. Note that if $P(0) = 0$, then \tilde{P} is continuous at 0 and the convergence obtains there as well; if $P(0) > 0$ the convergence at zero fails.

In light of the preceding paragraph, the continuity of P implies its concavity. Since P is concave and continuous on $[0, 1]$, it follows that $P(x) = P(0) + \int_0^x \hat{p}(r)dr$ for some non-increasing function \hat{p} , see Theorem 2.5 of [van Rooij, Rooij, and Schikhof \(1982\)](#). \square

Lemma 12. (1) $P(0) < 1$; (2) If $\rho \leq 1$ then $P(x) < 1$ for all $0 \leq x < 1$.

Proof. (1) Assume $P(0) = 1$. Then, there is a sequence i^n with $x^n := \sum_{j \leq i^n} \pi_j^n \rightarrow 0$ such that $P^n(x^n) \rightarrow 1$. Let c^n be the following consumption plan: $c^n(i) = a/3$ for all $i \leq i^n$ and

$$c^n(i) = \frac{a}{3(1 - P^n(x^n))}$$

for all $i > i^n$. This plan is crude and, for large n , in the budget set since $\sum_N c^n(i)p^n(i) \rightarrow 2a/3$. Since $(1 - P^n(x^n)) \rightarrow 0$ and $x^n \rightarrow 0$ the utility of this plan converges to $\lim_{x \rightarrow \infty} u(x) > u(b)$. Since equilibrium utility must be smaller than $u(b)$ this yields the desired contradiction.

(2) We show that if $P(x) = 1$ for some $x < 1$ then $\rho > 1$. Let c^n be the following consumption plan: $c^n(i) = a/3$ for all i such that $\sum_{j=1}^i \pi_j^n \leq x$ and

$$c^n(i) = \frac{a}{3(1 - P^n(x))}$$

for all i such that $\sum_{j=1}^i \pi_j^n \geq x$. This plan is crude and, for large n , in the budget set since $\sum_N c^n(i)p^n(i) \rightarrow 2a/3$ for all x . If $\rho \leq 1$ and $P^n(x) \rightarrow 1$ for some $x < 1$, then the utility of this

plan converges to infinity. Since the equilibrium utility must stay bounded, this implies that $\rho > 1$. \square

The vector $\gamma = (m, \beta) \in \mathbb{R}_+^k \times [0, 1]^k$ such that $m_1 \leq m_2 \leq \dots \leq m_k$ and $\beta_1 \leq \beta_2 \leq \dots \leq \beta_k = 1$ is a *cutoff plan*. Let Γ_k be the set of all cutoff plans. Let $\Gamma_k^\epsilon = \{(m, \beta) \in \Gamma_k \mid \epsilon \leq m_r \leq 1/\epsilon\}$ and let $\Gamma_k^0 = \{(m, \beta) \in \Gamma_k \mid m_r > 0\}$. We refer to Γ_k^0 as *regular* cutoff plans.

Each monotone consumption plan $c \in \mathcal{C}_k$ can be mapped to $\gamma_c \in \Gamma_k$ in the obvious way: $m_1 = c_1$; for $1 \leq r < k$ inductively define $m_{r+1} = m_r$ if m_r is the maximal consumption in c and let $m_{r+1} = \min_{\{j: c_j > m_r\}} c_j$ otherwise. Define $\beta_r = \sum_{i \leq j} \pi_i$ where j is the maximal state such that $c_j \leq m_r$.

Recall that F_{s^n} is the cdf of endowment in economy E^n . Let $w^n = \sum_j p_j^n s_j^n = \int F_{s^n}^{-1} dP^n$ be the equilibrium wealth of the consumer. Let F be the cdf of the limit endowment and let $F^{-1} : [0, 1] \rightarrow [a, b]$ denote the restriction of its inverse. Let $w = \int_0^1 F^{-1} dP$ be the *limit household wealth*.

For any cutoff plan $\gamma = (m, \beta)$ and cumulative price P let $\bar{w}(\gamma; P)$ denote the cost of γ at P . We have

$$\bar{w}(\gamma; P) := m_1 P(\beta_1) + \sum_{j=2}^k m_j (P(\beta_j) - P(\beta_{j-1})) = m_1 + \sum_{j=1}^{k-1} (m_{j+1} - m_j) (1 - P(\beta_j))$$

Define the limit budget set as follows: $\hat{B}_k^\epsilon(P) = \{\gamma \in \Gamma_k^\epsilon \mid \bar{w}(\gamma; P) \leq w\}$. We write $\hat{B}_k(P)$ instead of $\hat{B}_k^0(P)$. For $\gamma = (m, \beta) \in \Gamma_k^0$ we define

$$L(\gamma) := u(m_1)\beta_1 + \sum_{r=2}^k u(m_r)(\beta_r - \beta_{r-1})$$

to be the utility of plan γ . Define $L_k^\epsilon = \max\{L(\gamma) \mid \gamma \in \hat{B}_k^\epsilon\}$ and let $L_k^* = \lim_{\epsilon \rightarrow 0} L_k^\epsilon$ be the supremum of L over all regular plans in the limit budget set. The function $\gamma \in \Gamma_k$ is a *limit plan* if there is a sequence $c^n \in K(\mu^n)$ such that $\gamma_{c^n} \rightarrow \gamma$.

Lemma 13.

(1) $u(b) \geq L_k^* \geq L_{k-1}^* \geq u(a)$ for all $k \geq 2$.

(2) Let $P(x) = P(0) + \int_0^x \hat{p}(y)dy$. Suppose that $L_{k-1}^* = L_k^*$ and let $(m, \beta) \in \hat{B}_{k-1}$ be such that $L(m, \beta) = L_{k-1}^*$. Then \hat{p} is constant on intervals $(0, \beta_1), (\beta_1, \beta_2), \dots, (\beta_{k-1}, \beta_k)$.

(3) Let $\gamma^\epsilon \in \hat{B}_k$ be such that $L(\gamma^\epsilon) \rightarrow L_k^*$. Then, $m_r^\epsilon \rightarrow 0$ implies $\beta_r^\epsilon \rightarrow 0$.

(5) If $L(m, \beta) = L_k^*$ for some $(m, b) \in \hat{B}^k$ then there is $(m', \beta') \in \hat{B}^k$ such that $L(m', \beta') = L_k^*$ and $1 > \beta'_{k-1}, \beta'_1 > 0$.

Proof. (1) is obvious. For (2) pick an arbitrary interval (β_r, β_{r+1}) . For any $\alpha \in (\beta_r, \beta_{r+1})$ consider a collection of plans $(m', \beta') \in \Gamma_k$ such that $m'_j = m_j, \beta'_j = \beta_j$ for all $j = 1, \dots, r$, $\beta_{r+1} = \alpha, \beta'_j = \beta_{j-1}$ for all $j = r+2, \dots, k-1$, and $m'_j = m_{j-1}$ and for all $j = r+3, \dots, k-1$. Denote this collection Υ_k^α . Observe that for any value of α , the cutoff plan (m, β) can be represented by $(m', \beta') \in \Upsilon_k^\alpha$ with $m'_{r+1} = m'_{r+2} = m_{r+1}$.

Thus, for any $\alpha \in (\beta_r, \beta_{r+1})$ it follows that $L_k^* \geq \max_{\gamma' \in \Upsilon_k^\alpha \cap \hat{B}_k} L(\gamma') \geq L_{k-1}^*$. Any maximizer (m', β') of the above expression involves m'_{r+1}, m'_{r+2} that solve

$$\max u(m'_{r+1})(\alpha - \beta_r) + u(m'_{r+2})(\beta_{r+1} - \alpha)$$

s.t

$$m'_{r+1}[P(\alpha) - P(\beta_r)] + m'_{r+2}[P(\beta_{r+1}) - P(\alpha)] \leq \text{const.}$$

Since $L_k^* = L_{k-1}^*$ there is a solution that satisfies $m'_{r+1} = m'_{r+2}$. A routine argument shows that

$$\frac{1}{\alpha - \beta_r} \int_{\beta_r}^{\alpha} \hat{p}(y)dy = \frac{1}{\beta_{r+1} - \alpha} \int_{\alpha}^{\beta_{r+1}} \hat{p}(y)dy,$$

which is equivalent to \hat{p} being constant on (β_r, β_{r+1}) .

For part (3) pick the highest r with the property that $m_r^\epsilon \rightarrow 0$, which means that by passing to a subsequence we have $\lim_\epsilon m_{r+1}^\epsilon > 0$. Let $\beta_r := \limsup_\epsilon \beta_r^\epsilon$. Suppose toward contradiction that $\beta_r > 0$ and pass to that subsequence. For each ϵ consider a plan $\tilde{\gamma}^\epsilon = (\tilde{m}^\epsilon, \beta^\epsilon)$ with consumption higher by $\eta > 0$ for states in the interval $[0, \beta_r^\epsilon]$ and lower by $\eta \frac{P(\beta_r^\epsilon)}{1 - P(\beta_r^\epsilon)}$ for states

in the interval $(\beta_r^\epsilon, 1]$. Note that for any $\eta > 0$ we have $\bar{w}(\tilde{\gamma}^\epsilon) = \bar{w}(\gamma^\epsilon)$, so for values of $\eta \in (0, 0.5 \cdot \lim_\epsilon m_{r+1}^\epsilon)$ we have $\tilde{\gamma}^\epsilon \in \hat{B}_k$ for n large. Note that

$$L(\tilde{\gamma}^\epsilon) - L(\gamma^\epsilon) \geq \eta \left[\beta_r^\epsilon u'(m_r^\epsilon) - (1 - \beta_r^\epsilon) u'(m_{r+1}^\epsilon) \frac{P(\beta_r^\epsilon)}{1 - P(\beta_r^\epsilon)} \right],$$

and note that since $\beta_r > 0$, $u'(m_r^\epsilon) \rightarrow \infty$, and $u'(m_{r+1}^\epsilon)$ is bounded, the right hand side is unbounded; hence, $\lim_\epsilon L(\tilde{\gamma}^\epsilon) - L_k^* > 0$, a contradiction.

For part (5) first assume $P(0) > 0$. In that case $\beta_1 = 0, m_1 > 0$ is not optimal and therefore $\beta_1 > 0$. Let $\bar{r} = \max\{l \in \{1, \dots, k-1\} | \beta_l < 1\}$ and set $\beta'_l = \beta_{\bar{r}}, m'_r = m_{\bar{r}}$ for all $l \geq \bar{r}$. Set $\beta'_l = \beta_l, m'_l = m_l$ for $l \leq \bar{r}$. It is straightforward to verify that $L(m', \beta') = L(m, \beta)$ and $(m', \beta') \in \hat{B}_k$.

Next, assume $P(0) = 0$. Let $\underline{r} = \min\{l \in \{1, \dots, k-1\} | \beta_l > 0\}$ and let \bar{r} be defined as above. Set $\beta'_l = \beta_{\underline{r}}$ and $m'_l = m_{\underline{r}}$ for $l \leq \underline{r}$. Set $\beta'_l = \beta_{\bar{r}}, m'_r = m_{\bar{r}}$ for all $l \geq \bar{r}$. Set $\beta'_l = \beta_l, m'_l = m_l$ for $\underline{r} \leq l \leq \bar{r}$. It is straightforward to verify that $L(m', \beta') = L(m, \beta)$ and $(m', \beta') \in \hat{B}_k$. \square

We will need the following technical lemma.

Lemma 14. *Suppose that $-\infty < v < w < \infty$ and $f^n : [v, w] \rightarrow \mathbb{R}$ is a sequence of monotone functions that converges pointwise to a continuous function f . Then the convergence is uniform.*

Proof. To see that, note that since $[v, w]$ is compact and f is continuous, f is uniformly continuous. Fix $\epsilon > 0$ and find a partition of the interval $[v, w]$, $v = t_0 < t_1 < \dots < t_m = w$, such that for any $t_j \leq x, y \leq t_{j+1}$ we have $|f(x) - f(y)| < \epsilon/5$. Since f^n converges pointwise to f , for large enough n it must be that $|f^n(t_j) - f(t_j)| \leq \epsilon/5$. For any x let $t_j \leq x \leq t_{j+1}$ and observe that

$$\begin{aligned} |f^n(x) - f(x)| &\leq |f^n(x) - f^n(t_{j+1})| + |f^n(t_{j+1}) - f(t_{j+1})| + |f(t_{j+1}) - f(x)| \\ &\leq |f^n(t_j) - f^n(t_{j+1})| + \epsilon/5 + \epsilon/5 \\ &\leq |f^n(t_j) - f(t_j)| + |f(t_j) - f(t_{j+1})| + |f(t_{j+1}) - f^n(t_{j+1})| + 2/5\epsilon \\ &\leq \epsilon/5 + \epsilon/5 + \epsilon/5 + 2/5\epsilon = \epsilon. \end{aligned} \quad \square$$

Lemma 15. *If $z^n \rightarrow 0$, then $0 \leq \liminf_n P^n(z^n) \leq \limsup_n P^n(z^n) \leq P(0)$.*

Proof. To see that $0 \leq \liminf_n P^n(z^n)$, note that $0 \leq P^n(z^n)$ for all n . To prove that $\limsup_n P^n(z^n) \leq P(0)$, suppose toward contradiction that $\limsup_n P^n(z^n) > P(0)$. Then by passing to a subsequence we have $\lim_n P^n(z^n) > P(0)$ for some $z^n \rightarrow 0$. By continuity of P , there exists $\epsilon > 0$ such that $\lim_n P^n(z^n) > P(\epsilon)$. Thus, there exists $\delta > 0$ and N_1 such that $P^n(z^n) > P(\epsilon) + \delta$ for all $n \geq N_1$. Since $z^n \rightarrow 0$, there exists N_2 such that $z^n < \epsilon$ for $n > N_2$, so by monotonicity of P^n , $P^n(z^n) < P^n(\epsilon)$. Since $P^n(\epsilon) \rightarrow P(\epsilon)$, there exists N_3 such that $P^n(\epsilon) < P(\epsilon) + \delta$. Hence, for $n > \max\{N_1, N_2, N_3\}$ we have

$$P(\epsilon) + \delta > P^n(\epsilon) > P^n(z^n) > P(\epsilon) + \delta,$$

a contradiction. □

Lemma 16.

(i) *If $\gamma^n \rightarrow \gamma$, then $\limsup L(\gamma^n) \leq L(\gamma)$. If $\gamma \in \Gamma^0$, then $L(\gamma^n) \rightarrow L(\gamma)$.*

(ii) *If $\gamma^n \rightarrow \gamma = (m, \beta)$ with $\beta_1 > 0$, then $\bar{w}(\gamma^n, P^n) \rightarrow \bar{w}(\gamma, P)$.*

(iii) *If $\gamma^n \rightarrow \gamma = (m, \beta)$ with $\beta_1 = 0$, then $\liminf_n \bar{w}(\gamma^n, P^n) \geq \bar{w}(\gamma, P)$.*

Proof. To prove (i), let $\gamma = (m, \beta)$. Fix $\epsilon > 0$ and let $G := u(m_k + \epsilon)$. For n large enough we have $m_k^n \leq m_k + \epsilon$, hence $u(m_k^n) \leq G$ for all j and for all n large. Each plan γ^n can be naturally thought of as a function from $[0, 1]$ to \mathbb{R}_+ ; let f^n denote that function and let f denote the function induced by the plan γ . The Fatou lemma applied to the nonnegative functions $G - u(f^n)$ yields $\liminf \int_0^1 G - u(f^n(x))dx \geq \int_0^1 G - u(f(x))dx$; hence, the conclusion follows. If $\gamma \in \Gamma^0$, let $g := u(m_1 - \epsilon)$ and note that for ϵ small enough we have $g \leq u(f^n) \leq G$. Use the Lebesgue dominated convergence theorem to obtain the conclusion.

To prove (ii), note that since $m_1^n \rightarrow m_1$ and $(m_{j+1}^n - m_j^n) \rightarrow (m_{j+1} - m_j)$, it suffices to show that $P^n(\beta_j^n) \rightarrow P(\beta_j)$. To see that, note that there exists $\eta > 0$ such that $\beta_1^n > \eta$ for large enough n . Consider the sequence of functions P^n converging pointwise to P on the restricted domain $[\eta, 1]$.

By Lemma 14, P^n converges uniformly to P . Since $[\eta, 1]$ is compact, P is uniformly continuous. Thus, for n large enough we have $|P^n(x) - P(x)| \leq \epsilon/2$ for all $x \in [\eta, 1]$ and $|P(x) - P(y)| < \epsilon/2$ for $|x - y|$ close enough. To conclude the proof, note that

$$\begin{aligned} |P^n(\beta_j^n) - P(\beta_j)| &\leq |P^n(\beta_j^n) - P(\beta_j^n)| + |P(\beta_j^n) - P(\beta_j)| \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

given that there are only finitely many sequences $\beta_j^n \rightarrow \beta_j$.

To prove (iii) note that since $m_1^n \rightarrow m_1$ and $(m_{j+1}^n - m_j^n) \rightarrow (m_{j+1} - m_j)$, it suffices to show that $\limsup_n P^n(\beta_j^n) \leq P(\beta_j)$. For β_1^n this follows from Lemma 15, whereas the convergence $P^n(\beta_j^n) \rightarrow P(\beta_j)$ for $\beta_j > 0$ has been shown in the proof of part (ii). \square

Lemma 17. $w^n \rightarrow w$.

Proof. Note that

$$\begin{aligned} |w^n - w| &= \left| \int F_n^{-1} dP^n - \int F^{-1} dP \right| \\ &\leq \left| \int F_n^{-1} dP^n - \int F^{-1} dP^n \right| + \left| \int F^{-1} dP^n - \int F^{-1} dP \right| \\ &\leq \int |F_n^{-1} - F^{-1}| dP^n + \left| \int F^{-1} dP^n - \int F^{-1} dP \right|. \end{aligned}$$

The first term vanishes, as by Lemma 14 $F_n^{-1} \rightarrow F^{-1}$ uniformly, whereas the second term vanishes by continuity of F^{-1} and the weak convergence of P^n to P . \square

Lemma 18. (i) If $x \in (0, 1)$ with $P(x) < 1$ then, for all $\epsilon > 0$, there exists a limit plan $\gamma = (m, \beta)$ such that $\beta_r \in (x - \epsilon, x + \epsilon)$ and $m_r < m_{r+1}$. (ii) There exists a regular limit plan $\gamma = (m, \beta)$.

Proof. (i) Let $x \in (0, 1)$ be such that $P(x) < 1$. By continuity of P , there exists $\epsilon > 0$ such that $P(x + \epsilon) < 1$. Let $\delta = F^{-1}(x + \epsilon/2) - F^{-1}(x - \epsilon/2)$ and note that $\delta > 0$. Since s^n converges

in distribution to F , which is continuous, we can choose $i_n, j_n \in N$ with $\sum_{j \leq i_n} \pi_j^n \rightarrow x - \epsilon/2$ and $\sum_{j \leq j_n} \pi_j^n \rightarrow x + \epsilon/2$. Since $P(x + \epsilon) < 1$ it follows from Theorem 2 and Lemma 11 that $p_{i_n}^n, p_{j_n}^n > 0$ for large n . As established in the proof of Lemma 8, $\sigma_i(\mu) < s_i$ implies $p_i = 0$; hence, it follows that $\sigma_{i_n}(\mu^n) = s_{i_n}$ and $\sigma_{j_n}(\mu^n) = s_{j_n}$, and therefore $\sigma_{j_n}(\mu^n) - \sigma_{i_n}(\mu^n) \rightarrow \delta$. Since $P^n(x + \epsilon) \rightarrow P(x + \epsilon)$ and by Lemma 17 $w^n \rightarrow w$ the budget constraint implies that there is $M < \infty$ such that $c_{j_n} < M$ for all $c \in K(\mu^n)$ and n is sufficiently large. We have

$$\begin{aligned} \sigma_{j_n}(\mu^n) - \sigma_{i_n}(\mu^n) &= \sum_{K(\mu^n)} c_{j_n} \mu^n(c) - \sum_{K(\mu^n)} c_{i_n} \mu^n(c) \\ &\leq \sum_{K_R(\mu^n)} c_{j_n} \mu^n(c) - \sum_{K_R(\mu^n)} c_{i_n} \mu^n(c) + M \sum_{K(\mu^n) \setminus K_R(\mu^n)} \mu^n(c) \end{aligned}$$

Note that $\lim M \sum_{K(\mu^n) \setminus K_R(\mu^n)} \mu^n(c) \rightarrow 0$ as $R \rightarrow \infty$ by Lemma 10. Since $\sigma_{j_n}(\mu^n) - \sigma_{i_n}(\mu^n) \rightarrow \delta$ this implies that we may choose R large enough so that there exists $c^n \in K_R(\mu^n)$ with $c_{j_n}^n > c_{i_n}^n + \delta/2$. Let c^n be a sequence of consumption plans with this property and let $\gamma^n = (m^n, \beta^n)$ be the corresponding sequence of cutoff plans. Since $c_{j_n}^n > c_{i_n}^n + \delta/2$ there exists $\beta_r^n \in (x - \epsilon, x + \epsilon)$ such that $m_{r+1}^n > m_r^n + \delta/(2k)$. Since $c^n \in K_R(\mu^n)$ it follows that γ^n has a convergent subsequence. The limit of this sequence is a limit plan with the desired properties.

(ii) Observe that $c_1 \leq b$ for all $c \in K(\mu)$ by feasibility and fairness. Also note that if $p_1^n = 0$, then by Theorem 2, $p_i^n = 0$ for all i , a contradiction with $\sum_{i=1}^n p_i^n = 1$. Thus, $p_1^n > 0$ and as established in the proof of Lemma 8, $\sigma_1(\mu) < s_1$ implies $p_1 = 0$; hence, it follows that $\sigma_1(\mu^n) = s_1 \geq a$. Therefore,

$$\sum_{\{c \in K(\mu^n) | c_1 \leq a/2\}} \mu(c) \leq \frac{2b - 2a}{2b - a} < 1$$

It follows from Lemma 10 that for R large enough there is $c^n \in K_R(\mu^n)$ with $a/2 \leq c^n \leq R$. Let c^n be a sequence of such plans and let γ_{c^n} be the corresponding plans in Γ_k . This sequence has a convergent subsequence with limit γ , a regular limit plan. \square

Lemma 19. *If $\gamma = (m, \beta) \in \Gamma_k$ is a limit plan then $\bar{w}(\gamma; P) \leq w$ and $L(\gamma) = L_k^*$.*

Proof. Since γ is a limit plan, there exists $c^n \in K(\mu^n)$ such that $\gamma_{c^n} \rightarrow \gamma$. Since $c^n \in K(\mu^n)$

it follows that $\bar{w}(\gamma_{c^n}; P^n) = w^n$, so by Lemma 17, $\lim \bar{w}(\gamma_{c^n}; P^n) = w$. On the other hand $\liminf_n \bar{w}(\gamma^n; P^n) \geq \bar{w}(\gamma; P)$ by Lemma 16 (ii); hence, $\bar{w}(\gamma; P) \leq w$.

Take any $(\bar{m}, \bar{\beta}) \in \hat{B}_k$ with $\bar{w}(\bar{m}, \bar{\beta}; P) = w$ and $\beta_1 > 0$. Suppose toward contradiction that $L(\bar{m}, \bar{\beta}) > L(\gamma)$. Construct $i_1^n, i_2^n, \dots, i_k^n$ such that $\tilde{\beta}_r^n := \sum_{j \leq i_r^n} \pi_j^n \rightarrow \bar{\beta}_r$. Define $\tilde{m}_r^n := \bar{m}_r \frac{w^n}{\bar{w}(\bar{m}, \tilde{\beta}^n; P^n)}$ and note that $(\tilde{m}^n, \tilde{\beta}^n) \in \hat{B}_k^n$. Also note that by optimality of γ_{c^n} we have $L(\tilde{m}^n, \tilde{\beta}^n) \leq L(\gamma_{c^n})$. On the other hand, by Lemma 16 (ii) $\bar{w}(\bar{m}, \tilde{\beta}^n; P^n) \rightarrow \bar{w}(\bar{m}, \bar{\beta}; P)$, so $\lim \frac{w^n}{\bar{w}(\bar{m}, \tilde{\beta}^n; P^n)} = \frac{w}{\bar{w}(\bar{m}, \bar{\beta}; P)} = 1$. Hence, $(\tilde{m}^n, \tilde{\beta}^n) \rightarrow (\bar{m}, \bar{\beta})$ for all r . Thus, by Lemma 16 (i) we have $\lim L(\tilde{m}^n, \tilde{\beta}^n) = L(\bar{m}, \bar{\beta})$. By Lemma Lemma 16 (i), $\limsup L(\gamma_{c^n}) \leq L(\gamma)$; hence, $L(\bar{m}, \bar{\beta}) \leq L(\gamma)$.

It remains to show the same for any $(\bar{m}, \bar{\beta}) \in \hat{B}_k$ with $\bar{w}(\bar{m}, \bar{\beta}; P) = w$ and $\beta_1 = 0$. Suppose toward contradiction that there exists such $(\bar{m}, \bar{\beta})$ with $L(\bar{m}, \bar{\beta}) > L(\gamma)$. Wlog, $(\bar{m}, \bar{\beta}) \in \Gamma_k^0$ since otherwise consider $(\hat{m}, \bar{\beta})$ with $\hat{m}_1 := \eta$ and $\hat{m}_j := \bar{m}_j - \eta \frac{P(0)}{1-P(0)}$. Note that $\bar{w}(\hat{m}, \bar{\beta}; P) = w$ and $L(\hat{m}, \bar{\beta}) \geq L(\bar{m}, \bar{\beta}) - u'(\bar{m}_2)\eta > L(\gamma)$ for small η . For $\epsilon > 0$ let $(\tilde{m}^\epsilon, \tilde{\beta}^\epsilon)$ be defined as follows: $\tilde{\beta}_1^\epsilon := \epsilon$, $\tilde{\beta}_j^\epsilon := \bar{\beta}_j$ for $j > 1$, and $\tilde{m}_2^\epsilon := \frac{[P(\bar{\beta}_2) - P(0)]\bar{m}_2 - [P(\epsilon) - P(0)]\bar{m}_1}{P(\bar{\beta}_2) - P(\epsilon)}$, $\tilde{m}_j^\epsilon := \bar{m}_j$ for $j \neq 2$. Observe that $\bar{w}(\tilde{m}^\epsilon, \tilde{\beta}^\epsilon) = \bar{w}(\bar{m}, \bar{\beta}) = w$, which as proved in the paragraph above implies that $L(\tilde{m}^\epsilon, \tilde{\beta}^\epsilon) \leq L(\gamma)$. Moreover, $\lim_{\epsilon \rightarrow 0} (\tilde{m}^\epsilon, \tilde{\beta}^\epsilon) = (\bar{m}, \bar{\beta})$; hence, by Lemma 16 (i) we have $\lim_{\epsilon \rightarrow 0} L(\tilde{m}^\epsilon, \tilde{\beta}^\epsilon) = L(\bar{m}, \bar{\beta})$, a contradiction. \square

Lemma 20. *Suppose that P is a limit BCE price along the sequence (u, k, π^n, s^n) . Then $L_{k-1}^*(P) < L_k^*(P)$.*

Proof. Suppose toward contradiction that $L_k^*(P) = L_{k-1}^*(P)$. There are two cases:

Case 1: There exists $(m, \beta) \in \hat{B}_{k-1}$ with $L(m, \beta) = L_{k-1}^*(P)$. By Lemma 13 (5) we may assume that $\beta_1 > 0$. Lemma 13 (2) implies that $P(x)/x$ is constant on $(0, \beta_1]$. If $P(0) > 0$ then there exists $\eta > 0$ such that $P(x)/x$ is nonconstant on $(0, \eta]$, a contradiction. Hence, it follows that $P(0) = 0$.

By Lemma 18, there exists a limit plan (m', β') with $m'_{r+1} > m'_r$ and $0 < \beta'_r < \beta_1$ for some $r \geq 1$. By Lemma 19, $L(m', \beta') = L_{k-1}^*(P)$. Define $z = \min\{\beta_1, \beta'_{r+1}\}$ and define $R = m'_{r+1}(z - \beta'_r) + \sum_{l=1}^r m'_l[\beta'_l - \beta_{l-1}]$. Consider the following alternative plan $\Gamma_{k-1} \ni (m'', \beta'')$: $\beta''_1 = \beta''_2 = \dots = \beta''_r = z$; $m''_1 = m''_2 = \dots = m''_r = R$. For $\beta'_l > z$ set $\beta''_l = \beta'_l$ and $m''_l = m'_l$. This

plan is regular and satisfies the budget constraint because $P(x)/x$ is constant on $[0, z]$. Since u is strictly concave, $m'_{r+1} > m'_r$, and $P(x)/x$ is constant on $[0, z]$, it follows from Jensen's inequality that $L(m'', \beta'') > L(m', \beta')$, a contradiction.

Case 2: No regular plan attains L_{k-1}^* . Let $L(m^\epsilon, \beta^\epsilon) = L_{k-1}^\epsilon$ with $(m^\epsilon, \beta^\epsilon) \in \hat{B}_{k-1}^\epsilon$. Consider a subsequence of $(\beta^\epsilon, m^\epsilon)$ such that $\beta^\epsilon \rightarrow \beta$; $m_r^\epsilon \rightarrow m_r$ for $r \leq \bar{r}$; $m_r^\epsilon \rightarrow \infty$ for $r > \bar{r}$. Clearly such a sequence exists. (If m_r^ϵ stays bounded then $\bar{r} = k - 1$.) If $m_r = 0$ for some $r > 0$, let \underline{r} be the largest r such that $m_{\underline{r}} = 0$. Otherwise, let $\underline{r} = 0$.

We now show that if $\rho \leq 1$, then $\beta_{\bar{r}} = 1$. Suppose toward contradiction that $\beta_{\bar{r}} < 1$. It follows that $P(\beta_{\bar{r}}^\epsilon) < 1$. Otherwise for any $M > 0$ consider the plan that coincides with $(m^\epsilon, \beta^\epsilon)$ except for the states in the interval $(\beta_{\bar{r}}^\epsilon, 1]$, where it delivers M units of consumption more. Such a plan has the same expenditure, but yields unbounded utility as $M \rightarrow \infty$, a contradiction with $L(m^\epsilon, \beta^\epsilon) = L_{k-1}^\epsilon$. The same argument proves that $P(\beta_{\bar{r}}) < 1$, since otherwise the bound L_k^* can be exceeded.

Consider a plan $(\tilde{m}^\epsilon, \beta^\epsilon)$ with consumption lower by η than in $(m^\epsilon, \beta^\epsilon)$ for states in $(\beta_{\bar{r}}^\epsilon, 1]$ and higher by $\eta \frac{1-P(\beta_{\bar{r}}^\epsilon)}{P(\beta_{\bar{r}}^\epsilon)}$ in states $[0, \beta_{\bar{r}}^\epsilon]$. Notice that $\bar{w}(\tilde{m}^\epsilon, \beta^\epsilon) = \bar{w}(m^\epsilon, \beta^\epsilon)$ we have $(\tilde{m}^\epsilon, \beta^\epsilon) \in \hat{B}_k$. Observe that

$$L(\tilde{m}^\epsilon, \beta^\epsilon) - L(m^\epsilon, \beta^\epsilon) \geq \eta \left[\frac{1 - P(\beta_{\bar{r}}^\epsilon)}{P(\beta_{\bar{r}}^\epsilon)} \beta_{\bar{r}}^\epsilon u'(m_{\bar{r}}^\epsilon) - (1 - \beta_{\bar{r}}^\epsilon) u'(m_{\bar{r}+1}^\epsilon) \right].$$

Note that since $\frac{1-P(\beta_{\bar{r}}^\epsilon)}{P(\beta_{\bar{r}}^\epsilon)} \rightarrow \frac{1-P(\beta_{\bar{r}})}{P(\beta_{\bar{r}})} > 0$, $u'(m_{\bar{r}}) > 0$, $u'(m_{\bar{r}+1}^\epsilon) \rightarrow 0$ the right hand side is bounded away from zero; hence $\lim_\epsilon L(\tilde{m}^\epsilon, \beta^\epsilon) - L_k^* > 0$, a contradiction. Thus, if $\rho \leq 1$, then the plan (m, β) is well defined, which contradicts the unattainability of L_{k-1}^* .

On the other hand, if $\rho > 1$, then $P(\beta_r) = 1$ (otherwise $\bar{w}(m^\epsilon, \beta^\epsilon)$ is unbounded—a contradiction). Let $m_{\bar{r}+1} = m_{\bar{r}+2} = \dots, m_{k-1} = \infty$ and consider the extended $k - 1$ -crude plan (m, β) . Since u is bounded from above by zero, the utility of this plan is well defined and $L(m, \beta) = L_k^*$. The expenditure on this plan is well defined as well, since $P(\beta_r) = 1$. The remainder of the proof is analogous to Case 1. \square

Lemma 21. (i) $P(0) > 0$; (ii) $\hat{p}(x) \rightarrow 0$ as $x \rightarrow 1$.

Proof. part (i). By Lemma 18 (i) for all $\epsilon > 0$ there exists a limit plan with $\beta_1 < \epsilon$ and, therefore, there is a sequence of limit plans $(m^t, \beta^t) \in B_k$ with $\beta_1^t \rightarrow 0$ and by Lemma 19 $L(m^t, \beta^t) = L_k^*$. Let $(\tilde{m}^t, \tilde{\beta}^t)$ be the following $(k-1)$ -crude plan:

$$\begin{aligned}\tilde{m}_1^t &= m_2^t - (m_2^t - m_1^t)P(\beta_1^t) \\ \tilde{m}_2^t &= m_3^t - (m_3^t - m_2^t)P(\beta_1^t) \\ &\dots \\ \tilde{m}_{(k-1)}^t &= m_k^t - (m_k^t - m_1^t)P(\beta_1^t)\end{aligned}$$

and $\tilde{\beta}_{rt} = \beta_{(r+1)}^t$. Assume $P(0) = 0$. Then $P(\beta_1^t) \rightarrow 0$ and for all r , $\tilde{m}_r^t > 0$ for large t . Hence this plan is feasible. By construction we have $(\tilde{m}^t, \tilde{\beta}^t) \in \hat{B}_{k-1}$. Moreover, $L(\tilde{m}^t, \tilde{\beta}^t) - L(m^t, \beta^t) \rightarrow 0$. But this contradicts Lemma 20 and therefore proves (i).

Part (ii). If $P(x) = 1$ for some x then the lemma holds trivially. Hence, assume $P(x) < 1$ for all $x < 1$. By Lemma 18, for every $\epsilon > 0$ there is a limit plan (m, β) with $\beta_{k-1} > 1 - \epsilon$. Together with Lemma 19 this implies that there is a sequence of cutoff plans $(m^t, \beta^t) \in \hat{B}_k$ with $\beta_{k-1}^t \rightarrow 1$ and $L(m^t, \beta^t) = L_k^*$. Let $(\hat{m}^t, \hat{\beta}^t)$ be the plan such that $\hat{m}_r^t = m_r^t$ for all $r < k$ and $\hat{m}_k^t = m_{k-1}^t$. Set $\hat{\beta}_r^t = \beta_r^t$ for $r < k-1$ and set $\hat{\beta}_{(k-1)}^t = 1$. Clearly $(\hat{m}^t, \hat{\beta}^t) \in \hat{B}_{k-1}$ and. We now show that if $\lim_{x \rightarrow 1} \hat{p}(x) \neq 0$, then it follows that $L(\hat{m}^t, \hat{\beta}^t) \rightarrow L_k^*$, contradicting Lemma 20.

To see that, recall that by Lemma 11, \hat{p} is non-increasing and non-negative; hence, $\alpha := \lim_{x \rightarrow 1} \hat{p}(x)$ exists and $\alpha > 0$. Since (m^t, β^t) is the optimal solution given the partition $0 \leq \beta_1^t \leq \beta_2^t \leq \dots \leq \beta_{k-1}^t \leq \beta_k^t = 1$, it follows that by setting $\theta := 1/\rho$ and using the formula for

optimum with CES utility we have

$$\begin{aligned}
m_k^t &= \left(\frac{1 - \beta_{k-1}^t}{\int_{\beta_{k-1}^t}^1 \hat{p}(x) dx} \right)^\theta \frac{w}{(\beta_1^t)^\theta P(\beta_1^t)^{1-\theta} + \sum_{j=2}^k (\beta_j^t - \beta_{j-1}^t)^\theta \left(\int_{\beta_{j-1}^t}^{\beta_j^t} \hat{p}(x) dx \right)^{1-\theta}} \\
&\leq \left(\frac{1 - \beta_{k-1}^t}{\int_{\beta_{k-1}^t}^1 \hat{p}(x) dx} \right)^\theta \frac{w}{\sum_{j=1}^k (\beta_j^t - \beta_{j-1}^t)^\theta \left(\int_{\beta_{j-1}^t}^{\beta_j^t} \hat{p}(x) dx \right)^{1-\theta}} \\
&\leq \left(\frac{1 - \beta_{k-1}^t}{(1 - \beta_{k-1}^t)\alpha} \right)^\theta \frac{w}{\sum_{j=1}^k (\beta_j^t - \beta_{j-1}^t)(\alpha)^{1-\theta}} \\
&\leq \left(\frac{1}{\alpha} \right)^\theta \frac{w}{\alpha^{1-\theta}};
\end{aligned}$$

hence, m_k^t is bounded. Note that since u is concave, we have

$$\begin{aligned}
L(m^t, \beta^t) - L(\hat{m}^t, \hat{\beta}^t) &= (1 - \beta_{k-1}^t)(u(m_k^t) - u(m_{k-1}^t)) \\
&\leq (1 - \beta_{k-1}^t)u'(m_{k-1}^t)(m_k^t - m_{k-1}^t) \\
&\leq (1 - \beta_{k-1}^t)u'(m_{k-1}^t)m_k^t.
\end{aligned}$$

Since $u'(m_{k-1}^t)$ is bounded, it follows that $L(m^t, \beta^t) - L(\hat{m}^t, \hat{\beta}^t) \rightarrow 0$. \square

Lemma 22. *Assume $\hat{p} \leq K$ for some $K < \infty$. There is $\delta > 0$ such that for any sequence of limit plans (m^n, β^n) such that $0 < \beta_1^n \rightarrow 0$ we have $\beta_2^n - \beta_1^n > \delta$.*

Proof. Suppose toward contradiction that the opposite is true. It follows that there exists a sequence of limit plans (m^n, β^n) with $0 < \beta_1^n < \beta_2^n \rightarrow 0$. By Lemma 19, each $(m^\delta, \beta^\delta) \in \hat{B}_k$ and $L(m^\delta, \beta^\delta) = L_k$. Let j be the smallest index such that $\eta := \liminf_n \beta_j^n > 0$. Thus, for n large enough we have $\beta_j^n \geq \beta_j^n - \beta_{j-1}^n \geq \eta/2$. Extract a subsequence where the limit exists and define $\beta_j := \lim_n \beta_j^n$.

For each n we construct a $k-1$ -crude plan (\tilde{m}^n, β^n) by setting $\tilde{m}_l^n = m_l^n$ for $l \neq 2, j$ and $\tilde{m}_2^n = \tilde{m}_j^n = \alpha m_2^n + (1 - \alpha)m_j^n$, where

$$\alpha := \frac{P(\beta_2^n) - P(\beta_1^n)}{P(\beta_2^n) - P(\beta_1^n) + P(\beta_j^n) - P(\beta_{j-1}^n)}.$$

Notice that $(\tilde{m}^n, \beta^n) \in \hat{B}_{k-1}$. Define

$$\chi := \frac{\beta_2^n - \beta_1^n}{\beta_2^n - \beta_1^n + \beta_j^n - \beta_{j-1}^n}$$

and note that

$$\begin{aligned} \frac{L(m^n, \beta^n) - L(\tilde{m}^n, \beta^n)}{\beta_2^n - \beta_1^n + \beta_j^n - \beta_{j-1}^n} &= \chi u(m_2^n) + (1 - \chi)u(m_j^n) - u(\tilde{m}_j^n) \\ &= \chi u(m_2^n) + (1 - \chi)u(m_j^n) - u(\alpha m_2^n + (1 - \alpha)m_j^n) \\ &\leq \chi u(m_2^n) + (1 - \chi)u(m_j^n) - \alpha u(m_2^n) + (1 - \alpha)u(m_j^n) \\ &= (\alpha - \chi)[u(m_j^n) - u(m_2^n)]. \end{aligned}$$

We have

$$\begin{aligned} \alpha - \chi &\leq \frac{P(\beta_2^n) - P(\beta_1^n)}{P(\beta_2^n) - P(\beta_1^n) + P(\beta_j^n) - P(\beta_{j-1}^n)} - \frac{\beta_r^n - \beta_{r-1}^n}{\beta_2^n - \beta_1^n + \beta_j^n - \beta_{j-1}^n} \\ &\rightarrow \frac{P(0) - P(0)}{P(0) - P(0) + P(\beta_j) - P(0)} - \frac{0 - 0}{0 - 0 + \beta_j - 0} = 0. \end{aligned}$$

We also have

$$m_j^n \leq \frac{w}{P(\beta_j^n) - P(\beta_{j-1}^n)} \leq \frac{w}{P(\eta/2) - P(\beta_{j-1}^n)} \rightarrow \frac{w}{P(\eta/2) - P(0)}$$

Next, note that

$$\left(\frac{\beta_2^n - \beta_1^n}{\int_{\beta_1^n}^{\beta_2^n} \hat{p}(x) dx} \right) \geq \left(\frac{\beta_2^n - \beta_1^n}{(\beta_2^n - \beta_1^n) \hat{p}(\beta_1^n)} \right) \geq \left(\frac{1}{K} \right)^\theta$$

and

$$\sum_{l=1}^k (\beta_l^t - \beta_{l-1}^t)^\theta \left(\int_{\beta_{l-1}^t}^{\beta_l^t} \hat{p}(x) dx \right)^{1-\theta} \leq \sum_{l=1}^k (\beta_l^t - \beta_{l-1}^t)^\theta ((\beta_l^t - \beta_{l-1}^t) \hat{p}(\beta_{l-1}^t))^{1-\theta} \leq K^{1-\theta}$$

and, therefore,

$$\begin{aligned}
m_2^n &= \left(\frac{\beta_2^n - \beta_1^n}{\int_{\beta_1^n}^{\beta_2^n} \hat{p}(x) dx} \right)^\theta \frac{w}{(\beta_1^n)^\theta P(\beta_1^n)^{1-\theta} + \sum_{l=2}^k (\beta_l^n - \beta_{l-1}^n)^\theta (P(\beta_l^n) - P(\beta_{l-1}^n))^{1-\theta}} \\
&\geq \left(\frac{1}{K} \right)^\theta \frac{w}{\sum_{l=1}^k (\beta_l^n - \beta_{l-1}^n)^\theta (\int_{\beta_{l-1}^n}^{\beta_l^n} \hat{p}(x) dx)^{1-\theta} + (\beta_1^n)^\theta [(P(\beta_1^n)^{1-\theta} - (\int_0^{\beta_1^n} \hat{p}(x) dx)^{1-\theta})]} \\
&\geq \left(\frac{1}{K} \right)^\theta \frac{w}{K^{1-\theta} + (\beta_1^n)^\theta [(P(\beta_1^n)^{1-\theta} - (\int_0^{\beta_1^n} \hat{p}(x) dx)^{1-\theta})]} \\
&\rightarrow \left(\frac{1}{K} \right)^\theta \frac{w}{(K)^{1-\theta} + 0^\theta [(P(0)^{1-\theta})]} = \frac{w}{K}.
\end{aligned}$$

Thus, $L(m^n, \beta^n) - L(\tilde{m}^n, \beta^n) \rightarrow 0$, a contradiction to Lemma 20. \square

Lemma 23. *If $\rho \geq 1$ then $\hat{p}(x) \rightarrow \infty$ as $x \rightarrow 0$.*

Proof. Assume $\hat{p} \leq K$ for some $K < \infty$. By Lemma 18 (i) for any $x > 0$ there exists a limit plan $\gamma = (m, \beta)$ with $\beta_1 \in (0, x)$. Since x can be chosen arbitrarily close to zero, there exists a sequence of limit plans (m^t, β^t) such that $0 < \beta_1^t \rightarrow 0$. By Lemma 22 we can choose $\delta > 0$ so that $\beta_2^t - \beta_1^t \geq 2\delta$ for every t .

By Lemma 19, $L_k(m^t, \beta^t) = L_k^*$ and $(m^t, \beta^t) \in \hat{B}_k$. We have

$$m_2^t = \left(\frac{\beta_2^t - \beta_1^t}{P(\beta_2^t) - P(\beta_1^t)} \right)^\theta \frac{w}{(\beta_1^t)^\theta P(\beta_1^t)^{1-\theta} + \sum_{l=2}^k (\beta_l^t - \beta_{l-1}^t)^\theta (P(\beta_l^t) - P(\beta_{l-1}^t))^{1-\theta}}$$

Note that

$$\left(\frac{\beta_2^t - \beta_1^t}{P(\beta_2^t) - P(\beta_1^t)} \right) \geq \left(\frac{2\delta}{P(1) - P(\beta_1^t)} \right)$$

and

$$\begin{aligned}
& (\beta_1^t)^\theta P(\beta_1^t)^{1-\theta} + \sum_{l=2}^k (\beta_l^t - \beta_{l-1}^t)^\theta (P(\beta_l^t) - P(\beta_{l-1}^t))^{1-\theta} \\
&= \sum_{l=1}^k (\beta_l^t - \beta_{l-1}^t)^\theta \left(\int_{\beta_{l-1}^t}^{\beta_l^t} \hat{p}(x) dx \right)^{1-\theta} + (\beta_1^t)^\theta \left[(P(\beta_1^t))^{1-\theta} - \left(\int_0^{\beta_1^t} \hat{p}(x) dx \right)^{1-\theta} \right] \\
&\leq \sum_{l=1}^k (\beta_l^t - \beta_{l-1}^t) K^{1-\theta} + (\beta_1^t)^\theta \left[(P(\beta_1^t))^{1-\theta} - \left(\int_0^{\beta_1^t} \hat{p}(x) dx \right)^{1-\theta} \right] \\
&= K^{1-\theta} + (\beta_1^t)^\theta \left[(P(\beta_1^t))^{1-\theta} - (P(\beta_1^t) - P(0))^{1-\theta} \right]
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
m_2^t &\geq \left(\frac{2\delta}{P(1) - P(\beta_1^t)} \right)^\theta \frac{w}{K^{1-\theta} + (\beta_1^t)^\theta \left[(P(\beta_1^t))^{1-\theta} - (P(\beta_1^t) - P(0))^{1-\theta} \right]} \\
&\rightarrow \left(\frac{2\delta}{P(1) - P(0)} \right)^\theta \frac{w}{(K)^{1-\theta} + 0^\theta [(P(0))^{1-\theta}]} > 0
\end{aligned}$$

hence m_2^t is bounded away from zero.

Since $P(0) > 0$ it follows that $P(\beta_1^t)/\beta_1^t \rightarrow \infty$. Choose a subsequence such that $\beta_2 := \lim_t \beta_2^t$.

We have

$$\begin{aligned}
m_1^t &= \left(\frac{\beta_1^t}{P(\beta_1^t)} \right)^\theta \frac{w}{(\beta_1^t)^\theta P(\beta_1^t)^{1-\theta} + \sum_{j=2}^k (\beta_j^t - \beta_{j-1}^t)^\theta (P(\beta_j^t) - P(\beta_{j-1}^t))^{1-\theta}} \\
&\leq \left(\frac{\beta_1^t}{P(\beta_1^t)} \right)^\theta \frac{w}{(\beta_2^t - \beta_1^t)^\theta (P(\beta_2^t) - P(\beta_1^t))^{1-\theta}} \\
&\rightarrow (0)^\theta \frac{w}{(\beta_2)^\theta (P(\beta_2) - P(0))^{1-\theta}},
\end{aligned}$$

which implies that $m_1^t \rightarrow 0$.

Next, we show that (m^t, β^t) is not optimal if β_1^t is sufficiently close to zero. We prove this by constructing an alternative plan $(\hat{m}^t, \hat{\beta}^t)$ such that $\hat{m}_r^t = m_r^t$ for all $r \neq 2$ and $\hat{\beta}_r^t = \beta_r^t$ for

all $r \neq 1$. Specifically $\hat{\beta}_1^t < \beta_1^t$ and

$$\hat{m}_2^t = R^t(\hat{\beta}_1^t) := \frac{(P(\beta_1^t) - P(\hat{\beta}_1^t))m_1^t + (P(\beta_2^t) - P(\beta_1^t))m_2^t}{P(\beta_2^t) - P(\hat{\beta}_1^t)}$$

By construction $(\hat{m}^t, \hat{\beta}^t) \in \hat{B}_k$. Let $D^t(\hat{\beta}_1) := L(m^t, \beta^t) - L(\hat{m}^t, \hat{\beta}^t)$. We have

$$D^t(\hat{\beta}_1) = \hat{\beta}_1^t u(m_1^t) + (\beta_2^t - \hat{\beta}_1^t)u(R^t(\hat{\beta}_1)) - [\beta_1^t u(m_1^t) + (\beta_2^t - \beta_1^t)u(m_2^t)].$$

Since $P(2\delta) - P(0) > 0$ it follows that there is $\epsilon > 0$ such that $P(\beta_2^t) - P(\beta_1^t) > \epsilon$ for all t .

Note that D^t is well defined since $R^t(\hat{\beta}_1)$ is greater than zero for all $\hat{\beta}_1 \leq \beta_1^t$. Moreover, D is differentiable. Next, we show that $\partial D^t(\beta_1^t)/\partial \hat{\beta}_1 < 0$ if β_1^t is sufficiently close to zero. A straightforward calculation shows that

$$\frac{\partial D^t(\beta_1^t)}{\partial \hat{\beta}_1} = u(m_1^t) - u(m_2^t) + (\beta_2^t - \beta_1^t)u'(m_2^t) \frac{\partial R^t(\beta_1^t)}{\partial \hat{\beta}_1}$$

Note that $\frac{\partial R^t(\beta_1^t)}{\partial \hat{\beta}_1} \leq \frac{K\hat{m}_2^t}{\epsilon^2}$ and $(\beta_2^t - \beta_1^t)u'(m_2^t) \leq u'(m_2^t)$ which is bounded since m_2^t is bounded away from zero. Since $m_1^t \rightarrow 0$ we have $u(m_1^t) - u(m_2^t) \rightarrow -\infty$ and therefore the claim follows. \square

Lemma 24. *If $\rho > 1$ then $P(x) = 1$ for some $x < 1$.*

Proof. If $\rho > 1$ then $u(x) \leq 0$ for all $x > 0$. Let $\delta := L_k^* - L_{k-1}^* > 0$ and $R > 0$ be such that $u(R) = L_k^* - (1 - x)$. Suppose that $P(x) < 1$ for all $x < 1$. Then, by Lemma 18 and 19 there is $(m, \beta) \in \hat{B}_k$ such that $L(m, \beta) = L_k^*$ and $\beta_{k-1} > x$. Since m_r is increasing in r and $u \leq 0$ it follows that $m_{k-1} \geq R$. Consider an alternative plan $(\hat{m}, \hat{\beta})$ such that $\beta_r = \hat{\beta}_r$ and $m_r = \hat{m}_r$ for all $r \leq k-2$; $\hat{\beta}_{k-1} = 1$ and $\hat{m}_{k-1} = \hat{m}_k = m_{k-1}$. Since $(\hat{m}, \hat{\beta}) \in \hat{B}_{k-1}$ we have

$$\begin{aligned} \delta &\leq L(m, \beta) - L(\hat{m}, \hat{\beta}) = (1 - \beta_{k-1})[u(m_k) - u(m_{k-1})] \\ &\leq -(1 - x)u(R) = -(1 - x)L_k^* + (1 - x)^2. \end{aligned}$$

It follows that $-(1-x)L_k^* + (1-x)^2 > \delta$ and therefore $x < A$, where $A = 1 - \frac{L_k^*}{2} - \frac{1}{2}\sqrt{(L_k^*)^2 + 4\delta} < 1$, a contradiction. \square

A.5 Proof of Theorem 4

Let $E = (u, k, \pi, s)$ and let $E^* = (u, \delta, k, \pi, \phi, s)$. Let $W^*(\nu) = \sum_d V(d) \cdot \nu(d)$ and let $W_k^* = \sup_{M^*(\mathcal{D}_k)} W(\nu)$. We call $\nu \in M^*(\mathcal{D}_k)$ a solution to the planner's problem in the dynamic economy if $W^*(\nu) = W_k^*$. Let $W_k = \max_{M(\mathcal{C}_k)} U(\mu)$ be the value of the planner's problem for the static economy.

Consider the static economy $E^t = (u, s^t, \pi^t)$ where $\pi_h^t = \lambda_h$, $s_h^t = s_{\iota(h)}$ for all $h \in N^t$. Let W_k^t be the planner's value function for this economy. Note that $d \in \mathbb{R}^n \times \mathbb{R}^{n^2} \times \dots$. For any $\nu \in M^*(\mathcal{D}_k)$ let ν_t be the marginal of ν on \mathbb{R}^{n^t} .

Lemma 25. *The allocation ν is a solution to the planner's problem in the dynamic economy if and only if $W^t(\nu_t) = W_k$ for all t .*

Proof. If ν is a feasible allocation for the dynamic economy, then ν_t is a feasible allocation for E^t . By definition, $W^*(\nu) = \sum_{t \geq 1} (1 - \beta)\beta^{t-1} W^t(\nu_t)$. Then, since $F_{s^t} = F_s$, Lemma 6 implies that for any feasible ν , $W^t(\nu_t) \leq W_k$ for all t . Therefore, to conclude the proof, it suffices to show that $W_k^* \geq W_k$. Let μ be a solution to the planner's problem in the stationary economy, then the stationary allocation $T_3(\mu)$ is feasible for the dynamic economy and $W^*(T_3(\mu)) = W^t((T_3(\mu))_t) = W_k$ for all t . \square

Lemma 26. *Every solution to the planner's problem in the dynamic economy is stationary.*

Proof. Suppose ν is a nonstationary solution to the planner's problem. By Lemma 25, $W^t(\nu_t) = W_k$ and since $F_{s^t} = F_s$, we conclude that ν_t solves the planner's problem for E^t . It follows that ν_t is monotone and in particular, $\nu(d) > 0$ implies $d_h = d_{h'}$ for all $h, h' \in H_i^t$ and i, t . Then, since ν is not stationary, there must be a t, t', h, h', d, i such that $h \in N^t$ and $h' \in N^{t'}$ and $\iota(h) = \iota(h') = i$, $d_h \neq d_{h'}$ and $\nu(d) > 0$. Consider the economy $\hat{E} = (\hat{u}, \hat{s}, \hat{\pi})$ where $\hat{N} = N^t \cup N^{t'}$, $s_{\hat{h}} = s_{\iota(\hat{h})}$ for all $\hat{h} \in \hat{N}$ let $\hat{\pi}_{\hat{h}} = .5\lambda_{\hat{h}}$. Define the consumption plan \hat{c} for \hat{E} as follows $\hat{c}_{\hat{h}} = d_{\hat{h}}$ for all $\hat{h} \in \hat{N}$. Our choice of t, t', d ensures that \hat{c} fails monotonicity (ii).

Let \hat{W} be the planner's value function for the economy \hat{E} and note that since $F_{\hat{s}} = F_s$, by Lemma 6, $\hat{W} = W_k$. Let ν^2 be the marginal of ν on $N^t \times N^{t'}$ and note that ν^2 is an allocation for \hat{E} . Then, by Lemma 25, $\hat{W}(\nu^2) = .5W^t(\nu_t) + .5W^{t'}(\nu_{t'}) = W_k$. Therefore ν^2 solves the planner's problem in \hat{E} . But ν^2 fails monotonicity since $\nu^2(m) > 0$, contradicting Lemma 1. \square

Let $E = (u, s, \pi)$, $\hat{E} = (u, \hat{s}, \hat{\pi})$ be two static economies. We say that \hat{E} is noisier than E if there is a function $g : \hat{N} \rightarrow N$ such that $s_j = s_i$ whenever $g(j) = i$ and $\sum_{j:g(j)=i} \hat{\pi}_j = \pi_i$ for all i . We call such a g a *homomorphism*. The homomorphism g is rational (uniform) if $i = g(j)$ implies $\frac{\hat{\pi}_j}{\pi_i}$ is a rational number ($g(j) = g(j')$ implies $\hat{\pi}_j = \hat{\pi}_{j'}$). Clearly, if g is uniform, then it is rational. When \hat{E} is noisier than E , we write $[\hat{E}||E]$; if there exists a rational (uniform) homomorphism from \hat{E} to E , then we say $[\hat{E}||E]$ is rational (uniform).

For any consumption c , price p and allocation $\mu = (\mathbf{a}, \mathbf{c})$ in E , define the corresponding consumption $\hat{c} = \theta_1(c)$, price $\hat{p} = \theta_2(p)$ and allocation $\hat{\mu} = (\mathbf{a}, \hat{\mathbf{c}})$ for \hat{E} as follows: $\hat{c}_j = c_{g(j)}$ and $\hat{p}_j = \frac{\hat{\pi}_j p_{g(j)}}{\pi_{g(j)}}$ for all j and $\hat{c}^l = \theta_1(c^l)$ for all l and let $\Theta^g = (\theta_1, \theta_2, \theta_3)$. Also, let $D(p)$ be the set of solutions to a households utility maximization problems at price p in the economy E . We use \hat{D} , \bar{D} etc. for \hat{E} , \bar{E} etc.

Lemma 27. *If $[\hat{E}||E]$ is rational and $c \in D(p)$, then $\theta_1(c) \in \hat{D}(\theta_2(p))$.*

Proof. The following assertions are easy to verify: (1) $U(c) = \hat{U}(\theta_1(c))$ and (2) $c \in B(p)$ implies $\theta_1(c) \in \hat{B}(\hat{p})$.

Since $[\hat{E}||E]$ is rational, there exists \bar{E} such that $[\bar{E}||\hat{E}]$ and $[\bar{E}||E]$ are uniform. Let g be a $[\hat{E}||E]$ -homomorphism and $\Theta^g = (\theta_1, \theta_2, \theta_3)$; let \hat{g} be a uniform $[\bar{E}||\hat{E}]$ -homomorphism and $\Theta^{\hat{g}} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$. Then, $\bar{g} = g \circ \hat{g}$ is a uniform $[\bar{E}||E]$ -homomorphism and $\Theta^{\bar{g}} = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$, $\bar{\theta}_l = \hat{\theta}_l \circ \theta_l$ for $l = 1, 2, 3$. The two assertions above imply that we are done if we can show $\bar{c} \in D(\bar{\theta}_2(p))$ implies $\bar{c} = \bar{\theta}_1(c)$ for some $c \in D(p)$.

Suppose not and assume without loss of generality that for some $\bar{c} \in \bar{D}(\bar{\theta}_2(p))$, $\bar{c}_1 \neq \bar{c}_2$ despite $\bar{g}(1) = \bar{g}(2)$; that is, despite $\bar{p}_1 = \bar{p}_2$. Consider the endowment \tilde{s} such that $\tilde{s}_1 = \tilde{s}_2 = (\bar{c}_1 + \bar{c}_2)/2$ and $\tilde{s}_i = \bar{c}_i$ for all $i > 2$. Let $c'_1 = c_2$ and $c'_2 = c_1$, $c'_i = \hat{c}_i$ for all $i > 2$. Note that $\bar{U}(c') = \bar{U}(\bar{c})$ and therefore $c' \in \bar{D}(\bar{p})$. Hence, $\mu = .5\delta_{\bar{c}} + .5\delta_{c'}$ is a BCE for the economy $(u, \bar{\pi}, \tilde{s})$.

Therefore, μ is a solution to the planner's problem (by Theorem 1). But μ is not monotone, contradicting Lemma 1. \square

Lemma 28. *For any E, \hat{E} such that $[\hat{E}||E]$, $c \in D(p)$ implies $\theta_1(c) \in \hat{D}(\theta_2(p))$.*

Proof. Let $\hat{E} = (u, \hat{s}, \hat{\pi})$ where $\hat{\pi} \in \Delta(\hat{N})$ and let g be the $[\hat{E}||E]$ -homomorphism. We can construct a sequence $\hat{\pi}^m$ converging to $\hat{\pi}$ such that for all j , $\frac{\hat{\pi}_j}{\hat{\pi}_{g(j)}}$ is a rational number. Then, g is a rational homomorphism from $E^m = (u, \hat{s}, \hat{\pi}^m)$ to E . Let $c \in D(p)$ and $\hat{c} = \theta_1(c)$, $\hat{p} = \theta_2(p)$. Then, by Lemma 27,

$$\sum_{i \in \hat{N}} u(\hat{c}_i) \cdot \hat{\pi}_i^m \geq \sum_{i \in \hat{N}} u(c'_i) \cdot \hat{\pi}_i^m$$

for all $c' \in B(\hat{p})$ and therefore $\sum_{i \in \hat{N}} u(\hat{c}_i) \hat{\pi}_i \geq \sum_{i \in \hat{N}} u(c'_i) \hat{\pi}_i$ for all $c' \in B(\hat{p})$ as desired. \square

Assume (p, μ) is a BCE of E but $(T_2(p), T_3(\mu))$ is not a BCE of E^* . Hence, there exists $d \in B^*(T_2(p))$ such that $V(d) > W(\mu) = U(c)$ for $c \in D(p)$. Let $X = \{d_h : h \in H\}$ and for all $i \in N$, $x \in X$, let $H_i^t(x) = \{h \in H_i^t : d_h = x\}$ and

$$\chi_{ix} = \sum_{t \geq 1} \sum_{h \in H_i^t(x)} (1 - \beta) \beta^{t-1} \lambda_h.$$

Let $\hat{N} = N \times X$, $\hat{\pi}_{ix} = \chi_{ix}$ for all $i \in N$ and $x \in X$, $\hat{s}_{ix} = s_i$ for all i . Define $g(ix) = i$ and note it is a $[\hat{E}||E]$ -homomorphism. Therefore, by Lemma 28, $\theta_1(c) \in \hat{D}(\theta_2(p))$ whenever $c \in D(p)$. Define $\hat{c}_{ix} = x$ for all $ix \in \hat{N}$ and note that $U(\hat{c}) = V(d) > U(c) = \hat{U}(\theta_1(c))$ for any $c \in D(p)$, contradicting the fact that $\theta_1(c) \in D(\theta_2(p))$.

Finally, let (q, ν) be any BCE for E^* . Lemma 25 implies that there is a fair solution to the planner's problem. Standard arguments ensure that ν must be Pareto-efficient; since all households have the same endowment ν must also be fair. It follows that ν must solve the planner's problem. Then, Lemma 26 establishes that ν is stationary. If $T_3(\nu)$ is not a BCE allocation for E , then there exists an allocation μ for E such that $W(\mu) > W(T_3^{-1}(\nu))$, which implies that $W^*(T_3(\mu)) > W(\nu)$, so ν is not a solution to the planner's problem, a contradiction. \square

A.6 Proof of Corollary 1

Proof. Since there exist $0 < A < B < \infty$ such that $A \leq z_i^n \leq B$ for all i and n , we have

$$A \leq \liminf_n \sum p^n z^n \leq \limsup_n \sum p^n z^n \leq B.$$

Thus,

$$\frac{1}{\beta B} \frac{P(x)}{x} \leq \liminf \bar{Q}^n(x, z^n)$$

and

$$0 \leq \limsup \underline{Q}^n(x, z^n) \leq \frac{1}{\beta A} \frac{1 - P(1 - x)}{x}.$$

By Theorem 3, P has heavy high tails, i.e., $P(0) = 0$. Also, P is a continuous function; hence $P(x)/x \rightarrow \infty$, establishing the first conclusion of the Corollary. Likewise, $\lim_{x \rightarrow 0} \underline{Q}^n(x, z^n) = \chi(1 - P(1 - x))/x$. By Theorem 3, P has extreme lows, i.e., $\lim_{z \rightarrow 1} \hat{p}(z) = 0$. Thus, by l'Hopital's rule, $\lim_{x \rightarrow 0} \frac{1 - P(1 - x)}{x} = \lim_{x \rightarrow 0} \hat{p}(1 - x) \rightarrow 0$, which establishes the second conclusion of the Corollary. \square

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