Calibrated Uncertainty†

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Abstract

We define a binary relation (*qualitative uncertainty assessment*) that describes the shared likelihood assessments of decision makers with diverse ambiguity attitudes. Ambiguity renders this binary relation incomplete. Our axioms yield a representation according to which A is more likely than B if an only if a capacity, called *uncertainty measure*, assigns a higher value to A than to B and a higher value to B-complement than to A-complement. We modify Machina and Schmeidler's (1992) sophistication axiom to allow for ambiguity and show how our model can be extended to a full fledged model of choice under uncertainty.

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1. Introduction

A standard assumption in applied and theoretical information economics is that agents share a common prior but differ in their risk attitudes. The common prior describes the agents' common uncertainty perception while utility indices measure their idiosyncratic risk attitudes. The purpose of this paper is to provide an analogous separation for situations in which the agents may perceive ambiguity.

Subjective expected utility theory identifies uncertainty perception with betting behavior; that is, if a group of agents shares a common uncertainty perception, then the group deems event A more likely than event B if every agent would rather bet on A than on B. The resulting binary relation, defined on a collection of events, is a *qualitative probability* and Savage's theorem (Savage (1972)) gives conditions under which a probability represents it. Thus, Savage's theorem yields a cardinal measure of likelihood, a probability, that can be combined with different individual risk postures or even non-expected utility theories into a full-fledged theory of behavior.

One consequence of ambiguity is that the measure of uncertainty perception cannot be a probability. In the presence of ambiguity, an agent may be indifferent between betting on A or betting on A^c (the complement of A), indifferent between betting on B or betting on B^c, but may nevertheless prefer betting on A to betting on B. A second consequence of ambiguity is that even when all agents perceive the same uncertainty (and ambiguity), their betting preferences may differ. For example, it may be that ambiguity loving agents prefer betting on either A or A^c to betting on either B or B^c while ambiguity averse agents have the opposite preferences. Thus, a given uncertainty perception must be compatible with a range of betting behaviors to accommodate different ambiguity attitudes.

To deal with these issues, we modify Savage's model as follows: a binary relation describes the agents' likelihood assessments and $A \succeq B$ means that *all agents* in the population prefer to bet on A rather than B. Since agents may perceive some events as ambiguous and differ in their ambiguity attitude, the binary relation, $\succeq$, is incomplete.[1] We call this binary relation a *qualitative uncertainty assessment (QUA)* and interpret it as

[1] The idea to use an incomplete relation to describe consensus judgements is due to Gilboa, Maccheroni, Marinacci and Schmeidler (2010).

reflecting those comparisons that can be made for all ambiguity attitudes. Conversely, we interpret instances of incompleteness as reflecting situations in which an agent's ambiguity attitude affects her ranking of two bets. In Savage's model, we can elicit the qualitative probability from the betting behavior of a single agent. To elicit a QUA, we must observe the betting behavior of a population of agents that share a common uncertainty perception and exhibit a full range of ambiguity attitudes. Thus, the elicitation of a QUA imposes a greater burden on the available choice data than the Savage model. This added burden is unavoidable if we wish to separate the agents' common uncertainty perception from their uncertainty attitudes.

As an illustration, consider members of a PhD admissions committee who must choose candidates from a pool of applicants. Committee members base their beliefs about the relative strengths and weaknesses of the candidates on the same sources of information and, therefore, it is plausible that they share a common uncertainty perception. Moreover, some information sources may be ambiguous, leading to a shared perception of ambiguity. For example, suppose no candidate similar to candidate b was previously admitted and, therefore, committee members perceive ambiguity about b's prospects. In contrast, committee members agree that candidate a will succeed with probability .5 because many candidates with characteristics similar to candidate a were admitted in the past and 50% of those candidates succeeded. In this situation, ambiguity loving committee members prefer a bet on $B := [b \text{ will succeed}]$ to a bet on $A := [a \text{ will succeed }]$ while ambiguity averse committee members have the reverse ranking. The committee's QUA records all those rankings of bets that are shared by all committee members. Since there is no agreement on the ranking of A and B, the resulting binary relation is incomplete.

The hypothesis of a shared uncertainty perception is appropriate when members of the group draw from the same information sources or share their private information, as in the example of a PhD admissions committee.[2] By contrast, consider the uncertainty perception regarding the economic consequences of a tax change among all voters of a country. Since voters draw from different information sources we would not expect a common uncertainty perception. In particular, tax-experts may perceive no ambiguity because for them the

[2] Members of the admissions committee typically have different utility functions, and thus, their shared uncertainty perception does not imply they agree on who to admit.

uncertainty is akin to a draw from an urn with a known composition while non-experts may perceive ambiguity because for them the uncertainty resembles a draw from an urn with unknown composition. This issue does not arise when all members of the group share the same understanding of all information sources.

We assume that a subclass of events are unambiguous. For those events, ambiguity attitudes do not affect agents' betting behavior and, therefore, the qualitative uncertainty assessment is complete. Returning to the example of the admissions committee, assume that there is a collection of universities for which the committee has many examples of admitted students. All committee members assign the same probability to the event that a particular student from one of those universities will succeed. Bets on the success of any of these candidates are unambiguous. Suppose E^1 and E^2 are unambiguous events but A is not and $E^1 \succeq A \succeq E^2$. In that case, even the most ambiguity averse agent prefers a bet on A to a bet on E^2 while even the most ambiguity loving agent prefers E^1 to A. Thus, in the example above, suppose that all committee members prefer to bet on candidate c, whose success probability is .6, over a bet on candidate b, whose success probability is ambiguous. Assume also that every committee member prefers a bet on candidate b to a bet on candidate d whose success probability is .4. By identifying a "tight" window of unambiguous events of this form, we determine the range of probabilities that A might have. Our first theorem (Theorem 1) is a representation theorem for QUAs; it provides axioms that ensure the existence of a special type of capacity (*uncertainty measure*) representing the QUA. The uncertainty measure has the feature that $\pi(A) + \pi(A^c) \leq 1$ where equality holds if and only if the event is unambiguous. Thus, if we define $\bar{\pi}(A) = 1 - \pi(A^c)$, then the uncertainty measure identifies the range of probabilities $[\pi(A), \bar{\pi}(A)]$ for every event A. The uncertainty measure π represents the QUA $\succeq$ if

$$A \succeq B \text{ if and only if } \begin{cases} \pi(A) \geq \pi(B) \\ \bar{\pi}(A) \geq \bar{\pi}(B) \end{cases}$$

If $\pi(A) > \pi(B)$ but $\bar{\pi}(B) > \bar{\pi}(A)$, then the underlying QUA cannot compare A and B. In that case, a sufficiently ambiguity averse agent prefers a bet on A to a bet on B while a sufficiently ambiguity loving agent has the reverse preference.

In the second part of the paper, we show how our model of uncertainty perception can be combined with different uncertainty attitudes to form a full fledged model of choice. First, we consider betting behavior; that is, a setting with two fixed prizes. Specifically, we consider agents who combine their QUA and their ambiguity attitudes to form complete (and transitive) rankings of bets. Our model captures agents' ambiguity attitudes with a function that assigns a *risk equivalent* to each pair probability range. The risk equivalent of $[\pi(A), \bar{\pi}(A)]$ is the probability of the unambiguous event E that renders the decision maker indifferent between betting on A and betting on E. In Proposition 3, we provide a representation theorem for this betting model and show that the uncertainty measure is uniquely identified. Thus, Proposition 3 shows that even in a restricted setting with two prizes, we can separate an individual agent's uncertainty perception, quantified by the uncertainty measure, and her uncertainty attitude, quantified by the risk-equivalent function. We take advantage of this separation to define and characterize a measure of comparative ambiguity aversion.

Machina and Schmeidler (1992) introduce the notion of *probabilistic sophistication* and provide a setting in which agents may share a common uncertainty perception yet have very different uncertainty attitudes. Hence, probabilistically sophisticated agents may or may not be expected utility maximizers. However, Machina and Schmeidler's strong comparative probability axiom rules out ambiguity. We provide a weakening of their axiom – weak sophistication – that is consistent with ambiguity but does not require decision makers to be subjective expected utility maximizers when comparing unambiguous acts.

We then provide three nested models that extend QUAs to preferences over all acts. The first model can be thought of as a minimal departure from expected utility theory because it retains the expected utility hypothesis over unambiguous acts. The second extension is a generalization of Choquet Expected Utility theory that facilitates easy identification of decision makers' ambiguity perception, ambiguity attitude and risk attitude. When restricted to unambiguous prospects, this model coincides with the Machina-Schmeidler model. The third extension is the most general model consistent with QUAs. The second and third extensions show that any non-expected utility theory can be extended to a rich model of choice over ambiguous prospects within our framework.

Our paper is related to Nehring (2009) who axiomatizes a set of priors in a Savage setting. Like our model, Nehring takes as a primitive an incomplete comparative likelihood relation. Rather than betting behavior, Nehring's incomplete relation is meant to capture likelihood comparisons that are *coherent*, a normative requirement. To this end, Nehring requires that the incomplete binary relation satisfy Savage's additivity axiom. By contrast, QUA's represent betting behavior, albeit those of a group, and need not satisfy additivity.

Our paper is also related to a literature that develops measures of uncertainty perception in the Anscombe-Aumann setting. Examples are the papers by Ghirardato, Maccheroni and Marinacci (2004), Siniscalchi (2006), Ghirardato and Siniscalchi (2012), and Klibanoff, Mukerji and Seo (2014). The Anscombe-Aumann model describes a dynamic setting with a potentially ambiguous event occurring in the first stage followed by an unambiguous lottery in the second stage. The uncertainty measures developed for this setting rely on the assumption that the agent is an expected utility maximizer in the second stage. In other words, these papers postulate expected utility maximizers for unambiguous prospects. By contrast, our model is agnostic about the agent's uncertainty attitude over unambiguous prospects and applies to settings in which ambiguous and unambiguous uncertainty resolves concurrently.

We borrow from Gilboa et al. (2010) the interpretation that an incomplete binary relation describes judgements that are commonly agreed upon by a group of individuals. In Gilboa et al. (2010), an incomplete relation describes choices over acts that are "objectively" rational; that is, agreed upon by the a group of agents. In our model, an incomplete relation describes choices of bets that are objectively rational in the same sense.

Our main result (Theorem 1) characterizes agents whose uncertainty perception is described by a totally monotone capacity (or belief function). Dempster (1967) and Shafer (1976) introduced belief functions to model ambiguity.[3] Zhang (2002) and Gul and Pesendorfer (2014) axiomatize ambiguity models that yield inner probabilities, a special case of belief functions, as measures of uncertainty perception. The uncertainty measures analyzed in this paper are belief functions but are more general than inner probabilities.

[3] Wong, Yao, Bollmann, and Burger (1991) provide an axiomatization for belief functions for a finite state space. A key difficulty with the finite setting is that the representation is not unique. For example, the representation theorem for belief functions in Wong et al. could be restated as a representation theorem for supermodular capacities.

Casadesus-Masanell et al. (2000) axiomatize maxmin expected utility preferences over Savage acts. Gilboa (1987), Wakker (1989), Nakamura (1990), Sarin and Wakker (1992), and Chew and Karni (1994) axiomatize Choquet expected utility over Savage acts.

2. Cardinal Measurement of Uncertainty

Throughout this paper, Ω is the state space and Σ is a σ-algebra of events. Henceforth, *all sets* considered are elements of Σ. A *set function* $(q, \mathcal{A})$ is a mapping from a sub-σ-algebra $\mathcal{A}$ of Σ to $[0,1]$ such that $q(\emptyset) = 0$ and $q(A) \leq q(B)$ whenever $A \subset B$. Given a set function $(q, \mathcal{A})$, the event $A \in \mathcal{A}$ is *null* if $q(A^c) = q(\Omega)$ and it is whole if $B \subset A$ implies $B \in \mathcal{A}$. The following terminology will be used throughout the paper:

Definition: *The set function $(q, \mathcal{A})$ is*

(i) a capacity if $q(\Omega) = 1$.

(ii) additive if $q(A \cup B) + q(A \cap B) = q(A) + q(B)$;

(iii) continuous if $A_{n+1} \subset A_n$ for all n implies $q\left(\bigcap A_n\right) = \lim q(A_n)$.

(iv) complete if every null set is whole.

(v) a probability measure if it is complete, continuous, additive, and $q(\Omega) = 1$.

(vi) a subprobability measure if it is complete, continuous, additive and $q(\Omega) < 1$.

(vii) nonatomic if, for all A such that $q(A) > 0$, there is $B \subset A$ such that $0 < q(B) < q(A)$.

It is not difficult to verify that an additive set function is continuous if and only if it is countably additive. Moreover, a countably additive, nonatomic set function is convex ranged; that is, $q(A) > 0$ and $r \in [0,1]$ implies there is $B \subset A$ such that $q(B) = rq(A)$.[4]

Next, we will define a class of capacities, *uncertainty measures*, that represent the uncertainty perception of agents under the axioms provided in section 3. An uncertainty measure has two components: a probability measure $(\mu, \mathcal{E})$ that quantifies the uncertainty of unambiguous events and a subprobability measure (η, Σ) that quantifies the uncertainty of ambiguous events.

Definition: *A capacity π is an uncertainty measure if there exist a nonatomic probability measure $(\mu, \mathcal{E})$ and a non-atomic subprobability measure (η, Σ) such that*

(i) $\pi(A) = \max_{E \in [A]}\{\mu(E) + \eta(A \backslash E)\}$ where $[A] = \{E \in \mathcal{E}, E \subset A\}$

[4] See Billingsley (1995), p. 35.

(ii) $\mu(E) > \eta(E)$ *for every* μ*-nonnull* $E \in \mathcal{E}$*, and* $\eta(E) = 0$ *for every* μ*-whole* $E \in \mathcal{E}$*.*

We refer to $(\mu, \mathcal{E})$ in the above definition as the *risk measure* and to (η, Σ) as the *ambiguity measure*. When $(\mu, \mathcal{E})$ and (η, Σ) satisfy (2) above, we say the two measures are *compatible*. Notice that the risk measure is defined on $\mathcal{E}$, the sub-sigma-algebra of unambiguous events, while η is defined for all events in Σ.

Example 1: Let $S = [0,1]^2$ be the state space and let Σ be the Borel sets of S. Let λ denote the Lebesgue measure on Σ. Assume that the unambiguous events correspond to realizations of the first component of the state. More precisely, let $\mathcal{B}^1$ be the Borel sets of $[0,1]$, define $\mathcal{E}^0 = \{A \times [0,1] \mid A \in \mathcal{B}^1\}$ and let $\mathcal{E} = \{B \in \Sigma \mid \lambda(B \Delta B^0) = 0 \text{ for some } B^0 \in \mathcal{E}^0\}$ be the unambiguous events ($X \Delta Y$ is the symmetric difference of X and Y). Then, $\pi : \Sigma \to [0,1]$ such that

$$\pi(B) = \max_{\{E \subset B, E \in \mathcal{E}\}} \lambda(E) + \lambda(B \backslash E)/2$$

is an example of an uncertainty measure. In this case, the risk measure μ is Lebesgue measure on $\mathcal{E}$, while the ambiguity measure η is Lebesgue measure on Σ divided by 2. The risk and ambiguity measures are compatible since $E \in \mathcal{E}$ is $\mu-$whole if and only if $\mu(E) = 0$ and, therefore, $\eta(E) = 0$ for all μ-whole events.

In general, $\mu-$whole events are the completely unambiguous events; that is, all subsets of these events are unambiguous. For those events ambiguity plays no role and, thus, the ambiguity measure is zero. In Example 1 above, only null events are $\mu-$whole. If E is not $\mu-$whole, then some subsets of E are not $\mathcal{E}$-measurable. That is, there are $A, B \subset E$ such that $A \cup B = E$ but neither A nor B is unambiguous. For these events, the ambiguity measure (η, Σ) places bounds on the probabilities of A and B. For example, consider the unambiguous event $E = [0,1/2] \times [0,1]$. The events $A = [0,1/2] \times [0,1/2]$ and $B = [0,1/2] \times (1/2,1]$ partition E into two ambiguous events. Since $\eta(A) = \eta(B) = 1/8$, the ambiguity measure bounds the probabilities of A and B to be at least $1/8$. Since $\mu(E) = 1/2$, it follows that an upper bound for the probability of A is $1/2 - \eta(B) = 3/8$. Thus, $[1/8, 3/8]$ is the probability range for the event A.

Clearly, every compatible pair of risk and ambiguity measures give rise to a unique uncertainty measure via (1) in the above definition. Part (i) of Proposition 1, below, shows that the converse is true as well. Every uncertainty measure π can be uniquely decomposed into risk and ambiguity measures. Moreover, the σ-algebra of the risk measure coincides with the set of $\pi-$*unambiguous events*; that is, those events for which $\pi(A)+\pi(A^c)=1$. We let $\mathcal{E}_\pi$ denote the collection of all π-unambiguous sets. Finally, part (iii) shows that every uncertainty measure is a totally monotone capacity.[5] A capacity q is totally monotone if $q\left(\bigcup_{i\in N} A_i\right) \geq \sum_{I\subset N, I\neq\emptyset}(-1)^{|I|+1} q\left(\bigcap_I A_i\right)$ for all positive integers n and for any collection of sets $A_1, \ldots, A_n$.

Proposition 1: *(i) If $(\mu,\mathcal{E})$ is a risk measure for the uncertainty measure π, then $\mathcal{E}=\mathcal{E}_\pi$; (ii) every uncertainty measure has a unique risk measure and a unique ambiguity measure; (iii) every uncertainty measure is a belief function.*

In the following section, we consider a binary relation $\succeq$ on Σ. We interpret $A\in\Sigma$ as a bet; one that the decision-maker wins if and only if a state in A occurs. The ranking $A\succeq B$ means that the decision maker prefers to bet on A rather than B *regardless of her ambiguity attitude*. As we noted in the introduction, the binary relation $\succeq$ is observable through the betting behavior of a group of agents who have identical uncertainty perception and exhibit a full range of ambiguity attitudes. In that case, $A\succeq B$ reveals that all agents in the group prefer to bet on A rather than B. If agents disagree, then A and B are not comparable and, thus, the binary relation $\succeq$ may be incomplete. This incompleteness is resolved once the decision maker's ambiguity attitude is taken into account.

Since every uncertainty measure π is a belief function, it follows that $\pi(A)+\pi(A^c)\leq 1$. Define $\bar{\pi}(A)=1-\pi(A^c)$ and note that $\bar{\pi}(A)\geq\pi(A)$. The interval $[\pi(A),\bar{\pi}(A)]$ is the *probability range* of A. We interpret the probability range as a measure of ambiguity. Thus, $[\pi(A),\bar{\pi}(A)]\subset[\pi(B),\bar{\pi}(B)]$ means B is more ambiguous than A.

In the next section, we impose axioms on the binary relation $\succeq$ that yield the following representation:

$$A\succeq B \text{ if and only if } \begin{cases} \pi(A)\geq\pi(B) & \text{and} \\ \bar{\pi}(A)\geq\bar{\pi}(B) & \end{cases}$$

Thus, A is more likely than B if the lowest possible probability of A is higher than the lowest possible probability of B and the highest possible probability of A is higher than

[5] In Dempster-Shafer theory, a totally monotone capacity is called a belief function.

the highest possible probability of B. As an illustration, consider again Example 1, above. Let $A = [0, a] \times [0, b]$ be an ambiguous event and let $E = [0, c] \times [0, 1]$ be an unambiguous event. The uncertainty measure defined in that example yields the following:

$$E \succeq A \Leftrightarrow \; c \geq \frac{a(1+b)}{2}$$
$$A \succeq E \Leftrightarrow \; \frac{ab}{2} \geq c$$

Thus, if $a(1+b)/2 > c > ab/2$, then the two events cannot be ranked. In that case, agents who are sufficiently ambiguity loving prefer to bet on A rather than E while sufficiently ambiguity averse agents reverse this ranking.

3. Qualitative Uncertainty Assessments

In Ellsberg's thought experiments, the wording of the choice problem makes it clear which class of events are unambiguous and which ones are not. Events that correspond to draws from an urn with a known composition are unambiguous while events that are not are ambiguous. In our model, we do not assume that the distinction between ambiguous events and unambiguous events is self-evident or exogenous. Instead, we provide a criterion for identifying unambiguous events and calibrate the ambiguous events with them.

Implicitly or explicitly, there are at least two different notions of an unambiguous event in the literature. Some formal models and empirical/experimental work accept what we might call a "relative" notion of unambiguous events. In this approach, events are unambiguous if they belong to a collection of sets for which the more likely relationship is preserved whenever two unambiguous sets A, B are combined with a third disjoint unambiguous set C; that is, $A \succeq B$ if and only if $A \cup C \succeq B \cup C$ whenever $A \cap C = B \cap C = \emptyset$.[6] Other formulations assume (or imply) what we might call an "absolute" test for ambiguity, or lack thereof. Here, an event E is deemed unambiguous if the more likely relationship is preserved when *any* A, B such that $(A \cup B) \cap E = \emptyset$ is combined with E.[7] The first approach yields multiple collections of unambiguous events. Since our goal is to identify a

[6] This view is implicit in the notion of a source as in defined in Fox and Tversky (1995) and studied in Gul and Pesendorfer (2014).

[7] See for example, Epstein and Zhang's (2001) notion of an unambiguous event and Gul and Pesendorfer's (2014) notion of an ideal event.

unique collection of unambiguous events and use them to calibrate the uncertainty of all events, we take the second approach.

Definition: *An event E is unambiguous if $A \succeq B$ if and only if $A \cup E \succeq B \cup E$ whenever $(A \cup B) \cap E = \emptyset$.*

The set $\mathcal{E}$ denotes the set of all unambiguous events. In the following, whenever an event is referred to as E, F or G it is understood that the event is unambiguous and, therefore, we will write E, F, G instead of $E, F, G \in \mathcal{E}$. A generic event A, B, C or D is understood to be an element of Σ, the underlying sigma-algebra of events, and, therefore, we will write A, B, C, D instead of $A, B, C, D \in \Sigma$.

Our goal is to provide a version of Savage's Theorem for decision makers who perceive ambiguity. To illustrate our model and the axioms, we will refer to the following stylized version of an Ellsberg-type thought experiment: a ball will be chosen from an urn that has n balls, numbered from 1 to n. For each integer $i \in N := \{1, \ldots, n\}$, there is exactly one ball with number i. Each ball has one of k colors; that is, each ball i has some color $k \in K := \{c_1, \ldots, c_k\}$ and, therefore, the state space is $\Omega = N \times K$. Let $A_j \subset \Omega$ denote the event that a ball of color c_j is drawn and E_i denote the event that ball number i is drawn. Nothing is known about the composition of colors in the urn or about the color of any particular ball i. Let $A^j = \bigcup_{t=1}^{j} A_t$ and $E^i = \bigcup_{t=1}^{i} E_t$. Thus, E^i denotes the (unambiguous) event that one of the first i balls will be drawn, whereas A^j denotes the (ambiguous) event that the drawn ball will have one of the colors in $\{c_1, \ldots, c_j\}$. Below, we refer to an act that yields 1 if $\omega \in B$ and 0 if $\omega \in B^c$ as a bet on B.

For now, suppose that $n = k = 10$. How would we expect decision makers to rank a bet on A^j and a bet on E^i? If the agent is ambiguity averse, she might prefer E^5 to A^5; if she is ambiguity loving she may prefer A^5 to E^5. Even when comparing two ambiguity averse agents, it might be that one prefers a bet on E^5 to a bet on A^5 but would rather bet on A^5 than on E^4 while the other is more ambiguity averse and even prefers E^4 to A^5. Thus, simple rankings of bets by a single individual do not enable us to distinguish ambiguity perception from attitude towards ambiguity.

To overcome this difficulty, we take an approach analogous to that of Gilboa, Maccheroni, Marinacci and Schmeidler (2010) and interpret $\succeq$ as the common qualitative uncertainty assessment of a group of individuals. Thus, $E^7 \succeq A^5 \succeq E^3$ means that even the most ambiguity averse person in this group prefers A^5 to E^3 while even the most ambiguity loving member of the group prefers E^7 to A^5. By identifying a "tight" window of unambiguous events of this form, we determine the range of probabilities that A^5 might have. Specifically, assume that $E^7 \succeq A^5 \succeq E^3$ and $A^5 \not\succeq E^4$ and $E^6 \not\succeq A^5$. Thus, we conclude that the range of probabilities for A^5 is $[.3, .7]$ and, therefore, the comparison between A^5 and E^5 is indeterminate; i.e., $A^5 \not\succeq E^5$ and $E^5 \not\succeq A^5$. This indeterminacy does not reflect the incompleteness of a single agent's preferences; rather it reflects the fact that the group's common uncertainty perception by itself is not sufficient to rank the two bets E^5 and A^5.

Our main hypothesis, *calibration*, imposes two consistency properties on the qualitative uncertainty assessment. The first of these is the complements property: $A \succeq B$ implies $B^c \succeq A^c$. Hence, if all decision makers with a common perception of ambiguity agree that A is a better bet than B irrespective of their attitude towards ambiguity, then they must also agree that B^c is a better bet than A^c. To see why this is reasonable, suppose for example that the range of probabilities for A is $[a, a^*]$ and the range of probabilities for B is $[b, b^*]$. Then, the range of probabilities for A^c is $[1 - a^*, 1 - a]$ and for B^c, the range is $[1 - b^*, 1 - b]$. If a bet on A is preferred to a bet on B for all ambiguity attitudes, we must have $a^* \geq b^*$ and $a \geq b$. But then it follows that $1 - b \geq 1 - a$ and $1 - b^* \geq 1 - a^*$ and, therefore, a bet on B^c is preferred to a bet on A^c for all ambiguity attitudes. Of course, it may be the case that A and B are not comparable, for example, if $a > b$ but $b^* > a^*$. In that case, an ambiguity averse decision maker may prefer both a bet on A to a bet on B and a bet on A^c to a bet on B^c. However, this particular pair of preferences emerges only if we "complete" the QUA by considering the decision maker's attitude to ambiguity (as we do in the next section).

To understand the second consistency requirement, *range dependence*, consider two events A and B such that their range of probabilities, as defined above, are the same. Range dependence implies that all agents in the group are indifferent between betting on A and betting on B. For agents who are maximally ambiguity averse (or maximally ambiguity loving) this assumption requires no justification since for those agents only the lower (upper) bound affects betting behavior. More generally, this assumption means that the range of probabilities associated with an event captures all that is relevant for betting behavior. The main idea of this paper is to investigate the extent to which the ambiguity of an arbitrary set can be calibrated; i.e., quantified with unambiguous events. Range dependence makes this idea formal.

Definition: *Let $W(A) = \{E \mid A \succeq E\}$ and $S(A) = \{E \mid E \succeq A\}$. Then, $\succeq$ satisfies range dependence if*

$$A \succeq B \text{ if and only if } \begin{cases} W(B) \subset W(A) & \text{and} \\ S(A) \subset S(B) \end{cases}$$

Our main hypothesis, *calibration*, combines range dependence with the complements property and ensures that this range of probabilities is sufficient to fully describe the decision maker's betting behavior. In particular, shifting the range upward makes the bet more attractive and shifting downward makes it less attractive. It is easy to verify that calibration, defined below, is equivalent to the conjunction of the complements property and range dependence.

Definition: *The binary relation $\succeq$ is calibrated if*

$$A \succeq B \text{ if and only if } \begin{cases} W(B) \subset W(A) & \text{and} \\ W(A^c) \subset W(B^c) \end{cases}$$

We write $A \succ B$ when both of the calibration inclusions are strict.[8]

Definition: *An event A is null if $B \succeq B \cup A$ for all B.*

The binary relation is *monotone* if $A \subset B$ implies $B \succeq A$ and $B \succ A$ if, in addition, $B \backslash A$ is not null. The binary relation $\succeq$ is *non-degenerate* if Ω is not null.

Axiom 1: *The binary relation $\succeq$ is monotone, non-degenerate and calibrated.*

Clearly, a calibrated binary relation must be transitive. As we noted above, $\succeq$ may be incomplete, reflecting a dependence of the decision maker's betting behavior on her

[8] Note that the strict relation $\succ$ is defined independently of $\succeq$. We show in Lemma B1(i) that $A \succ B$ if and only if $A \succeq B$ and $B \not\succeq A$. Thus, the strict preference derived from $\succeq$ coincides with $\succ$.

ambiguity attitude. However, in some cases the ranking of bets is independent of the agents' ambiguity attitude. One such case is the comparison of bets on unambiguous events. A second case is the comparison of 'color events' in our stylized version of the Ellsberg thought experiment: the comparison of $A = A_1 \cup A_2 \cup A_3$ and $B = A_4 \cup A_5$ is independent of the agents' ambiguity attitude since both A and B are equally ambiguous. More generally, we assume two events are comparable if they preserve the same collection of unambiguous events. For any event A, let

$$\langle A \rangle = \{E \,|\, E \cap A \in \mathcal{E}, E \cap A^c \in \mathcal{E}\}$$

denote the sets of unambiguous events that A preserves.

Definition: *A, B are similar ($A \cong B$) if $\langle A \rangle = \langle B \rangle$.*

When two *arbitrary* events, A, B are similar, Axiom 2(i), below, requires that the decision maker be able to compare them solely based on her perception of ambiguity. The second part of Axiom 2 requires *unambiguous* events, E, F, to be similar. The two parts together ensure that the restriction of $\succeq$ to the set of unambiguous events is complete. A second consequence of Axiom 2 is that the set of unambiguous events is closed under unions and complements; that is, it is an algebra.

Axiom 2: *(i) $A \cong B$ implies $A \succeq B$ or $B \succeq A$. (ii) $E \cong F$.*

The familiar additivity axiom of qualitative probability requires that $A \succeq B$ if and only if $A \cup C \succeq B \cup C$ whenever A, C and B, C are both disjoint. Axiom 3 below weakens the additivity requirement of qualitative probability to permit ambiguity.

To see why we need a weaker form of additivity, consider again our Ellsberg experiment. Assume the decision maker finds the events $A = A_1 \cap E_1$ ("ball 1 will be drawn and it will be red") and $B = A_1 \cap E_2$ ("ball 2 will be drawn and it will be red") equally likely. Consider the event $C = E_1 \backslash A_1$ ("ball 1 will be drawn and it will not be red.") Then $A \cup C = E_1$ ("ball one is drawn") is an unambiguous event while $B \cup C$ is not. Therefore, ambiguity averse agents might prefer $A \cup C$ to $B \cup C$ while an ambiguity loving agent might have the opposite preference. Such a situation can arise when $A \cup C$ is unambiguous while A and C are not. This situation cannot arise if $A \cup C$ is similar to A or if A and

C are contained in disjoint unambiguous events, that is $A \subset E, C \subset E^c$ for some $E \in \mathcal{E}$. This motivates the following definition:

Definition: *The pair (A, B) conforms if $A \cup B \cong A$ or if $A \subset E, B \subset E^c$ for some $E \in \mathcal{E}$.*

Ellsberg-style thought experiments suggest violations of additivity do not occur when A, C and B, C are conforming pairs. For example, suppose $A = (E_1 \cup E_2) \cap A_1$ ("ball 1 or 2 will be drawn and it will be red") while $B = E_1 \cap (A_1 \cup A_2)$ ("ball 1 will be drawn and it will be red or green.") Notice that these two events are not similar and, therefore, they need not be ranked. However, if there is agreement and, for example, all decision makers agree that A and B are equally likely, then we require that they consider $A \cup C$ and $B \cup C$ equally likely as well, where $C = E_3 \cap A_2$ ("ball 3 will be drawn and it is green.")

Axiom 3: *If (A, C) and (B, C) conform and $A \cap C = B \cap C = \emptyset$, then $A \succeq B$ if and only if $A \cup C \succeq B \cup C$.*

Our axioms allow but do not require the decision-maker to perceive ambiguity. However, we do impose the existence of a rich set of unambiguous events. This assumption is implicit in any uncertainty model stated in the Anscombe-Aumann framework and in Savage's formulation of subjective expected utility. We also assume that every event, unambiguous or not, can be divided into finer, similar events. The following version of Savage's fineness assumption ensures that this is so.

Axiom 4: *$A \succ B$ implies there is a partition $C_1, \ldots, C_k$ such that $C_n \backslash B \cong B$ and $A \succ B \cup C_n$ for all n.*

The following axiom is the counterpart of the monotone continuity axiom from the Savage framework. It ensures that unambiguous events are closed under countable (not just finite) unions. It also ensures countably additivity of the risk measure.

Axiom 5: *$A_{n+1} \subset A_n$, $A_n \succeq B$ for all n and either $B \in \mathcal{E}$ or $A_n \backslash A_{n+1} \in \mathcal{E}$ for all n, implies $\bigcap A_n \succeq B$.*

We call a binary relation that satisfies the five axioms above a *qualitative uncertainty assessment* (QUA). Let $\succeq$ be any binary relation on Σ. A capacity π represents $\succeq$ if

$$A \succeq B \text{ if and only if } \begin{cases} \pi(A) \geq \pi(B) & \text{and} \\ \bar{\pi}(A) \geq \bar{\pi}(B) \end{cases}$$

where $\bar{\pi}(C) = 1 - \pi(C^c)$.

Theorem 1: *A binary relation $\succeq$ on Σ can be represented by a nonatomic uncertainty measure if and only if it is a qualitative uncertainty assessment.*

Theorem 1 formalizes our notion of separating ambiguity attitude from ambiguity perception. It establishes that every QUA can be represented by a nonatomic uncertainty measure. The proposition below establishes that the uncertainty measure that represents the QUA is unique.

Proposition 2: *If π with risk measure $(\mu, \mathcal{E})$ and ambiguity measure (η, Σ) and $\hat{\pi}$ with risk measure $(\hat{\mu}, \hat{\mathcal{E}})$ and ambiguity measure $(\hat{\eta}, \Sigma)$ both represent $\succeq$, then $(\mu, \mathcal{E}) = (\hat{\mu}, \hat{\mathcal{E}})$, $\eta = \hat{\eta}$ and hence $\pi = \hat{\pi}$.*

4. Betting Behavior and Uncertainty Attitude

In the previous section, we assumed that the observed primitive is the betting behavior of a group of individuals with a broad range of ambiguity attitudes. From this primitive we derived the common uncertainty perception of the group. In this section, we analyze extensions of this common uncertainty perception to complete and transitive preferences over bets; that is, we consider an individual who combines her uncertainty perception and her uncertainty attitude to form a complete ranking of bets.

Recall that an uncertainty measure π assigns to each event A an interval, $[\pi(A), \bar{\pi}(A)]$ (where $\bar{\pi}(A) = 1 - \pi(A^c)$), of probabilities called the probability range which depicts the decision maker's ambiguity *perception.* The function ρ assigns a value in $[a, b]$ to each such interval $[a, b]$ and describes the agent's ambiguity *attitude.* Values close to the lower bound of the range indicate ambiguity aversion while values close to the upper bound indicate ambiguity loving. The *risk equivalent* of any bet A with probability range $[a, b]$ is the probability $\rho(a, b)$ such that the decision maker is indifferent between betting on A and betting on any unambiguous event E such $\pi(E) = \rho(a, b)$.

For $I \subset [0, 1]^2$, we say that the function $\rho : I \to I\!R$ is *strict* if $(a, b) \geq (a', b'), (a, b) \neq (a', b')$ implies $\rho(a, b) > \rho(a', b')$. Thus, a strict ρ increases if either of the two probability bounds increases. The function ρ is *maximally ambiguity averse (loving)* if $\rho(a, b) = a$

($\rho(a,b) = b$) for all $(a,b) \in I$. Maximally ambiguity averse functions depend only on the lower bound and maximally ambiguity loving functions depend only on the upper bound.

Let π be the unique uncertainty measure that represents the QUA $\succeq$ and let $I = \{(\pi(A), \bar{\pi}(A)) : A \in \Sigma\}$.

Definition: *The function $\rho : I \to I\!R$ is a risk equivalent function for $\succeq$ if (i) ρ is continuous; (ii) ρ is either strict, maximally ambiguity averse, or maximally ambiguity loving; (iii) $a \leq \rho(a,b) \leq b$.*

Not included in the set risk equivalent functions are functions that are sometimes maximally ambiguity loving and sometimes maximally ambiguity averse or functions that remain constant unless both bounds increase. This restriction notwithstanding, the set permits a wide range of ambiguity attitudes.

In Proposition 3, below, we characterize betting behavior that can be represented by an uncertainty measure and a risk equivalent function ρ. The pair (π, ρ) represent the betting preference $\succeq^o$ if

$$A \succeq^o B \text{ if and only if } \rho\left(\pi(A), \bar{\pi}(A)\right) \geq \rho\left(\pi(B), \bar{\pi}(B)\right)$$

To obtain this representation, we assume that the agent's uncertainty perception is a QUA and that the agent's ranking of bets extends it. Specifically, the complete and transitive binary relation $\succeq^o$ is a *completion* of the QUA $\succeq$ if

$$B \succeq A \text{ implies } B \succeq^o A \tag{C}$$

We say that the completion $\succeq^o$ is *strict* if, for all A, B, $A \succ B$ implies $A \succ^o B$. In that case, a bet on A is strictly preferred to a bet on B if both probability bounds are weakly greater and one is strictly greater for A than for B. The completion is *maximally ambiguity averse* if, for all A, E, $A \not\succeq E$ implies $E \succ^o B$; it is *maximally ambiguity loving* if, for all A, E, $E \not\succeq A$ implies $A \succ^o E$. Thus, a maximally ambiguity averse agent prefers a bet on an unambiguous event E over a bet on A whenever E and A are not ranked by the common uncertainty perception. In the same situation, a maximally ambiguity loving agent prefers a bet on A.

Definition: *The binary relation $\succeq^o$ is a regular completion if there exists a QUA $\succeq$ such that $\succeq^o$ is a strict, maximally ambiguity averse, or maximally ambiguity loving completion of $\succeq$.*

The betting preference is Archimedean if it satisfies the following continuity property:

$$B \succ^o A \text{ implies there are } E, F \text{ such that } A \sim^o B \cap E \text{ and } B \sim^o A \cup F \qquad (A^o)$$

Proposition 3: *The binary relation $\succeq^o$ is a regular Archimedean completion if and only if some (π, ρ) represents it. If (π, ρ) and (π', ρ') both represent $\succeq^o$, then $\pi = \pi'$ and $\rho = \rho'$.*

Proposition 3 establishes that the underlying uncertainty measure can be recovered uniquely from the completed preference. One implication of this result is that the assumption of a common uncertainty perception among a group of agents is testable; that is, if an analyst incorrectly assumes that two agents share a common uncertainty perception π, then some choices (of bets) can falsify this assumption. The result also implies that the agent's uncertainty perception can be separated from her uncertainty attitude even in the context of bets; that is, in a setting with only two prizes. This separation justifies our interpretation of ρ as measuring ambiguity attitude, allows us to compare the ambiguity attitudes of agents even if their uncertainty perceptions are different and facilitates the following definition of comparative ambiguity aversion.

Definition: *Let $\hat{\rho}$ and ρ be two risk equivalent functions for $\succeq$. Then, $\hat{\rho}$ is more ambiguity averse than ρ if $\rho(a, b) \geq \rho(a', b')$ implies $\hat{\rho}(a, b) \geq \hat{\rho}(a', b')$ for all $(a, b), (a', b')$ such that $0 \leq a' \leq a \leq b \leq b' \leq 1$.*

The definition above asserts that the more ambiguity averse agent is the one more inclined to favor tighter probability bounds. As we show in Proposition 4, below, this notion of "more ambiguity averse" facilitates a local characterization when the utility functions are differentiable. We let ρ_1 and ρ_2 denote the partial derivatives of ρ with respect to the first and second arguments. Let $\mathbf{R}$ be the set of risk equivalent functions that are continuously differentiable with partial derivatives $\rho_1(\cdot, b)$ and $\rho_2(a, \cdot)$ such that: (i) $\rho_1 > 0, \rho_2 > 0$ (if ρ is strict); (ii) $\rho_1 > 0, \rho_2 = 0$ (if ρ is maximally ambiguity averse);

(iii) $\rho_1 = 0, \rho_2 > 0$ (if ρ is maximally ambiguity loving). Proposition 4 below, characterizes our relative measure of ambiguity aversion in terms of the partial derivatives. In the proposition, we let $y/x = \infty$ whenever $y = 0 \neq x$. We define the measure $L_\rho(a, b)$ of local ambiguity aversion: for all $\rho \in \mathbf{R}$ and $(a, b) \in I$

$$L_\rho(a, b) := \frac{\rho_1(a, b)}{\rho_2(a, b)}$$

Proposition 4: *Let $\hat{\rho}$ and ρ be two risk equivalent functions for $\succeq$. Then, $\hat{\rho}$ is more ambiguity averse than ρ if and only if $L_{\hat{\rho}} \geq L_\rho$ at every (a, b).*

Note that L_ρ is a local measure of the relative weight the agent places on the lower bound versus the upper bound. If ρ is linear; that is, $\rho(a, b) = \alpha a + (1 - \alpha)b$ for all (a, b), then L_ρ is constant and

$$\alpha = \begin{cases} \frac{L_\rho}{1+L_\rho} & \text{if } L_\rho < \infty \\ 1 & \text{otherwise.} \end{cases}$$

Existing comparative measures (Epstein (1999), Ghirardato and Marinacci (2002)) differ from the one above in two ways: first, they require a shared uncertainty perception, at least for unambiguous events.[9] Second, the existing measures are weaker since they rely only on comparisons between ambiguous and unambiguous acts (or bets). The analogous measure in our setting is the following:

Definition: *Let ρ and $\hat{\rho}$ be two risk equivalent functions for $\succeq$. Then, $\hat{\rho}$ is weakly more ambiguity averse than ρ if $\rho(c, c) \geq \rho(a, b)$ implies $\hat{\rho}(c, c) \geq \hat{\rho}(a, b)$.*

Clearly, $\hat{\rho}$ more ambiguity averse than ρ implies that $\hat{\rho}$ is weakly more ambiguity averse than ρ. Moreover, for linear ρ the two notions coincide. However, if ρ or $\hat{\rho}$ are nonlinear, then one can be weakly more ambiguity averse without being more ambiguity averse as the following example illustrates: Let $\rho(a, b) = \min\{(a+b)/2, a+b/4\}$ and $\hat{\rho}(a, b) = (3a+b)/4$. Then, $\hat{\rho}$ is weakly more ambiguity averse than ρ since $\rho(a, a) = \hat{\rho}(a, a)$ and $\rho \geq \hat{\rho}$. However, $\hat{\rho}$ is not more ambiguity averse than ρ.[10] To see this, note that $\rho(0, 1) = \rho(1/8, 1/2)$ while

[9] In Gul and Pesendorfer (2014), we also derive a measure of ambiguity attitude that does not require a common uncertainty perception. However, that paper considers a more restrictive model so that the ambiguity attitude over bets can be described by a single parameter.

[10] For ease of exposition, we have chosen ρ^2 so that it is not differentiable at $b = 2a$; however, it is easy to see that this is not essential.

$\hat{\rho}(0,1) > \hat{\rho}(1/8,1/2)$. Thus, the increase in ambiguity from $(1/8,1/2)$ to $(0,1)$ is acceptable to $\hat{\rho}$ but not to ρ. The relation between these two notions of ambiguity aversion is analogous to the relationship between notions of risk aversion based on mean preserving spreads and those based on certainty equivalents.

5. Preferences over Acts and Weak Sophistication

Machina and Schmeidler (1992) introduce the notion of probabilistic sophistication in the course of establishing Savage-type foundations for the theory of choice among risky prospects without expected utility maximization. That is, they provide a theory that identifies an agent's uncertainty perception without imposing the expected utility hypothesis on her uncertainty attitude. The goal of this section is to provide an analogous separation for uncertain prospects. In particular, we will identify the most permissive theory that can be interpreted as an extension of QUAs and the extension of QUAs that can be interpreted as the smallest deviation from expected utility theory.

We assume uncertain prospects yield a monetary prize in every state of the world. The restriction to monetary prizes simplifies the exposition below but is inessential for our results. It is straightforward to adapt the results below for a general prize space. Let $X = [w, z]$ be a nondegenerate compact interval. Let $\mathcal{F}$ be the set of all simple, Σ-measurable functions on Ω; that is,

$$\mathcal{F} = \{f : \Omega \to X \,|\, f^{-1}(x) \in \Sigma \text{ and } f(\Omega) \text{ finite}\}$$

where $f(A) = \{f(s) \,|\, s \in A\}$ for all $A \subset \Omega$. We call $\mathcal{F}$ the set of all acts. For any $f \in \mathcal{F}$, $x \in X$ and $A \in \Sigma$, let fAg denote $h \in \mathcal{F}$ such that $h(s) = f(s)$ for all $s \in A$ and $h(s) = g(s)$ for all $s \notin A$. We identify a constant act that always yields x with x and hence write $x \in \mathcal{F}$.

An *act preference* is a complete, transitive and continuous binary relation $\succeq^*$ on $\mathcal{F}$. An act preference $\succeq^*$ is an *extension* of the QUA $\succeq$ if

$$\begin{aligned} &B \succeq A \text{ implies } zBw \succeq^* zAw \text{ and} \\ &B \succ A \text{ implies } zBw \succ^* zAw \end{aligned} \tag{X}$$

Condition (X) extends condition (C) in section 4 to all prize pairs.

Machina and Schmeidler provide axioms that yield a subjective probability ν such that the decision maker prefers f to g whenever she prefers G_f^ν, the cumulative distribution of the act f, to G_g^ν, the cumulative distribution of g. In particular, the decision maker is indifferent between two acts whenever the acts have the same cumulative distribution; that is, the decision maker is probabilistically sophisticated. Machina-Schmeidler's key axiom, *strong comparative probability*, asserts the following: if $y \succ^* x$ and $\hat{y} \succ^* \hat{x}$, then

$$\begin{aligned}&yBx(A \cup B)h \succeq^* yAx(A \cup B)h \text{ implies}\\&\hat{y}B\hat{x}(A \cup B)\hat{h} \succeq^* \hat{y}A\hat{x}(A \cup B)\hat{h}\end{aligned}$$

The act $f = yBx(A \cup B)h$ yields the more desirable prize y on event B and the less desirable prize x on A; by contrast, $g = yAx(A \cup B)h$ yields the more desirable prize on A and the less desirable prize on B. At each state not in $A \cup B$ both f, g yield the same prize. The decision maker prefers f to g means that she considers B at least as likely as A; that is, $B \succeq A$. Thus, the axiom requires this comparative probability judgement to remain valid regardless of what the act yields on $(A \cup B)^c$ and the choice of more/less desirable prizes. Since we are interested in extending a QUA rather than a qualitative probability, we will need to identify a weaker axiom that permits ambiguity. Also, since we are assuming that more is better, we will replace $y \succ^* x$ with $y > x$.

Weak probabilistic sophistication, formally defined below, requires that Machina and Schmeidler's separability axiom hold only for certain events: consider the three color Ellsberg urn; that is, a partition of the state space into three events C, D, E with C and D representing the two ambiguous colors and E representing the unambiguous color. Since E is unambiguous, the agent's preference for getting the better prize on C or D should be independent of the act given E. Therefore, the Machina-Schmeidler axiom should hold for $A = C, B = D$. By contrast, an agent's preference for getting the prize on C or E need not be independent of the act given D and, thus, we would not expect the Machina-Schmeidler axiom to hold for $A = C, B = E$. More generally, Machina and Schmeidler's axiom should hold whenever the underlying QUA satisfies the standard separability assumption; that is, when Axiom 3 applies to the pairs $(A, (A \cup B)^c)$ and $(B, (A \cup B)^c)$:

$$A \succeq B \text{ if and only if } A \cup (A \cup B)^c \succeq B \cup (A \cup B)^c$$

Axiom 3 ensures that the above relationship holds if $(A, (A \cup B)^c)$ and $(B, (A \cup B)^c)$ conform. Thus, we obtain the following definition:

Definition: *An extension $\succeq^*$ of $\succeq$ satisfies weak sophistication (WS) if, for all $y > x, \hat{y} > \hat{x}$ and for all A, B such that either $B = \emptyset$ or $(A, (A \cup B)^c)$ and $(B, (A \cup B)^c)$ conform,*

$$\begin{aligned} &yBx(A \cup B)h \succeq^* yAx(A \cup B)h \text{ implies} \\ &\hat{y}B\hat{x}(A \cup B)\hat{h} \succeq^* \hat{y}Ax(A \cup B)\hat{h} \end{aligned}$$

The $B = \emptyset$ case simply ensures monotonicity; that is $yAh \succ^* xAh$ whenever A is nonnull and $y > x$. We refer to an extension that satisfies (WS) as a *sophisticated extension.* Weak probabilistic sophistication asserts that events that are separable for the underlying QUA are also separable for general acts. Thus, we can interpret (WS) as an implication of the hypothesis that the QUA fully describes the agent's uncertainty perception.

In the following two subsections, we provide three sophisticated extensions. The first, *Double Expected Utilities*, can be thought of as a minimal departure from expected utility theory. It retains the expected utility hypothesis for unambiguous acts. The second, *Double Lottery Utilities*, is the most general extension of QUAs; it does not impose the expected utility hypothesis on unambiguous acts; instead, when restricted to unambiguous act, it coincides with the Machina-Schmeidler model. The final extension, *Capacity Utilities*, identifies a subclass of Double Lottery Utilities that generalizes the Choquet Expected Utility model. We show that Capacity Utilities facilitate an easy identification of the decision maker's ambiguity perception, ambiguity attitude and risk attitude.[11]

5.1 Double Expected Utility

Theorem 2, below, identifies a class of utility functions for which ambiguity is the only source of deviation from expected utility theory. In addition to weak sophistication, we assume that $\succeq^*$ satisfies the following two assumptions. (In the statements below, A, B, C

[11] Note that all our extensions maintain Savage's P4; that is, the assumption that betting preferences are independent of the stakes, since this assumption is embedded in the definition of weak sophistication.

represent arbitrary, possibly ambiguous, events whereas the events E and F represent unambiguous events of the QUA.) The preference $\succeq^*$ is *separable* if for all E

$$fEh \succeq^* gEh \text{ implies } fE\hat{h} \succeq^* gE\hat{h} \tag{S}$$

The preference $\succeq^*$ is *Archimedean* if for all $y > x' > x$ and $A \subset F$, there is E, x^* such that

$$yEx \sim^* x'Fx \text{ and } x^*Fx \sim^* yAx \tag{A}$$

Assumption (S) is Savage's sure thing principle restricted to unambiguous prospects. Assumption (A) requires that $\succeq^*$ satisfies an Archimedean property both in terms of prizes and in terms of events; the first part of the axioms states that for any bet $x'Fx$ on an unambiguous event F and any prize $y > x'$, we can find a (less likely) unambiguous event E such that the agent is indifferent between yEx and $x'Fx$. The second part states that if A is a subset of the unambiguous event F, then for a suitable choice of x^*, the decision maker can be made indifferent between yAx and x^*Fx.

Theorem 2 below shows that if a weakly sophisticated extension satisfies (S) and (A), then it has a Dual Expected Utility (DEU) representation. For any capacity π and act f, define the cumulative G_f^π and the *dual cumulative* F_f^π as follows:

$$G_f^\pi(x) = \sum_{y \leq x} \pi(f^{-1}(y))$$
$$F_f^\pi(x) = 1 - \sum_{y > x} \pi(f^{-1}(y))$$

Note that the definition of a cumulative distribution function for an uncertainty measure is the same as the corresponding definition for a probability. Moreover, when π is a probability, $F_f^\pi = G_f^\pi$ for all f. Then, V is a *Dual Expected Utility* (DEU) function if there is some uncertainty measure π, continuous function $u : X \to I\!R$ and $\alpha \in [0, 1]$ such that

$$V(f) = \alpha \int u dF_f^\pi + (1 - \alpha) \int u dG_f^\pi \tag{DEU}$$

for all f. We say that an extension $\succeq^*$ of a QUA, $\succeq$, is DEU extension if there exists an $\alpha \in [0, 1]$, a continuous utility index u and π representing $\succeq$ such that the V defined by (DEU) represents $\succeq^*$.

Theorem 2: *A sophisticated extension satisfies (S) and (A) if and only if it is a DEU extension.*

Since Proposition 1 ensures that every uncertainty measure is a belief function and hence convex (i.e., supermodular), we can write a DEU utility V equivalently as an $\alpha-$MEU utility. Let Δ be the set of all probability measures on (S, Σ). For any uncertainty measure π, let

$$C^\pi := \{q \in \Delta : q(A) \geq \pi(A), \forall A \subset \Sigma\}$$

be the core of π. Then, V, the DEU representing the extension $\succeq^*$ with parameters (α, π, u) satisfies

$$V(f) = \alpha \min_{q \in C^\pi} \int u \circ f dq + (1-\alpha) \max_{q \in C^\pi} \int u \circ f dq \qquad (\alpha MM)$$

The equivalence of the above two representations, αMM and DEU follows from known results relating Choquet expected utility theory and maxmin expected utility theory.

5.2 Non-Expected Utility Theories and Ambiguity

Machina and Schmeidler's notion of sophistication implies that the DM identifies each act with a lottery and ranks acts according to the corresponding ranking of lotteries. The DEU representation above provides a weaker interpretation of sophistication that can accommodate ambiguity: the decision-maker prefers f to g if and only if she prefers (F^π_f, G^π_f) to (F^π_g, G^π_g); that is, the decision maker identifies acts with pairs of lotteries and ranks them accordingly. This interpretation suggests a generalization of DEU, which we call *Double Lottery Utility*, that can accommodate non-expected utility theories for unambiguous prospects.

Let $\mathcal{L}$ be the collection of cumulative distributions on $[w, z]$. Given an uncertainty measure π, let $\phi : \mathcal{F} \to \mathcal{L} \times \mathcal{L}$ be the function $\phi(f) = (F^\pi_f, G^\pi_f)$ and let $\Phi = \phi(\mathcal{F})$ be the range of ϕ. We say that f *dominates* g if F^π_f first order stochastically dominates F^π_g, G^π_f first order stochastically dominates G^π_g and $F^\pi_f \neq F^\pi_g, G^\pi_f \neq G^\pi_g$. The function $V : \Phi \to I\!R$ is increasing if f dominates g implies $V(F_f, G_f) > V(F_g, G_g)$. We say that $\succeq^*$ has a double-lottery representation if there is an uncertainty measure π and a continuous increasing function $V : \Phi \to I\!R$ such that $f \succeq^* g$ if and only if $V(F^\pi_f, G^\pi_f) \geq V(F^\pi_g, G^\pi_g)$.

Utility functions of the form $U(f) = V(F_f^\pi, G_f^\pi)$ represent the most general theories consistent with QUAs. For unambiguous acts, the two lotteries F_f^π and G_f^π coincide and the double lottery representation simplifies to a Machina-Schmeidler representation. Note that the convexity of π implies G_f^π (weakly) stochastically dominates F_f^π; that is, $G_f^\pi(x) \leq F_f^\pi(x)$ for all x. Next, we define a notion of ambiguity aversion for Double Lottery Utilities:

Definition: *Double Lottery Utility $\hat{V}$ is more ambiguity averse than V if $V(F,G) \geq V(G',G')$ implies $\hat{V}(F,G) \geq \hat{V}(G',G')$ for all $G \leq F$ and G'.*

It is easy to verify that this comparative measure, when restricted to bets (i.e., binary acts with a fixed pair of prizes, $y > x$), coincides with the definition of "weakly more ambiguity averse" as defined in the previous section. Next, we will define a rich set of utility functions over acts, which we call Capacity Utilities that enables a clear separation of ambiguity perception, ambiguity attitude and risk attitude. Let $\rho : I \to I\!R$ be a risk equivalent function and, for any cumulative F, let $\overline{F}(x) := 1 - F(x)$ denote the complementary cumulative distribution function. Then, for any pair of lotteries (F, G), define the cumulative $F\rho G$ as follows:

$$(F\rho G)(x) = 1 - \rho(\overline{F}(x), \overline{G}(x))$$

We call $F\rho G$ the risk equivalent of (F, G). To see that this is the appropriate generalization of the risk equivalent notion of section 4, it is enough to note that if $(a, b) = (\pi(A), \bar{\pi}(A))$ for the event $A = \{y > x\}$, then $\rho(\overline{F}(x), \overline{G}(x)) = \rho(a, b)$.

Let W be any monotone utility function (i.e., if G (strictly) stochastically dominates G', then $W(G) \geq (>) W(G')$) on the set of all lotteries. Fix π, ρ and define the following utility function, U, on the set of all acts:

$$U(f) = W(F_f^\pi \rho G_f^\pi)$$

That π describes the decision maker's ambiguity (or uncertainty) perception and W describes her risk attitude is clear. The following proposition establishes that ρ represents her ambiguity attitude:

Proposition 5: *The capacity utility $(\hat{\pi}, \hat{\rho}, W)$ is more ambiguity averse than the capacity utility (π, ρ, W) if and only if $\hat{\rho}$ is weakly more ambiguity averse than ρ.*

The proof of Proposition 5 is straightforward and omitted. If W is not linear; that is, if it is not an integral, then the restriction of the utility function U above to unambiguous acts will be a non-expected utility capable of accommodating the Allais paradox and other known violations of the expected utility hypothesis. If W is linear, then the U above is a Choquet Expected Utility function with capacity η defined by:

$$\eta(A) = \rho(\pi(A), \bar{\pi}(A))$$

Note that η need not be totally monotone but this failure of monotonicity arises from the fact that it conflates the decision maker's ambiguity perception and ambiguity attitude.

Next, we provide an example of a Capacity Utility (π, ρ, W) that reveals that a similar conflation of ambiguity perception, ambiguity attitude and risk attitude occurs in Maxmin Expected Utility theory. Assume the decision maker is maximally ambiguity averse and, therefore, $\rho(a, b) = a$ for all (a, b). Let $\tau : [0, 1] \to [0, 1]$ be convex, strictly increasing and onto. Define $\tau * H$ as follows: $\tau * H(x) = 1 - \tau(\overline{H}(x))$ for all x. Finally, we define W as follows:

$$W(H) = \int u(x) d\tau * H$$

where u is a continuous, strictly increasing utility function. Note that for the Capacity Utility (π, ρ, W), the function τ is a parameter of W and, therefore, is a part of the agent's risk attitude. This becomes clear if we restrict attention to risky prospects; that is, unambiguous lotteries, and τ become the familiar preference parameter of rank-dependent expected utility theory.

Clearly, the composition τ and π, $\tau\pi$, is also a capacity. In fact, it is not difficult to show that $\tau\pi$ is convex capacity whenever τ is convex function and π is convex capacity. It is also not difficult to show that we can restate the utility function U as follows:

$$U(f) = \min_{\nu \in C^{\tau\pi}} \int u \circ f d\nu$$

where $C^{\tau\pi}$ is the core of $\tau\pi$. The core of π is a strict subset of the core of $\tau\pi$. Thus, the maxmin representation of this agent yields a set of priors that is strictly larger than the set of priors implied by the uncertainty measure π because this set reflects both the uncertainty perception and the non-linearity of the decision makers risk preferences τ.[12]

Motivated by Gilboa and Schmeidler's (1989) maxmin expected utility theory, uncertainty perception is often identified with a set of priors (see for example Ghirardato and Siniscalchi (2012)). The preceding example shows that the relevant set of priors may not capture uncertainty perception when agents are not expected utility maximizers for unambiguous prospects. More generally, while the uncertainty measure π remains unchanged as we vary the agent's (non-expected) utility function over unambiguous prospects, the corresponding set of relevant priors (as defined by Ghirardato and Siniscalchi (2012)) will typically change. This example illustrates the usefulness of separating ambiguity perception from ambiguity and risk attitudes, as we have done in this paper, when agents simultaneously exhibit Allais- and Ellsberg-style behavior.

6. Appendix A

6.1 Preliminaries

Throughout, Ω is the state space and all sets are contained in the $\sigma-$algebra Σ. Thus, if we refer to a probability measure $(\mu, \mathcal{E})$ it is understood that $\mathcal{E} \subset \Sigma$. We let E, F, G denote unambiguous events; that is, elements of $\mathcal{E}$ and A, B, C, D denote arbitrary events; that is, elements of Σ.

Definition: *Given any probability measure $(\mu, \mathcal{E})$ and A, let $[A] = \{E \subset A\}$. The inner probability of A is $\mu_*(A) = \sup_{E \in [A]} \mu(E)$. Let $\mathcal{E}(A) = \{E \in [A] \mid \mu(E) = \mu_*(A)\}$.*

Note that for all A, $\mathcal{E}(A)$ is non-empty. To see this, let $E^i \in [A]$ be a sequence such that $\lim \mu(E^i) = \mu_*(A)$. By definition such a sequence exists. Then, since μ is a probability measure, $E = \bigcup E^i$ is the desired set. Note also that we can replace the sup in the definition of the inner probability with a max. Finally, note that $\mathcal{E}(A)$ is unique up to a set of measure zero. Let $E(A)$ be this unique element and refer to it as "the" *core* of A. We define the *boundary* of A to be the set $F(A) := [E(A) \cup E(A^c)]^c$. Since $E(A)$ is unique up to a set of measure zero, so is $F(A)$. Also note that $F(A) = F(A^c)$.

[12] In particular, $\tau\pi$ need not be totally monotone; that is, a belief function. The example shows that some Maxmin Expected Utility functions with a set of priors that does not correspond to the core of a belief function are also Capacity Utility functions.

For any $n \geq 1$, let $N = \{1, \ldots, n\}$ and let $\mathcal{N}$ denote the set of all nonempty subsets of N. Then, let $(\mu, \mathcal{E})$ be any probability measure and $\mathcal{A} = \{A_i \,|\, i \in N\}$ be any partition of Ω. The partition $\{E_\kappa \,|\, \emptyset \neq \kappa \subset N\}$ is a μ-split of $\mathcal{A}$ if $E_\kappa \in \mathcal{E}$ for all κ and

$$\begin{aligned} \bigcup_{\kappa \in \mathcal{K}} E_\kappa &\subset \bigcup_{i \in K} A_i \\ \mu\left(\bigcup_{\kappa \in \mathcal{K}} E_\kappa\right) &= \mu_*\left(\bigcup_{i \in K} A_i\right) \end{aligned} \tag{A1}$$

for all $\emptyset \neq K \subset N$ and $\mathcal{K} = \{\kappa \in \mathcal{N} \,|\, \kappa \subset K\}$.

Note that $\{E(A), E(A^c), F(A)\}$ is a μ-split of the binary partition $\{A, A^c\}$.

Lemma A1: *Let $(\mu, \mathcal{E})$ be a probability measure. Then, every partition $\mathcal{A}$ has a μ-split. Moreover, if $\{E_\kappa \,|\, \emptyset \neq \kappa \subset N\}$ is a μ-split of $\mathcal{A}$, then the partition $\{F_\kappa \,|\, \emptyset \neq \kappa \subset N\}$ is also a μ-split of $\mathcal{A}$ if and only if $\mu(E_\kappa \backslash F_\kappa) = 0$ for all κ.*

Proof: Suppose $(\mu, \mathcal{E})$, $N = \{1, \ldots, n\}$ and $\mathcal{A} = \{A_i \,|\, i \in N\}$ satisfy the hypotheses of the Lemma. Let $E^\emptyset = \emptyset$ and choose $E^\kappa \in E\left(\bigcup_{i \in \kappa} A_i\right)$ for all $\kappa \in \mathcal{N}$ and let

$$E_\kappa = E^\kappa \backslash \bigcup_{\kappa \neq \hat{\kappa} \subset \kappa} E^{\hat{\kappa}}$$

for all $\kappa \in \mathcal{N}$. It is straightforward but somewhat tedious to verify that $\{E_\kappa \,|\, \kappa \in \mathcal{N}\}$ is the desired partition. Now, if $\{E_\kappa\} \subset \mathcal{E}$ and $\{F_\kappa\} \subset \mathcal{E}$ are two partitions as defined above and $\mu(E_\kappa \backslash F_\kappa) > 0$, then $\mu_*(A_\kappa) \geq \mu(E_\kappa \cup F_\kappa) > \mu(E_\kappa) = \mu_*(A_\kappa)$, a contradiction. $\square$

Lemma A2:

(i) *The partition $\{E_1, E_{12}, E_2\}$ is a μ-split of (A, A^c) if and only if $E_1 \subset A, E_2 \subset A^c$ and $\mu_*(A \cap E_{12}) = 0 = \mu_*(E_{12} \cap A^c)$.*

(ii) *If $A \subset E$, then $E(A) \subset E, F(A) \subset E$.*

(iii) *Let $A_1 \subset E_1, A_2 \subset E_2$, $E_1 \cap E_2 = \emptyset$, then $E(A_1 \cup A_2) = E(A_1) \cup E(A_2), F(A_1 \cup A_2) = F(A_1) \cup F(A_2)$.*

(iv) *Let $\{E_\kappa\}$ be a $\mu-$split of the partition $\{A_1, A_2, A_3\}$ and let $A = A_1 \cup A_2$. Then, $E(A) = E_1 \cup E_2 \cup E_{12}$ and $F(A) = E_{13} \cup E_{23}$.*

Proof: (i) Let E_1, E_{12}, E_2 be a μ-split of (A, A^c). Then, for all $E \subset A$, $\mu(E \cap E_{12}) = 0$ and, similarly, for all $E \subset A^c$, $\mu(E \cap E_{12}) = 0$. It follows that $\mu_*(A \cap E_{12}) = 0 = \mu_*(A^c \cap E_{12})$. For the converse, let $E_1 \subset A, E_2 \subset A^c$ and $\mu_*(A \cap E_{12}) = \mu_*(A^c \cap E_{12}) = 0$. Then, for all $E \subset A$, $\mu(E_1 \cap E^c) = 0$ and, therefore, $\mu(E_1) = \mu_*(A)$. An analogous argument shows that $\mu_*(A^c) = \mu(E_2)$.

(ii) Since $E(A) \subset A$ it follows that $E(A) \subset E$. To see that $F(A) \subset E$, note that if E_1, E_{12}, E_2 is a μ-split of (A, A^c) then $E^c \subset A^c$ and, therefore, $E^c \subset E_2$. Thus, $E_{12} \subset E$.

(iii) First note that $[A_1 \cup A_2] = [A_1] \cup [A_2]$. This follows since $E' \in [A_1 \cup A_2]$ implies $E_1 \cap E' \in [A_1]$, $E_2 \cap E' \in [A_2]$ and, therefore, $E' \in [A_1] \cup [A_2]$. The converse is obvious. Since $A_1^c \cap A_2^c = (E_1 \backslash A_1) \cup (E_2 \backslash A_2)$ the same argument shows that $[A_1^c \cap A_2^c] = [E_1 \backslash A_1] \cup [E_2 \backslash A_2]$. The last two observations imply

$$E(A_1 \cup A_2) = E(A_1) \cup E(A_2)$$
$$E(A_1^c \cap A_2^c) = E(A_1^c) \cap E(A_2^c)$$

It follows that $F(A_1 \cup A_2) = (E(A_1 \cup A_2) \cup E(A_1^c \cap A_2^c))^c = (E(A_1) \cup E(A_2) \cup E(A_1^c))^c \cup (E(A_1) \cup E(A_2) \cup E(A_2^c))^c = (E(A_1) \cup E(A_1^c))^c \cup (E(A_2) \cup E(A_2^c))^c = F(A_1) \cup F(A_2)$. The penultimate equality follows from $E(A_2) \subset E_2 \subset E(A_1^c)$ and $E(A_1) \subset E_1 \subset E(A_2^c)$.

(iv) By definition of a $\mu-$split $E_1 \cup E_2 \cup E_{12} \subset A$ and $\mu(E_1 \cup E_2 \cup E_{12}) = \mu_*(A)$. Similarly, $E_3 \subset A_3 = A^c$ and $\mu(E_3) = \mu_*(A^c)$. It follows that $E(A) = E_1 \cup E_2 \cup E_{12}$ and $E(A^c) = E_3$. Since $F(A) = (E(A) \cup E(A^c))^c$, the result follows. $\square$

Lemma A3: *If $(\mu, \mathcal{E})$ is a probability measure, then μ_* is a belief function.*

Proof: Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be any partition of Ω and let $\{\mathcal{E}_\kappa\}$ be the corresponding $\mu-$split of $\mathcal{A}$. Assume without loss of generality that $\emptyset \notin \mathcal{A}$. Let $\hat{\mathcal{A}}$ be the finite collection of sets that can be expressed as the unions of elements in $\mathcal{A}$. Let $\gamma(A) = \mu(E_\kappa)$ for all $A = \bigcup_{i \in \kappa \in \mathcal{N}} A_i$. Clearly, $\gamma(A) \geq 0$ for all $A \in \hat{\mathcal{A}}$. Lemma A1 ensures that

$$\mu_*(A) = \sum_{\substack{B \in \hat{\mathcal{A}} \\ B \subset A}} \gamma(B)$$

for all $A \in \mathcal{A}$. Dempster (1967) shows that a capacity on the algebra of sets generated by some partition $\mathcal{A}$ is a belief function if and only if there exists a $\gamma \geq 0$ satisfying the above display equation. Hence, the restriction of μ_* to $\hat{\mathcal{A}} \cup \{\emptyset\}$ is a belief function. Since the partition $\mathcal{A}$ was arbitrary, this proves that μ_* is a belief function. $\square$.

Definition: *(i) E is blank if there exists $A \subset E$ such that $\mu_*(A) = \mu_*(E \backslash A) = 0$; (ii) (E, E^c) is an essential partition for μ if $E \in \mathcal{E}$ and is whole and E^c is blank.*

Lemma A4: *Every non-atomic probability measure has an essential partition. If (E, E^c) and (F, F^c) are two essential partitions for μ, then $\mu(E \Delta F) = 0$.*

Proof: Let $\mathcal{B} = \{E \in \mathcal{E} \,|\, E \text{ is blank}\}$ and $b = \sup_{E \in \mathcal{B}} \mu(E)$. First, we will show that a countable union of blank sets is a blank set. Suppose $E = \bigcup E_n$ and define $F_1 = E_1$, $F_n = E_n \backslash \bigcup_{i<n} E_n$. Hence, the sets F_n are pairwise disjoint and $\bigcup F_n = F$. Choose $A_n \subset F_n$ for all n so that $\mu_*(A_n) = \mu_*(F_n \backslash A_n) = 0$. Suppose there exists $E' \subset F$ such that $\mu(E') > 0$ and (i) $E' \cap (\bigcup A_n) = \emptyset$ or (ii) $E' \cap (E \backslash (\bigcup A_n)) = \emptyset$. Then, the countable additivity of μ ensures that we have $\mu(E' \cap F_n) > 0$ for some n. Then, (i) yields $\mu_*(F_n \backslash A_n) > 0$ while (ii) yields $\mu_*(A_n) > 0$. In either case we have a contradiction.

Let $E_n \in \mathcal{B}$ be a sequence such that $\lim \mu(E_n) = b$. Then, $\mu(\bigcup E_n) = b$ and by the argument above, $\bigcup E_n \in \mathcal{B}$. Set $G = (\bigcup E_n)^c$. Assume there exists $A \subset G$ such that $A \notin \mathcal{E}$. Since μ is complete, part (i) implies that $\mu(F(A)) > 0$. Thus, $F(A)$ is a blank set and, by the above argument, so is $F(A) \cup G^c$. But, $\mu(F(A)) \cup G^c) > b$, which contradicts the definition of b and proves that F is whole. So, (G, G^c) is the desired partition. The proof of the second part is straightforward and omitted. $\square$

6.2 Proof of Proposition 1

Proof of parts (i) and (ii): Suppose, π is an uncertainty measure let $(\mu, \mathcal{E})$ and (η, Σ) satisfy the appropriate properties. In particular,

$$\pi(A) = \max_{E \in [A]} \mu(E) + \min_{E \in [A]} \eta(A \backslash E)$$

for all A.

Since μ is complete, every μ-null set is μ-whole. Hence, $\eta(E) = 0$ whenever $\mu(E) = 0$ and therefore $\mu(E) \geq \eta(E)$ for all $E \in \mathcal{E}$. Next, we claim that for all $E \in [A]$, $\mu(E) = \max_{F \in [A]} \mu(F)$ implies $\pi(A) = \mu(E) + \eta(A \backslash E)$.

To see this, choose any $E \in [A]$ such that $\mu(E) = \max_{F\in[A]} \mu(F)$. Suppose there is $\hat{E} \in [A]$ such that $\eta(A\backslash\hat{E}) < \eta(A\backslash E)$. Hence, $\eta(A\backslash(\hat{E} \cup E)) < \eta(A\backslash E)$ and therefore $\eta(\hat{E} \cup E) > \eta(E)$ which means $\eta(\hat{E}\backslash E) > 0$ and hence $\mu(\hat{E}\backslash E) > 0$, contradicting the fact that $\mu(E) = \max_{F\in[A]} \mu(F)$.

Next, we will show that π is superadditive; that is, $\pi(A \cup B) \geq \pi(A) + \pi(B)$ whenever $A \cap B = \emptyset$. To see this, suppose $\pi(A) = \mu(E) + \eta(A\backslash E)$ and $\pi(B) = \mu(E) + \eta(A\backslash F)$ for some $E \in [A]$ and $F \in [B]$. Then, since $E \cup F \in [A \cup B]$, we have

$$\pi(A\cup B) \geq \mu(E\cup F)+\eta((A\cup B)\backslash(E\cup F)) = \mu(E)+\mu(F)+\eta(A\backslash F)+\eta(B\backslash F) = \pi(A)+\pi(B)$$

as desired.

Next, we prove that $\mathcal{E} = \mathcal{E}_\pi$. To see this, suppose $A \in \mathcal{E}_\pi$, then

$$1 = \pi(A) + \pi(A^c) = \mu(E) + \eta(A\backslash E) + \mu(F) + \eta(A^c\backslash F)$$

for some $E \in [A]$ and $F \in [A^c]$. Hence, $1 = \mu(\hat{E}) + \eta(\hat{E}^c)$ where $\hat{E} = (E \cup F)$. If $\eta(\hat{E}^c) > 0$, we have $\mu(\hat{E}^c) > \eta(\hat{E}^c)$ and hence $\mu(\hat{E}) + \mu(\hat{E}^c) > 1$ contradicting the fact that μ is a probability. So, we must have $\mu(\hat{E}^c) = \eta(\hat{E}) = 0$. Then, $\mu(E) = 1$ and the completeness of μ ensures that $A\backslash E \in \mathcal{E}$ and hence $A \in \mathcal{E}$.

Suppose $E \in \mathcal{E}$. Then, $\pi(E) \geq \mu(E)$ and $\pi(E^c) \geq \mu(E^c)$. Since π is superadditive, $1 = \pi(E \cup E^c) \geq \pi(E) + \pi(E^c) \geq \mu(E) + \mu(E^c) = 1$. Hence, $E \in \mathcal{E}_\pi$ proving that $\mathcal{E} = \mathcal{E}_\pi$.

It follows that $\mu(E) = \pi(E)$ for all $E \in \mathcal{E}_\pi$ proving that the risk measure of any uncertainty measure is unique. Let (F, F^c) be any essential partition and choose $B \subset F^c$ such that $\mu_*(B) = \mu_*(F^c\backslash B) = 0$. Hence, for any $E \subset F^c$, $\pi(E \cap B) = \eta(E \cap B)$ and $\pi(E\backslash B) = \eta(E \cap B)$. Hence, $\eta(E) = \pi(E \cap B) + \pi(E\backslash B)$. Then, for any $E \in \mathcal{E}$, $\eta(E) = \eta(E \cap F) + \eta(E \cap F^c) = \eta(E \cap F^c) = \pi(E \cap B) + \pi(E\backslash B)$. Hence, any two ambiguity measures for π must agree on $\mathcal{E}$.

Take any A and $E \in [A]$ such that $\mu(E) = \mu_*(E)$ and note that $\pi(A) = \mu(E) + \eta(A\backslash E) = \mu(E) + \eta(A) - \eta(E)$. Hence, $\eta(A) = \pi(A) - \eta(E)$. Since any two ambiguity measures for π must agree on $\mathcal{E}$, it follows that they must agree everywhere.

Proof of part (iii): Let π be an uncertainty measure with risk measure μ and ambiguity measure η. If $\eta(\Omega) = 0$, then $\pi = \mu_*$ and hence, Lemma A3 ensures that π is a belief function. If $\eta(\Omega) > 0$, then let $a = 1 - \eta(\Omega)$. Since, μ dominates η, $\eta(\Omega) < \mu(\Omega)$ and hence $a \in (0, 1)$. Also, $\mu(E) = 0$ implies $\eta(E) = 0$. To see this, note that if $\mu(E) = 0$ and $\eta(E) > 0$ for some E, then $\pi(\Omega) > \mu(E^c) + \eta(E) = 1 + \eta(A) > 1$, contradicting the fact that π is a capacity.

Let $\hat{\eta}(A) = \eta(A)/(1 - a)$ and note that η is a probability measure. Since μ dominate η, the countable additivity of μ ensures countable additivity of the restriction of η to $\mathcal{E}$. Hence, $\hat{\mu} = (\mu - \eta)/a$ is a probability measure on $\mathcal{E}$ and therefore, by Lemma A3, its inner probability, $\hat{\mu}_*$, is a belief function.

Clearly $\mu(E \backslash F) = 0$ for all $E, F \in \mathcal{E}(A)$ and hence $\eta(E \backslash F) = 0$ for all $E, F \in \mathcal{E}(A)$ and therefore,

$$\pi(A) = \mu(E) + \eta(A \backslash E) = \mu_*(A) + \eta(A) - \eta(E)$$

for all $E \in \mathcal{E}(A)$. Then,

$$\pi = a\hat{\mu}_* + (1 - a)\hat{\eta}$$

Clearly a convex combination of a belief function and a probability is a belief function. This completes the proof of part(iii). $\square$

7. Appendix B: Proof of Theorem 1 and Proposition 2

Assume $\succeq$ is a qualitative uncertainty assessment. We will prove that it can be represented by a non-atomic uncertainty measure. The converse is straightforward to verify and, therefore, omitted. For the remainder of this section, we assume that $\succeq$ satisfies all properties of a qualitative uncertainty assessment, that is, it is calibrated and satisfies Axioms 1-5. Calibration implies *transitivity* ($A \succeq B$ and $B \succeq C$ implies $A \succeq C$) and *consistency* ($A \succeq B$ if and only if $B^c \succeq A^c$). We will use these properties without explicit reference to calibration below.

For any two sets X, Y, let $X \Delta Y := (X \cup Y) \backslash (X \cap Y)$. For any two collection of events $\mathcal{A}$, $\mathcal{B}$, let $\mathcal{A} \sqcup \mathcal{B} = \{A \cup B \,|\, A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\mathcal{A} \sqcap \mathcal{B} = \{A \cap B \,|\, A \in \mathcal{A}, B \in \mathcal{B}\}$. Finally, let $[A] = \{E \subset A\}$.

Lemma B1:

(i) $A \succeq B$ *and* $B \not\succeq A$ *implies* $A \succ B$.

(ii) $\langle A\rangle = [A] \sqcup [A^c]$.

(iii) *If* $B \succeq A_n$, $A_n \subset A_{n+1}$, $A_{n+1}\backslash A_n \in \mathcal{E}$ *for all* n, *then* $B \succeq \bigcup A_n$.

(iv) $\mathcal{E}$ *is a* $\sigma-$*algebra.*

(v) $A \in \mathcal{E}$ *if and only if* $A \cong \Omega$.

(vi) $A \cap E = \emptyset$ *implies* $A \cup E \cong A$.

Proof: (i) Note that $A \succeq B, B \not\succeq A$ implies that $W(A) \supset W(B), W(B^c) \supset W(A^c)$ with at least one inclusion strict. It follows that there exists E such that $A \succeq E, E^c \succeq A^c$ and $B \not\succeq E$ or $E^c \not\succeq B^c$. Consistency then implies that both $B \not\succeq E$ and $E^c \not\succeq B^c$ must hold. Thus, $W(A) \subset W(B), W(B^c) \subset W(A^c)$ with both inclusions strict and, hence, $A \succ B$.

(ii) Suppose $E \in \langle A\rangle$. Then, $E \cap A \in [A]$ and $E \cap A^c \in [A^c]$ and hence $E = (E\cap A)\cup(E\cap A^c) \in [A]\sqcup[A^c]$. To prove $[A]\sqcup[A^c] \subset \langle A\rangle$, it is enough to show that $E, F \in \mathcal{E}$ and $E \cap F = \emptyset$ implies $E \cup F \in \mathcal{E}$. Suppose $A \succeq B$ and $(A \cup B) \cap (E \cup F) = E \cap F = \emptyset$. Then, $A \cup E \succeq B \cup E$ and hence $(A \cup E) \cup F \succeq (B \cup E) \cup F$ as desired.

(iii) Suppose $B \succeq A_n$, $A_n \subset A_{n+1}$ and $A_{n+1}\backslash A_n \in \mathcal{E}$ for all n. Then, by consistency, $A_n^c \succeq B^c$ and also $A_{n+1}^c \subset A_n^c$, $A_n^c\backslash A_{n+1}^c \in \mathcal{E}$ for all n. Hence, A_n^c converges to $\bigcap A_n^c$ and, by Axiom 5, $\bigcap A_n^c \succeq B^c$. Hence, again by consistency, $B \succeq \bigcup A_n$.

(iv) We first show that $\mathcal{E}$ is an algebra. Clearly, $\Omega \in \mathcal{E}$ and $\emptyset \in \mathcal{E}$. Then, Axiom 2(ii) and part (ii) imply $[E] \sqcup [E^c] = [\Omega] \sqcup [\emptyset]$ and hence $\Omega \in [E] \sqcup [E^c]$ which yields $E^c \in \mathcal{E}$. To complete the proof that $\mathcal{E}$ is an algebra, we need to show that $E \cap F \in \mathcal{E}$. By Axiom 2(ii) and part (ii), $[E] \sqcup [E^c] = [F] \sqcup [F^c]$. Clearly, $E \in [E] \cup [E^c]$ and therefore $E \in [F] \cap [F^c]$ which implies $E \cap F \in [F]$ and hence $E \cap F \in \mathcal{E}$.

To show that $\mathcal{E}$ is a $\sigma-$algebra, let $E = \bigcup F_i$. Define $E_n = \bigcup_{i=1}^n F_i$ and note that $E = \bigcup F_i$. Since $\mathcal{E}$ is an algebra, $E_n \in \mathcal{E}$ for all n. Suppose $A \succeq B$ and $A\cap E = B\cap E = \emptyset$. Then, $A \cup E_n \succeq B \cup E_n$ for all n and, by monotonicity, $A \cup E \succeq A \cup E_n \succeq B \cup E_n$ for all n. Hence, transitivity implies $A \cup E \succeq B \cup E_n$ for all n. Then, Lemma B1 (iii) yields $A \cup E \succeq B \cup E$.

For the converse, assume $A \cup E \succeq B \cup E$ and $(A \cup B) \cap E = \emptyset$. By consistency, $B^c\cap E^c \succeq A^c\cap E^c$. Hence, $(B^c\cap E^c)\cup E_n \succeq (A^c\cap E^c)\cup E_n$. Monotonicity and transitivity yield $(B^c \cap E^c) \cup E \succeq (A^c \cap E^c) \cup E_n$. Then Lemma B1 (iii) implies $(B^c \cap E^c) \cup E \succeq$

$(A^c \cap E^c) \cup E$; that is, $B^c \succeq A^c$ and hence by consistency $A \succeq B$ as desired. This proves that $\mathcal{E}$ is a σ-algebra.

(v) Since Ω is unambiguous, one direction follows from Axiom 2(ii). For the other direction, assume $A \cong \Omega$ and, since Ω is unambiguous, $\Omega \in [A] \sqcup [A^c]$ by part (ii); that is, $A \in [A]$ and therefore $A \in \mathcal{E}$.

(vi) We will show that when $A \cap E = \emptyset$, $[A] \sqcup [A^c] = [A \cup E] \sqcup [A^c \cap E^c]$ and appeal to part (ii). Suppose $F_1 \in [A]$ and $F_2 \in [A^c]$. Then, $F_1 \subset A \cup E$, $F_2 = F_3 \cup F_4$ where $F_3 := F_2 \cap E$ and $F_4 = F_2 \cap E^c$. Since $\mathcal{E}$ is an algebra (by part (ii)), $F_1 \cup F_3 \in [A \cup E]$ and $F_4 \in [A^c \cap E^c]$ and hence $F_1 \cup F_2 = F_1 \cup F_3 \cup F_4 \in [A \cup E] \sqcup [A^c \cap E^c]$.

For the converse, take $F_1 \in [A \cup E]$ and $F_2 \in [A^c \cap E^c]$. Then, let $F_3 = F_1 \cap A$ and $F_4 = F_1 \cap E$. Since $\mathcal{E}$ is an algebra, $F_4 \in \mathcal{E}$ and therefore, $F_3 = F_1 \backslash F_4 \in \mathcal{E}$ and, therefore, $F_3 \in [A]$ and $F_2, F_4 \in [A^c]$. Since $\mathcal{E}$ is an algebra, $F_2 \cup F_4 \in [A^c]$ and therefore $F_1 \cup F_2 = F_3 \cup F_4 \cup F_2 \in [A] \sqcup [A^c]$.

$\square$

Lemma B2: *There exists a nonatomic probability measure μ on $\mathcal{E}$ such that $E \succeq F$ if and only if $\mu(E) \geq \mu(F)$.*

Proof: Let $\succeq_{\mathcal{E}}$ be the restriction of $\succeq$ to $\mathcal{E}$. Note that $\succeq$ is transitive and Axiom 2 ensures that $\succeq_{\mathcal{E}}$ is complete. Moreover, by Lemma B1(i), $F \succeq E$, $E \not\succeq F$ implies $F \succ E$, and, thus, $\succ$ coincides with the strict preference derived from $\succeq$. By definition, $E \succeq F$ if and only if $E \cup F' \succeq F \cup F'$ whenever $(E \cup F) \cap F' = \emptyset$. Hence, $\succeq$ restricted to $\succeq_{\mathcal{E}}$ is a qualitative probability. By Lemma B1(iii), Axiom 4 restricted to $\succeq_{\mathcal{E}}$ yields Savage's small event continuity axiom. Repeating Savage's proof for the σ-algebra $\mathcal{E}$ instead of the σ-algebra of all subsets of Ω yields a finitely additive, convex valued probability μ that represents $\succeq_{\mathcal{E}}$. Then, the restriction of Axiom 5 to sets in $\mathcal{E}$ establishes that μ is countably additive.

Next, we will show that μ is complete; that is, $C \subset E$ and $\mu(E) = 0$ implies $C \in \mathcal{E}$. Assume $C \subset E$ and $\mu(E) = 0$. Since μ represents $\succeq_{\mathcal{E}}$, $E \sim \emptyset$. To show that E is null, note that $E \sim \emptyset$ implies $A \cup E \sim A \backslash E$ (by Axiom 3) and hence, by monotonicity, $A \succeq A \cup E$, proving that E is null. Monotonicity implies that any subset of a null set is also null and hence C is null. Then, for any A, B such that $A \succeq B$, we have $A \cup C \sim A \succeq B \sim B \cup C$ and hence $C \in \mathcal{E}$.

Finally, we will show that μ is nonatomic. Suppose $\mu(E) > 0$. Then, since μ represents $\succeq_{\mathcal{E}}$, $E \succ \emptyset$. Then, by Axiom 4, there is $F \subset E$ such that $F \cong E$, $E \succ F \succ \emptyset$. By Lemma B1(v), $F \in \mathcal{E}$. Again, since μ represents $\succeq_{\mathcal{E}}$, we have $\mu(E) > \mu(F) > 0$ as desired. $\square$

For the remainder of this proof we fix the nonatomic probability measure $(\mu, \mathcal{E})$ and let $\mathcal{N}(A)$ denote the set of all null subsets of A, $\mathcal{E}_o(A) = \{E \subset A \mid \mu(E) = 0\}$. Let $E(A)$ be the core of A, $F(A)$ the boundary of A and let $(E(A), F(A), E(A^c))$ be a $\mu-$split of A.

Lemma B3: $\mathcal{N}(A) = \mathcal{E}_o(A)$

Proof: While proving that μ is complete (Lemma B2), we have shown that $\mu(E) = 0$ implies E is null. It follows that $\mathcal{E}_o(A) \subset \mathcal{N}(A)$. For the converse, assume C is null. Then, $A' \cup C \sim A'$ for all A' and hence $A' \succeq B'$ implies $A' \cup C \sim A' \succeq B' \sim B' \cup C$ and by transitivity $A' \cup C \succeq B' \cup C$ proving that $C \in \mathcal{E}$. But if $C \in \mathcal{E}$ is null then clearly, $\mu(C) = 0$. It follows that $\mathcal{N}(A) \subset \mathcal{E}_o(A)$.

$\square$

If $\mathcal{E} = \Sigma$. Then, set $\pi = \mu$ and $\eta(A) = 0$ for all $A \in \mathcal{E}$ and note that π is the desired uncertainty measure. So, from now on, we will assume $\mathcal{E} \neq \Sigma$. We recall the definition of a blank set and an essential partition. Their existence and uniqueness was established in Lemma A4:

Definition: *(i) E is blank if there exists $A \subset E$ such that $\mu_*(A) = \mu_*(E \backslash A) = 0$; (ii) (E, E^c) is an essential partition for μ if $E \in \mathcal{E}$ and is whole and E^c is blank.*

Assume $\mathcal{E} \neq \Sigma$ and let (G, G^c) be an essential partition for μ. Since μ is complete and $\mathcal{E} \neq \Sigma$, $\mu(G^c) > 0$. For any subset E of G^c such that $\mu(E) > 0$ and let

$$\mathcal{C}_E = \{A \subset E \mid \mu_*(A) = \mu_*(E \backslash A) = 0\}$$

Since (G, G^c) is an essential partition, there is $B \subset G^c$ such that $\mu_*(B) = \mu_*(G^c \backslash B) = 0$. Setting $A_1 = B \cap F$ and verifying that $A_1 \in \mathcal{C}_E$ establishes that $\mathcal{C}_E \neq \emptyset$. Note that, by Lemma A2 (iii), $A \in C_E$ if and only if $\{E_1 = \emptyset, E_2 = E^c, E_{12} = E\}$ is a $\mu-$split of $\{A, A^c\}$. Thus, $\mu(E \Delta F(A)) = 0$ and $\mu(E(A^c) \Delta E^c) = 0$.

Lemma B4: *Let $A, B \in \mathcal{C}_E$ and $F, F' \subset E^c$. Then,*

(i) $E \succ A \succ \emptyset$

(ii) $A \cup F \cong C$ *if and only if* $C = B' \cup E'$ *for some* $B' \in \mathcal{C}_E$ *and some* $E' \subset E^c$;

(iii) $A \subset C \subset E$, $B \cap C = \emptyset$ *implies* $C \in \mathcal{C}_E$;

(iv) $A \cup F \succ B \cup F'$ *implies there is* $C \in \mathcal{C}_E$ *such that* $B \cap C = \emptyset$, $A \cup F \succ B \cup F' \cup C$ *and* $B \cup C \in \mathcal{C}_E$.

(v) $A \cup F \succ B \cup F'$, $\mu(E) \leq 1/3$, *and either* $F = \emptyset$ *or* $F' = \emptyset$ *implies there is* $E' \subset E^c$ *such that* $E' \cap (F \cup F') = \emptyset$, $\mu(E') > 0$ *and* $A \cup F \succ B \cup F' \cup E'$.

(vi) If $E_1 \ldots, E_n$ *are pairwise disjoint and* $E = \bigcup_{i=1}^n E_i$, *then,* $\mathcal{C}_E = \mathcal{C}_{E_1} \sqcup \ldots \sqcup \mathcal{C}_{E_n}$.

Proof: (i) If A is null then $A \in \mathcal{E}$ by Lemma B3. Then, $A^c \cap E_1 \in \mathcal{E}$ and hence $\mu(A^c \cap E_1) = \mu(E_1) > 0$ contradicting the fact that $A \in \mathcal{C}_E$. Therefore, monotonicity implies that $A \succ \emptyset$. From the definition of $\mathcal{C}_E$ it follows that $E \backslash A \in \mathcal{C}_E$ and, therefore, $E \backslash A$ is not null. Monotonicity then implies $E \succ A$.

(ii) First we show that $A \cong B$ if $B \in \mathcal{C}_E$. As we noted above, $\mu(E^c \Delta E(A^c)) = 0 = \mu(E^c \Delta E(B^c))$. By Lemma B1(ii), it suffices to show that $[A] \sqcup [A^c] = [B] \sqcup [B^c]$. Suppose $F_1 \in [A]$ and $F_2 \in [A^c]$. Let $F_3 = F_1 \cap B$, $F_4 = F_1 \cap B^c$, $F_5 = F_2 \cap E \cap B$, $F_6 = F_2 \cap E \cap B^c$, $F_7 = F_2 \cap E^c \cap B$ and $F_8 = F_2 \cap E^c \cap B^c$. Clearly,

$$F_1 \cup F_2 = F_3 \cup F_4 \cup F_5 \cup F_6 \cup F_7 \cup F_8$$

Since μ is complete and $\mu_*(A) = \mu_*(B) = 0$, we have $F_3, F_4, F_5, F_7 \in \mathcal{E}$. Since $F_2 \in [A^c]$, $\mu(F_2 \backslash E(A^c)) = 0$. Then, since $\mu(E^c \Delta E(A^c)) = 0$, we have $\mu(F_2 \backslash E^c) = 0$ and therefore $F_6 \in \mathcal{E}$. Also, since $\mu(E^c \Delta E(B^c)) = 0$, we can write $E^c = (E(B^c) \backslash F_9) \cup F_{10}$ for $F_9, F_{10} \in \mathcal{E}$ such that $\mu(F_9 \cup F_{10}) = 0$. Then, $E^c \cap B^c = (E(B^c) \cap F_9^c \cap B^c) \cup (F_{10} \cap B^c)$. Since $E(B^c) \subset B^c$, we have $E^c \cap B^c = (E(B^c) \cap F_9^c) \cup (F_{10} \cap B^c)$. Clearly, $E(B^c) \cap F_9^c \in \mathcal{E}$. Since μ is complete, we have $(F_{10} \cap B^c) \in \mathcal{E}$ and therefore, $E^c \cap B^c \in \mathcal{E}$ and hence $F_8 \in \mathcal{E}$. It follows that $F_1 \cup F_3 \cup F_5 \cup F_7 \in [B]$ and $F_2 \cup F_4 \cup F_6 \cup F_8 \in [B^c]$ and therefore $F_1 \cup F_2 \in [B] \sqcup [B^c]$ proving that $A \cong B$.

Next, we show that $C \cong A$ if and only if $C = B \cup F$ for some $B \in \mathcal{C}_E$. First, assume $C = B \cup F$ for some $F \subset E^c$. Then, by Lemma B1(vi), $B \cong B \cup E$ and, since $A \cong B$, it follows that $A \cong B \cup F = C$.

For the converse, we first show that $[A] \sqcup [A^c] = \mathcal{N}(\Omega) \sqcup [E^c]$. Note that if $F_1 \in [A]$ and $F_2 \in [A^c]$, then $\mu(F_1) = 0$ and $\mu(F_2 \cap E) = 0$. Hence, F_1 and $F_2 \cap E$ are in $\mathcal{N}(\Omega)$ by Lemma B3. Since the union of null sets is null, $F_1 \cup (F_2 \cap E) \in \mathcal{N}(\Omega)$. Let $F_3 = F_2 \backslash E$ and note that $F_2 \in [E^c]$ and therefore, $F_1 \cup F_2 \in \mathcal{N}(\Omega) \sqcup [E^c]$. Similarly, if $F_1 \in [E^c]$ and $F_2 \in \mathcal{N}(\Omega)$, then, since μ is complete, $F_3 = F_2 \cap A \in [A]$ and $F_4 = F_2 \cap A^c \in [A^c]$ and also $F_1 \in [A^c]$. Hence, $F_1 \cup F_2 = F_3 \cup (F_1 \cup F_4) \in [A] \sqcup [A^c]$ as desired.

Now, suppose $C \cap E \notin \mathcal{C}_E$. Then, either there is $F \in [C \cap E]$ such that $\mu(F) > 0$ or there is $F \in [C^c \cap E]$ such that $\mu(F) > 0$. Since, $[A] \sqcup [A^c] = \mathcal{N}(\Omega) \sqcup [E^c]$, either of these establishes that $[A] \sqcup [A^c] \neq [C] \sqcup [C^c]$. If $C \cap E^c \notin \mathcal{E}$, then, let E_1, E_2, E_{12} be a μ-split of the partition A_1, A_2 where $A_1 = C \cap E^c$ and $A_2 = A_1^c$. Then, $\mu(E_{12} \backslash E) > 0$. Hence, $E_{12} \backslash E \in [A] \sqcup [A^c]$ and $E_{12} \backslash E \notin [C] \cup [C^c]$.

It remains to show that $A \cup F \cong B \cup F'$ whenever $F, F' \subset E^c$. By Lemma B1(vi), $A \cong A \cup E$ and $B \cong B \cup F$. The result now follows from $A \cong B$.

(iii) Note since $B \in \mathcal{C}_E$, $C \subset E$ and $B \cap C = \emptyset$, then $C \subset E \backslash B$ and hence $\mu_*(C) \leq \mu_*(E \backslash B) = 0$. Similarly, $E \backslash C \subset E \backslash A$ and therefore, $\mu_*(E \backslash C) \leq \mu_*(E \backslash A) = 0$. Hence, $C \in \mathcal{C}_E$.

(iv) Assume $A \cup F \succ B \cup F'$. By Axiom 4 there exists a partition $C_1, \ldots, C_k$ of Ω such that $B \cup F \cong C_n \backslash (B \cup F)$ and $A \cup F \succ B \cup F' \cup C_n$ for all n. By part (ii) this implies that $B \cong C_n \backslash (B \cup F)$ for all n and that there are B_n, F_n such that $B_n \cup F_n = C_n \backslash (B \cup F)$ for $B_n \in \mathcal{C}_E$ and $F_n \cap E = \emptyset$. Note that $B_2 \cap B_1 = \emptyset$ and therefore $B \cup B_1 \in \mathcal{C}_E$ by part (iii). Since $B_1 \in \mathcal{C}_E$, part(i) implies it is non-null; then, monotonicity implies that $B_1 \cup B \cup F' \succ B \cup F'$ and, therefore, B_1 is the desired set.

(v) Argue as in the proof of part (iv) to get $B_n \cup F_n = C_n \backslash (B \cup F)$ such that $B_n \in \mathcal{C}_E$, $F_n \cap E = \emptyset$. Since F_n is a partition of E^c and $\mu(E^c) > 0$ we must have $\mu(F_n) > 0$ for some n. Then if $F' = \emptyset$, $E' = F_n$ is the desired set. Similarly, if $F = \emptyset$, then $A \succ B \cup F'$ implies $\mu(F') \leq \mu(E) < \mu(E^c)$. Hence, there must be F_n such that $\mu(F_n \backslash F') > 0$ and therefore, $E' = F_n \backslash F'$ is the desired set.

(vi) We will prove the results for $n = 2$. Then, the general case follows from an inductive argument. Suppose $A \in \mathcal{C}_E$. Then, for $i = 1, 2$, $\mu_*(A \cap E_i) \leq \mu_*(A) = 0$ and $\mu_*(E_i \backslash A) = \mu_*((E \backslash A) \cap E_i) \leq \mu_*(E \backslash A) = 0$ and hence $A \in \mathcal{C}_{E_1} \sqcup \mathcal{C}_{E_2}$. Conversely, if

$A_i \in \mathcal{C}_{E_i}$ for $i = 1, 2$, then for any $F \subset A_1 \cup A_2$, $F \cap E_i \subset A_i$ and hence $\mu(F \cap E_i) = 0$. Therefore, $\mu(F) = \mu(F \cap E_1) + \mu(F \cap E_2) = 0$ and, similarly, $\mu(F) = 0$ whenever $F \subset A_1^c \cup A_2^c$ and therefore $A_1 \cup A_2 \in \mathcal{C}_E$. □

For $E \subset G^c$ such that $\mu(E) > 0$ and $A \in \mathcal{C}_E$, define

$$\eta_E(A) = \sup\{\mu(E') \mid A \succeq E'\}$$

$$\eta_E(E) = \sup_{A \in \mathcal{C}_E} \eta_E(A)$$

For E such that $\mu(E) = 0$, define $\eta_E(A \cap E) = 0$ for all A. Next, observe that if $E \Delta E^*$ is null, and $E \cup E^*$ is not null, the definition of $\mathcal{C}_E$ implies that $A \cap E \in \mathcal{C}_E$ if and only if $A \cap E^* \in \mathcal{C}_{E^*}$. Then, the definition of η_E implies that $\eta_E(A \cap E) = \eta_{E^*}(A \cap E^*)$ for all $A \in C_{E \cup E^*}$. If $E \cup E^*$ is null, then $\eta_E(A \cap E) = \eta_{E^*}(A \cap E^*) = 0$.

Let (G, G^c) be an essential partition of Ω. Define the set function η on Σ as follows: for any $A \in \Sigma$, $E = E(A \cap G^c), F = F(A \cap G^c)$, let

$$\eta(A) := \eta_E(E) + \eta_F(A \cap F)$$

If $\mu(F) > 0$, then $A \cap F \in C_F$. Moreover, by the argument above, η_E is unaffected by the addition of null events. It follows that $\eta(A)$ is well defined and $\eta(A \cup B) = \eta(A)$ if B is null. Since $\mu(F(A) \Delta G^c) = 0$, we can assume (wlog) that $F(A) \subset G^c$ whenever $\mu(F(A)) > 0$.

Lemma B5: *Let $A \subset E = F(A)$. Then,*

(i) $\eta(A) \geq \mu(F)$ if and only if $A \succeq F$;

(ii) For all $\epsilon > 0$ there is C such that $\eta(C) < \epsilon$, $C \subset F(C) = E$, $A \cap C = \emptyset$.

Proof: (i) Let $E = F(A)$. If $\mu(E) = 0$, the result is immediate. Thus, assume $\mu(E) > 0$ and $A \in \mathcal{C}_E$. If $\eta(A) \geq \mu(F)$ we can find a sequence $F_n \in \mathcal{E}$ such that $A \succeq F_n$ and $\lim \mu(F_n) \geq \mu(F)$. Since μ is nonatomic, we can choose this sequence so that $F_n \subset F_{n+1}$. Hence, by Lemma B1(iii), $A \succeq \bigcup F_n \succeq F$, establishing $A \succeq F$. The converse follows from the definition of η_E.

(ii) If $\mu(E) = 0$ the result is immediate. Hence, assume $\mu(E) > 0$. Then, by Lemma B4(i), $E \succ A$. By Axiom 4, there is a partition $C_1, \ldots, C_n$ of Ω such that $C_n \backslash A \cong A$. By Lemma B4(ii), it follows that $\hat{B}_n := (C_n \cap E) \backslash A \in \mathcal{C}_E$, and, by Lemma B4(iii), $A \cup \hat{B}_n \in \mathcal{C}_E$.

Let $B_1 = \hat{B}_1$. Replacing A with $A \cup B_1$ and repeating the argument yields $B_2 \in \mathcal{C}_E$ such that $(A \cup B_1) \cap B_2 = \emptyset$, $B_2 \in \mathcal{C}_E$ and $A \cup B_1 \cup B_2 \in \mathcal{C}_E$. Continuing in this fashion, we get a sequence of pairwise disjoints set B_n such that $B_n \in \mathcal{C}_E$ and $B_n \in E_1 \backslash A$ for all n. Since $B_n \in \mathcal{C}_E$, it follows that $E = F(B_n)$ for all n. By part (i) the sequence $\eta(B_n)$ is bounded. If $\lim \eta(B_n) = 0$, we are done. Otherwise, $\eta(B_{n_j}) \geq \epsilon > 0$ for some subsequence B_{n_j}. Assume, without loss of generality, that this subsequence is B_n and let $A_n = \bigcup_{i \geq n} B_n$. Then, by Lemma B4(iii), $A_n \in \mathcal{C}_E$. Also, by monotonicity, $A_n \succeq E$ for any E such that $\mu(E) \in (0, \epsilon)$ and, therefore, by Axiom 5, $\bigcap A_n \succeq E$. But $\bigcap A_n = \emptyset$ and hence we have a contradiction.

□

Lemma B6: *Let $A, B \subset F(B) = F(A) = E$, $1/3 \geq \mu(E)$, and let $F, F' \in E^c$. Then, $F, F' \subset E^c$ implies $A \cup F \succeq B \cup F'$ if and only if $\eta(A) + \mu(F) \geq \eta(B) + \mu(F')$.*

Proof: If E is null, the Lemma follows from the fact that μ represents $\succeq$ on $\mathcal{E}$. Assume, therefore, E is not null. Then, $E \subset G^c$ and $A, B \in \mathcal{C}_E$. First, we show that $A \cup F \succeq B$ implies $\eta(A) + \mu(F) \geq \eta(B)$. Choose $F^* \subset E^c$ such that $\eta(B) = \mu(F^*)$. If $\mu(F) \geq \mu(F^*)$, we get $\eta(A) + \mu(F) \geq \eta(B)$ immediately. Otherwise, since μ is non-atomic, we may assume $F \subset F^*$. Since $F \backslash F^* \in E^c, A \cup F \succeq B \succeq F^*$ (by Lemma B5(i)), it follows from Axiom 3 that $A \succeq F^* \backslash F^{**}$. By Lemma B5(i) this implies $\eta(A) \geq \mu(F^* \backslash F^{**}) = \eta(B) - \mu(F^{**})$ as desired.

Next, we show that $A \succeq B \cup F'$ implies $\eta(A) \geq \eta(B) + \mu(F')$. By Lemmas B4(i) and B5(i), it follows that $\eta(A) < 1/3$. Therefore $\mu(F') < 1/3$ and, since μ is non-atomic, we can choose $F^* \subset E^c, F^* \cap F' = \emptyset$ such that $\eta(B) = \mu(F^*)$. Then, $B \sim F^*$ and hence $B \cup F' \sim F^* \cup F'$ and therefore $\eta(A) \geq \mu(F^*) + \mu(F') = \eta(B) + \mu(F')$.

Next, we show that $\eta(A) + \mu(F) \geq \eta(B)$ implies $A \cup F \succeq B$. Suppose $\eta(A) + \mu(F) \geq \eta(B)$ and $A \cup F \not\succeq B$. By Lemma B4(ii), $A \cup F \cong B$, and, therefore, we must have $B \succ A \cup F$. Then, by Lemma B4(v), there is $F'' > 0$ such that $(A \cup F) \cap F'' = \emptyset$, $\mu(F'') > 0$ and $B \succ A \cup F' \cup F''$. Then, the part of the this lemma that we have already proven yields $\eta(B) \geq \eta(A) + \mu(F \cup F'')$. Hence, $\eta(B) > \eta(A) + \mu(F)$, a contradiction.

Next, we show that $\eta(A) \geq \eta(B) + \mu(F')$ implies $A \succeq B \cup F'$. Suppose $\eta(A) \geq \eta(B) + \mu(F')$ and $A \not\succeq B \cup F'$. Then, arguing as above, we conclude that $B \cup F' \succ A$ and hence, by Lemma B4(v), $B \cup F' \succ A \cup F''$ for some $F'' \subset E^c$ such that $\mu(F'') > 0$. Then, since $\mu(F') < 1/3$ by the argument above, we can assume that F' and F'' are nested. If $F' \subset F''$, then we get $B \succ A \cup (F'' \backslash F')$ and then, part of the this lemma that we have already proven yields $\eta(B) > \eta(A)$, a contradiction. Similarly, if $F'' \subset F'$, then we have $B \cup (F' \backslash F'') \succ A$ and hence $\eta(B) + \mu(F') > \eta(B) + \mu(F' \backslash F'') \geq \eta(A)$, a contradiction.

Finally, to complete the proof assume F, F' are nested. Since μ is nonatomic, we can do so without loss of generality. Then, if $F \subset F'$, $A \cup F \succeq B \cup F'$ if and only if $A \succeq B \cup (F' \backslash F)$. If $F' \subset F''$ then $A \cup F \succeq B \cup F'$ if and only if $A \cup (F \backslash F') \succeq B$. Then, the arguments given above prove the Lemma. □

Lemma B7: *Let $A_i \in F(A_i)$ for $i = 1, \ldots, n$ such that $F(A_i) \cap F(A_j) = \emptyset$ for $i \neq j$. Then, $\eta(\bigcup A_i) = \sum \eta(A_i)$.*

Proof: We will prove the result for $n = 2$; then the general case follows from an inductive argument. Let $E_i = F(A_i)$. If $\mu(E_1 \cap G^c)\mu(E_2 \cap G^c) = 0$ the result is immediate. Thus, assume $\mu(E_1)\mu(E_2) > 0$ and $E_1, E_2 \subset G^c$. Since μ is non-atomic and $\mu(E_i) \geq \eta(A_i)$ by Lemma B4(i) we may choose $E_i^* \subset E_i$ such that $\eta(A_i) = \mu(E_i^*)$. Let $B_i = E_i \backslash A_i$ and $F_i = E \backslash E_i^*$.

To prove $\eta(A_1 \cup A_2) \geq \eta(A_1) + \eta(A_2)$, note that $F(A_1 \cup A_2) = F(A_1) \cup F(A_2)$ by Lemma A2. By Lemma B5(i), $A_i \succeq E_i^*$. Note that A_1, A_2 and E_1^*, A_2 are conforming pairs and, therefore, Axiom 3 implies that $A_1 \cup A_2 \succeq E_1^* \cup A_2$. Since E_2^* is unambiguous it follows that $E_1^* \cup A_2 \succeq E_1^* \cup E_2^*$ and, by transitivity, $A_1 \cup A_2 \succeq E_1^* \cup E_2^*$. We conclude that $A_1 \cup A_2 \succeq A_1 \cup E_2^* \succeq E_1^* \cup E_2^*$. Lemma B5(i) therefore implies $\eta(A_1 \cup A_2) \geq \mu(E_1^*) + \mu(E_2^*) = \eta(A_1) + \eta(A_2)$.

To prove the converse, note that (by Axiom 3) for $E_i' \subset E_i$, $A_1 \cup E_2' \succeq E_1' \cup E_2'$ if and only if $A_1 \succeq E_1'$. The definition of E_i^* therefore implies that $W(A_1 \cup E_2^*) = W(E_1^* \cup E_2^*)$. If $A_1 \cup A_2 \succ E_1^* \cup E_2^*$, then, by calibration, $W(A_1 \cup A_2) \supset W(E_1^* \cup E_2^*) = W(A_1 \cup E_2^*)$ with a strict inclusion and, therefore, $A_1 \cup E_2^* \not\succeq A_1 \cup A_2$. Since $A_1 \cup A_2 \succeq A_1 \cup E_2^*$, it follows (from Lemma B1(i)) that $A_1 \cup A_2 \succ A_1 \cup E_2^*$. By Axiom 3, this, in turn, implies that $A_2 \succ E_2^*$. Then, by calibration, there exists E' such that $\mu(E') > \mu(E_2^*)$ and $A_2 \succeq E'$, contradicting $\eta(A_2) = \mu(E_2^*)$. □

Lemma B8: *If $A \cap B = \emptyset$ then $\eta(A) + \eta(B) = \eta(A \cup B)$.*

Proof: If A or B are null the result is trivial. Thus, assume that A and B are non-null.

Claim 1: If $0 < \mu(E) \leq 1/3$ and $F(A) = F(B) = E, A \cup B \in \mathcal{C}_E$ then $\eta(A \cup B) = \eta(A) + \eta(B)$.

To prove that $\eta(A) + \eta(B) \geq \eta(A \cup B)$. Choose $F^* \subset E^c$ such that $\mu(F^*) = \eta(B)$. By part (i), $A \cup F^* \succeq A \cup B$ if and only if $\eta(A) + \eta(B) = \eta(A) + \mu(F^*) \geq \eta(A \cup B)$. Hence, it suffices to show that $A \cup F^* \succeq A \cup B$. If $A \cup F^* \not\succeq A \cup B$ then, since $A \cup F^* \cong A \cup B$ (by Lemma B4(ii)) it follows from Lemma B1(v) that $A \cup B \succ A \cup F^*$. Then, by Lemma B4(iv), $A \cup B \succ A \cup C \cup F^*$ for some $C \in \mathcal{C}_E$ such that $A \cap C = \emptyset$ and $A \cup C \in \mathcal{C}_E$. Let $\hat{A} = (A \cup B) \backslash B$ and note that $C \cup \hat{A} \in \mathcal{C}_E$. Therefore, $B \cong A \cup B \cong (A \backslash \hat{A}) \cup C$. Axiom 3 then implies that $(A \cup B) \backslash \hat{A} = B \succ (A \backslash \hat{A}) \cup C \cup F^* \succeq C \cup F^*$ (by monotonicity). From part (i) it now follows that $\eta(B) > \mu(F^*)$, a contradiction.

Next, we show that $\eta(A \cup B) \geq \eta(A) + \eta(B)$ if $A \cup B = \emptyset$. If not, then choose $F^* \subset E^c_E$ such that $\eta(A \cup B) = \mu(F^*) + \eta(A)$. It follows that $A \cup F^* \succeq A \cup B$ by part(i). Since $A \cap B = \emptyset, A \cup B \cong A$, Axiom 3 implies that $F^* \succeq B$. Choose $C \in \mathcal{C}_E$ such that $\eta(C) < \eta(B) + \eta(A) - \eta(A \cup B)$, $A \cap C = \emptyset$ and $A \cup C \in \mathcal{C}_E$. By Lemma B5(iii) this is possible. By monotonicity, $A \cup F^* \not\succeq A \cup C \cup F^*$ and, since $A \cup F^* \cong A \cup C \cup F^*$, it follows that $A \cup C \cup F^* \succeq A \cup F^*$ and, therefore, Lemma B1(i) implies that $A \cup C \cup F^* \succ A \cup F^*$. Thus, (again by Axiom 3), $C \cup F^* \succ F^* \succeq B$. Then, part (i) implies $\eta(C) + \mu(F^*) \geq \eta(B) = \mu(F^*) + \eta(A) + \eta(B) - \eta(A \cup B) > \mu(F^*) + \eta(C)$, a contradiction.

Claim 2: If $0 < \mu(E) \leq 1/3$ and $F(A) = F(B) = E, A \cup B = E$ then $\eta(A \cup B) = \eta(A) + \eta(B)$.

First, we show that $\eta(A) + \eta(B) \leq \eta(E)$. Note that $A \in \mathcal{C}_E$ implies $A^c \cap E \in \mathcal{C}_E$ and, therefore, by monotonicity we may assume that $A \cup B = E$. By Lemma B5(ii), we can choose a set $C \in \mathcal{C}_E$ such that $C \subset B$, $A \cup C \in \mathcal{C}_E$ and $\eta(C) < \epsilon$. Hence, $E \cap (A \cup C)^c = B \backslash C \in \mathcal{C}_E$ and $E \cap C^c = A \cup (B \backslash C)) \in \mathcal{C}_E$. By part (ii) of this lemma, $\eta(B) = \eta(B \backslash C) + \eta(C)$ and $\eta(A \cup (B \backslash C) = \eta(A) + \eta(B \backslash C)$. Hence, $\eta(A) + \eta(B) < \eta(A \cup (B \backslash C)) + \epsilon$. Therefore, $\eta(A) + \eta(B) - \epsilon < \sup_{D \in \mathcal{C}_E} \eta(D)$. Since ϵ is arbitrary, the result follows.

Next, choose any $D \in \mathcal{C}_E$ such that $\eta(D) > \eta(E) - \epsilon$. Let $C_1 = A \cap D, C_2 = B \cap D, C_3 = A \cap (E \backslash D), C_4 = B \cap (E \backslash D)$ and $C_5 = E^c$ and let $\{E_\kappa\}$ for $\emptyset \neq \kappa \subset \{1, \ldots, 5\}$ be a μ-split of $C_1, \ldots, C_5$. Since $E^c \in \mathcal{E}$, E_κ is null for all κ such that $5 \in \kappa, \kappa \neq \{5\}$. Hence, we ignore these elements of the μ-split and assume $E_5 = E^c$; that is, we can find an alternative μ-split in which all these sets are empty and $E_5 = E^c$. Let $F_1, \ldots, F_m$ be the nonnull elements of the μ-split E_κ for κ such that $5 \notin \kappa$ and, therefore, $F_1, \ldots, F_m$ is a partition of E. Since $A, B, D \in \mathcal{C}_E$, Lemma B4(v) implies that $A \cap F_i, B \cap F_i, D \cap F_i \in \mathcal{C}_{F_i}$ for all i. By Lemma B7, $\eta(A) = \sum_i \eta(A \cap F_i)$, $\eta(B) = \sum_i \eta(A \cap F_i)$ and $\eta(D) = \sum_{F_i} \eta(D \cap F_i)$. Since $D \subset A \cup B$ it follows that $D \cap F_i = (D \cap A \cap F_i) \cup (D \cap B \cap F_i)$ for all i. So, $\eta(D \cap F_i) \leq \eta(A \cap F_i) + \eta(B \cap F_i)$ for all i. Hence, $\eta(D) \leq \eta(A) + \eta(B)$ and therefore, $\sup_{D \in \mathcal{C}_E} \eta(D) \leq \eta(A) + \eta(B)$. This proves claim 2.

To complete the proof of the Lemma, let $A = A_1 \cup A_2$, $A_3 = A^c$ and let $\{E_\kappa\}$ be a μ-split of A_1, A_2, A_3. By Lemma A2(iv), $E(A) = E(A_1) \cup E(A_2) \cup E_{12} = E_1 \cup E_2 \cup E_{12}$, and $F(A) = E_{13} \cup E_{23} \cup E_{123}$. Let F_1, F_2, F_3 be a partition such that $\mu(F_i) = 1/3$ for all i and define $\{E^i_\kappa\} = E_\kappa \cap F^i \cap G^c$. Then, for all $i = 1, 2, 3$, Claim 2 implies that

$$\eta(E^i_{12}) = \eta(E^i_{12} \cap A_1) + \eta(E^i_{12} \cap A_2)$$

and, similarly, for all $i = 1, 2, 3$, Claim 1 implies that

$$\eta(E^i_{123} \cap A) = \eta(E^i_{123} \cap A_1) + \eta(E^i_{123} \cap A_2)$$

Note that $E^i_1 \subset A_1, E_2 \subset E^i_2, B \cap E_{13} = A \cap E_{23} = \emptyset$. Therefore, Lemma B7 and the two display equations above imply that

$$\begin{aligned}\eta(A \cap F_i) &= \eta(E^i_1) + \eta(E^i_{12} \cap A_1) + \eta(E^i_{123} \cap A_1) + \eta(E_2) + \eta(E^i_{12} \cap A_2) + \eta(E^i_{123} \cap A_2) \\ &= \eta(A_1 \cap F_i) + \eta(A_2 \cap F_i)\end{aligned}$$

where the second equality follows from Lemma A2 (iii). A further application of Lemmas A2 (iii) and B7 then yields

$$\eta(A) = \sum_{i=1}^{3} \eta(A \cap F_i) = \sum_{i=1}^{3} \eta(A_1 \cap F_i) + \eta(A_2 \cap F_i) = \eta(A_1) + \eta(A_2)$$

as desired. $\square$

Lemma B9: *η is countably additive and non-atomic.*

Proof: It is enough the establish that $A_{n+1} \subset A_n$, $\bigcap A_n = \emptyset$ implies $\lim \eta(A_n) = 0$. Let $E_n = E(A_n)$ and $F_n = F(A_n)$. Clearly, $E_{n+1} \subset E_n, F_{n+1} \subset F_n$ for all n. If $\lim \mu(E_n) > 0$, then countable additivity of μ yields $\mu(\bigcap E_n) = \lim \mu(E_n) > 0$. Hence, $\emptyset \neq \bigcap E_n \subset \bigcap A_n$, a contradiction. Since $\eta(E) \leq \mu(E)$, it follows that $\lim \eta(E_n) = 0$. It remains to show that $\lim \eta(A_n \cap F_n) = 0$.

Let $F = \bigcap_n F_n$. If $\mu(F) = 0$ then, since $\eta(A_n \cap F_n) \leq \mu(F_n)$ and $\lim \mu(F_n) = 0$ the result follows. Thus, assume that $\mu(F) > 0$. Since $\eta(A) = \eta(A \cap G^c)$ we may assume that $F \subset G^c$. By Lemma B8, $\eta(A_n \cap F_n) = \eta(A_n \cap F) + \eta(A_n \cap (F \backslash F_n) \leq \eta(A_n \cap F) + \eta(F \backslash F_n)$. Let N be such that $\mu(F_N \backslash F) < \epsilon$. Then, $\eta(A_n \cap F_n) \leq \eta(A_n \cap F) + \epsilon$ for $n > N$. Let $G_n \in \mathcal{E}$ be such that $\mu(G_n) = \eta(A_n \cap F)$. Then, $A_n \cap F \succeq G_n$ for all n. Let $\mu(G) = \delta > 0$. If $\eta(A_n \cap F) \geq \delta$, for all n, then, by Axiom 5, $\bigcap_n (A_n \cap F) = \emptyset \succeq G$. It follows that $\eta(A_n \cap F) \to 0$.

To see that η is nonatomic, note that by definition, $\eta(A) > 0$ implies $\eta(E(A)) = \eta(E(A) \cap G^c) > 0$ or $\eta(F(A)) > 0$ (and $F(A) \subset G^c$). In the first case, nonatomicity follows from Lemma B5(ii). In the second case, let $B = E \backslash A$ and note that $B \in \mathcal{C}_{F(A)}$. Then, Lemma B5(ii) implies the result also in this case. $\square$

Let

$$\pi(A) = \max_{E \in [A]} \mu(A) + \min_{E \in [A]} \eta(A \backslash E)$$

Lemma B10:

(i) If $A \subset E$ and $B \cap E = \emptyset$, then $\pi(A \cup B) = \pi(A) + \pi(B)$.

(ii) If $\mu(E) = \eta(E)$, then $\mu(F) = \eta(F)$ for all $F \subset E$.

(iii) If $\mu(E) = \eta(E)$, $A_1 \cup A_2 \subset E$, $A_1 \cap A_2 = \emptyset$, then $\pi(A_1 \cup A_2) = \pi(A_1) + \pi(A_2)$.

Proof: (i) This follows from Lemma B9 and the additivity of μ.

(ii) By Lemma B4(i), $\eta(E') \leq \mu(E')$ for all E'. By definition, $\eta(E) = \eta(E \cap G^c)$. Then, $\eta(E \cap G^c) \leq \mu(E \cap G^c) = \mu(E) - \mu(E \backslash G^c)$. So, $\mu(E \backslash G^c) = 0$. Hence, we may assume without loss of generality that $E \subset G^c$. Let $E^* = E \backslash F$. Since μ is additive, $\mu(E) =$

$\mu(F) + \mu(E^*)$. Lemma B9 implies that $\eta(E) = \mu(E) = \eta(F) + \eta(E^*) \leq \eta(F) + \mu(E^*)$. Therefore, $\mu(F) = \eta(F)$ as desired.

(iii) Without loss of generality we may assume that $A \cup B \subset E \subset G^c$. Let $A = A_1 \cup A_2$, $A_3 = A^c$ and let $\{E_\kappa\}$ be a μ-split of A_1, A_2, A_3. Then, by Lemma A2(iii), $E(A) = E(A_1) \cup E(A_2) \cup E_{12} = E_1 \cup E_2 \cup E_{12}$ and $F(A) = E_{13} \cup E_{23} \cup E_{123}$. From the definition of π and the additivity of μ and η it follows that

$$\pi(A) = \mu(E_1) + \mu(E_2) + \mu(E_{12}) + \eta(A \cap E_{13}) + \eta(A \cap E_{23}) + \eta(A \cap E_{123})$$

$$\pi(A_1) = \mu(E_1) + \eta(A_1 \cap E_{12}) + \eta(A_1 \cap E_{13}) + \eta(A_1 \cap E_{123})$$

$$\pi(A_2) = \mu(E_2) + \eta(A_2 \cap E_{12}) + \eta(A_2 \cap E_{23}) + \eta(A_2 \cap E_{123})$$

Since $(A_1 \cap E_{12}) \cup (A_2 \cap E_{12}) = E_{12}$ it follows from Lemma B8(ii) that $\eta(A_1 \cap E_{12}) + \eta(A_2 \cap E_{12}) = \eta(E_{12})$ and, by part (ii), $\eta(A_1 \cap E_{12}) + \eta(A_2 \cap E_{12}) = \mu(E_{12})$. The additivity of η then implies the result. $\square$

Lemma B11:

(i) $\pi(A) = \max\{\mu(E) \,|\, A \succeq E\}$.

(ii) π *represents* $\succeq$.

(iii) $(\mu, \mathcal{E})$ *and* (η, Σ) *are compatible.*

Proof: (i) If $A \in \mathcal{E}$, then $\pi(A) = \mu(E)$ and hence the result is obvious. Suppose $A \notin \mathcal{E}$ Then, $A = E(A) \cup (A \cap F(A))$ and $\pi(A) = \mu(E(A)) + \eta(A \cap F(A))$ and $F(A) \subset G^c$. Let $a = \sup\{\mu(E) \,|\, A \succeq E\}$ let E_n be such that $E_n \subset E_{n+1}$, $\bigcup E_n = E$, $\mu\left(\bigcup_{n=1}^k E_n\right) < a$ and $\lim\left(\bigcup E_n\right) = a$. Then, $A \succeq \bigcup_{n=1}^k E_n$ for all k and hence $\left(\bigcup_{n=1}^k E_n\right)^c \succeq A^c$ by consistency and therefore $E^c = \left(\bigcap E_n\right)^c \succeq A^c$ by Axiom 5. Hence, $A \succeq E$, again by consistency. Note that $\mu(E(A) \cup F(A)) > \mu(E) \geq \mu(E(A))$. Therefore, we may choose E such that $E(A) \subset E \subset E(A) \cup F(A)$. Since $A \succeq E$, we have $A \cap F(A) \succeq E \backslash E(A)$. Therefore, $\eta(A \cap F(A)) \geq \mu(E) - \mu(E(A))$ and $\pi(A) \geq a$.

If $\pi(A) > a$, then we can find F such that $F \subset E(A) \cup F(A)$, $\mu(F) > a$ and $E(A) \cup (A \cap F(A)) \succeq F$. Hence $A \cap F(A) \succeq F \backslash E(A)$. That is, $\eta(A) > \pi(A) - \mu(E(A))$, contradicting the definition of π. This completes the proof of (i).

For (ii), suppose $A \succeq B$, then by consistency $W(B) \subset W(A)$ and $W(A^c) \subset W(B^c)$. Take E, F such that $\pi(B) = \mu(E)$ and $\pi(A^c) = \mu(F)$. By part (i), E and F are well-defined. Then, by calibration, $A \succeq E$ and $B^c \succeq F$ and hence $\pi(A) \geq \mu(E) = \pi(B)$ and $\pi(B^c) \geq \mu(F) = \pi(A^c)$. For the converse, suppose $\pi(A) \geq \pi(B)$ and $\pi(B^c) \geq \pi(A^c)$. Then, take F such that $\mu(F) = \pi(A)$. By part (i), $A \succeq F$. Then, $B \succeq E$ implies $\mu(F) \geq \mu(E)$ and hence $A \succeq F \succeq E$. Hence, $W(B) \subset W(A)$. The same argument establishes that $W(A^c) \subset W(B^c)$ and hence $A \succeq B$ by calibration.

For (iii), note that $\eta(E) = \eta(E \cap G^c)$ by construction. If $\eta(E \cap G^c) = 0$, we are done. Otherwise, let $E_0 = E \cap G^c$. By definition, $\eta(E \cap G^c) = \eta(E_0)$. Lemma B4(i) implies that $\eta(E_0) \leq \mu(E_0)$. To conclude the proof, we will show that this inequality is strict. Pick any $A \in \mathcal{C}_{E_0}$. Then, by Lemma B9, $\eta(E_0) = \eta(A) + \eta(B)$ for $B := E_0 \backslash A$. To conclude the proof, we will show that if $\eta(A) = \mu(E_0) - \eta(B)$, then $A \in \mathcal{E}$, which would contradict the fact that $A \in \mathcal{C}_{E_0}$. To prove this, we will show that $\eta_{E_0}(A) = \mu(E_0) - \eta_{E_0}(B)$ and $(C \cup D) \cap A = \emptyset$ implies $[\pi(C) \geq \pi(D), \pi(D^c) \geq \pi(C^c)]$ if and only if $[\pi(C \cup A) \geq \pi(D \cup A)$, $\pi(D^c \cap A^c) \geq \pi(C^c \cap A^c)]$ and appeal to part (ii) of this lemma.

Note that $\pi(C \cup A) = \pi(((C \cup A) \cap E_0) \cup (C \cup A) \cap E_0^c) = \pi((C \cup A) \cap E_0) + \pi((C \cup A) \cap E_0^c)$ by Lemma B10(i). By Lemma B10(iii), $\pi(C \cup A) = \pi(C \cap E_0^c) + \pi(C \cap E_0) + \pi(A \cap E_0)$. Thus, by Lemma B10(i), we have (a) $\pi(C \cup A) = \pi(C) + \pi(A)$. Similarly, $\pi(C^c) = \pi(E_0^c \cap C^c) + \pi(C^c \backslash E^c) = \pi(A) + \pi(B \cap C^c) + \pi(C^c \backslash E^c) = \pi(A) + \pi(A^c \cap C^c)$; that is, (b) $\pi(C^c) = \pi(A) + \pi(A^c \cap C^c)$. But, (a) and (b) imply $[\pi(C) \geq \pi(D), \pi(D^c) \geq \pi(C^c)]$ if and only if $[\pi(C \cup A) \geq \pi(D \cup A), \pi(D^c \cap A) \geq \pi(C^c \cap A^c)]$ as desired. $\square$

Proof of Proposition 2: Let $\mathcal{E}_o$ be the set of $\succeq$-unambiguous events. Since π and $\hat{\pi}$ both represent $\succeq$, $\mathcal{E} =\subset \mathcal{E}_o$ and $\hat{\mathcal{E}} \subset \mathcal{E}_o$. By Proposition 1, $\mathcal{E}_\pi = \mathcal{E}$ and $\mathcal{E}_{\hat{\pi}} = \hat{\mathcal{E}}$. Suppose there is $A \in \mathcal{E} \backslash \hat{\mathcal{E}}$. Then $\hat{\pi}(A) < 1 - \hat{\pi}(A^c)$. Choose $E \in \hat{\mathcal{E}}$ such that $\hat{\pi}(A) < \hat{\pi}(E) < 1 - \hat{\pi}(A^c)$. Then, $A \in \mathcal{E}_o$, $A \not\succeq E$ and $E \not\succeq A$ contradicting Axiom 2. Hence $\mathcal{E} = \mathcal{E}_*$. Then, familiar arguments from the uniqueness of subjective probability in the Savage setting ensure that $\mu = \mu_*$.

Next, assume w.l.o.g. that $\eta(A) > \hat{\eta}(A)$ for some A. Choose E such that $\eta(A) > \mu(E) = \hat{\mu}(E) > \hat{\eta}(A)$. Since π represents $\succeq$, we $A \succeq E$. Since $\hat{\pi}$ represents $\succeq$, we have $A \not\succeq E$, a contradiction. $\square$

8. Appendix C: Proof of Propositions 3 and 4

Proof of Proposition 3: The 'only of' part of the representation theorem is straightforward and, therefore, omitted. Let $\succeq$ be a QUA and let π be the capacity that represents it. Let $\bar{\pi}$ be the 'dual' of π, that is, $\bar{\pi}(A) = 1 - \pi(A^c)$. Note that $\Omega \succeq A$ for all A and, therefore, by (A^o), for every A there exists E such that $E \sim^o A$. Define $\rho : I \to I\!R$ as follows:

$$\rho(a, b) = \pi(E) \text{ such that } E \sim^o A \text{ and } (a, b) = (\pi(A), \bar{\pi}(A))$$

First, we will show that ρ is well defined. Let A, B be such that $\pi(A) = \pi(B), \bar{\pi}(A) = \bar{\pi}(B)$. Then, since π represents $\succeq$, we have $A \sim B$ and since $\succeq^o$ is an extension of $\succeq$, we have $A \sim^o B$. Hence, ρ is well defined.

Next, we will show that ρ. Let $(a_n, b_n) \in I$ be a convergent sequence and let $(a, b) \in I$ be its limit. Consider a convergent subsequence of $\rho(a_n, b_n)$ and let r be its limit. If $r > \rho(a, b)$, then, since $\succeq$ is represented by an uncertainty measure, there is E^* such that $\pi(E^*) = \rho(a, b)/2 + r/2$. Then, by (A^o), there exists E, A such that $(\pi(A), \bar{\pi}(A)) = (a, b)$ and $A \cup E \sim^o E^*$ and therefore, $\rho(A \cup E) = \rho(a, b)/2 + r/2$. Since $E \cap A^c$ is not null, it follows that $\pi(A \cup E) > a, \bar{\pi}(A \cup E) > b$. Then (C) implies that $\rho(a_n, b_n) \leq \rho(a, b)/2 + r/2$ for large n, yielding the desired contradiction. If $\rho(a, b) > r$, then, arguing as above, we can find E, A such that $(\pi(A), \bar{\pi}(A)) = (a, b)$ and $\rho(A \cap E) = \rho(a, b)/2 + r/2$. Since $E \cap A^c$ is not null, it follows that $\pi(A \cap E) < a, \bar{\pi}(A \cap E) < b$. Then (C) implies that $\rho(a_n, b_n) \geq \rho(a, b) + r/2$ for large n, again yielding the desired contradiction.

To conclude the proof of sufficiency, we will prove that ρ is either strict, maximally ambiguity averse, or maximally ambiguity loving. If $\succeq$ is maximally ambiguity averse (loving), then (C) implies $\rho(\pi(A), \bar{\pi}(A)) = \pi(A)$ $(\bar{\pi}(A))$. Hence, ρ is maximally ambiguity averse (loving). That ρ is strict if $\succeq^o$ is a strict is immediate.

Finally, to prove the uniqueness of the representation, let (π, ρ) and (π', ρ') be two representations of the preference $\succeq^o$ such that $\pi = (\mu, \eta)$ and $\pi' = (\mu', \eta')$. We will first show that $\mu = \mu'$ and $\eta = \eta'$. Let $\mathcal{E} = \mathcal{E}_\mu, \mathcal{E}' := \mathcal{E}_{\mu'}$. First, we show that $\mathcal{E} = \mathcal{E}'$.

Assume $A \in \mathcal{E}', A \notin \mathcal{E}$. Let E_1, E_{12}, E_2 be a μ-split of A. That is, $E_1 \subset A$ is the maximal (μ, η)-unambiguous subset of A; $\mu(A \cap E_{12}) = \mu(E_{12} \backslash A) = 0$, and $A \subset E_1 \cup E_{12}$.

Let $B := E_{12} \setminus A$. Partition E_{12} into two subsets $F_1, F_2 \in \mathcal{E}$ and partition B into two subsets B_1, B_2 such that $\eta(B_i \cap F_k) = \eta(B_j \cap F_l) = \alpha > 0$ for all $i, j \in \{1,2\}, k, l \in \{1,2\}$. Since μ and η are non-atomic, it is straightforward to show that such a partition exists. Then, let

$$C_1 := (B_1 \cap F_1) \cup (B_2 \cap F_1)$$

$$C_2 := (B_1 \cap F_1) \cup (B_2 \cap F_2)$$

Note that

$$\pi(C_1) = \pi(C_2) = 2\alpha$$

$$1 - \pi(C_1^c) = 2\alpha + \mu(F_1) - \eta(F_1)$$

$$1 - \pi(C_2^c) = 1 - \pi(C_1^c) + \mu(F_2) - \eta(F_2)$$

Therefore, $\pi(C_1) = \pi(C_2)$ and $1 - \pi(C_2^c) > 1 - \pi(C_1^c)$. Next, note that

$$\pi(A \cup C_1) = \pi(A \cup C_2) + (\mu(F_1) - \eta(F_1))$$

$$\pi([A \cup C_1]^c) = \pi([A \cup C_2]^c)$$

Thus, if ρ is strictly increasing in the lower bound, then $A \cup C_1 \succ^o A \cup C_2$ while $C_2 \succeq^o C_1$ whereas if ρ is strictly increasing in the upper bound, then $C_2 \succ^o C_1$ but $A \cup C_1 \succeq^o A \cup C_2$. In either case, we obtain a contradiction to the fact that $A \in \mathcal{E}'$ and, hence, unambiguous. We conclude that $\mathcal{E} = \mathcal{E}'$. For all $E, F \in \mathcal{E}$, we have $E \succeq^o F$ if and only if $\mu(E) \geq \mu(F)$ if and only if $\mu'(E) \geq \mu'(F)$. The uniqueness of the probability representation in Savage's theorem then implies that $\mu = \mu'$.

Given the unique μ, let (G, G^c) be an essential partition of Ω; recall that $G \in \mathcal{E}$ is whole and G^c is blank. If $\mu(G^c) = 0$, then we are done since $\eta = \eta' = 0$. Thus, assume $\mu(G^c) > 0$ and let $\mathcal{C} = \{D : \mu_*(D) = 0 = \mu_*(G^c \setminus D)\}$. By the definition of an essential partition, $\mathcal{C}$ is not empty. Partition G^c into two subsets G_1, G_2 such that $\mu(G_1) = \mu(G_2)$. Consider any $A \subset G_1$ such that $\mu_*(A) = 0$. Choose $D \in \mathcal{C}$ such that $D \cap A = \emptyset$. Let $D_1 = G_1 \cap D$. Let $F \in \mathcal{E}, F \subset G_2$ be such that $F \cup D_1 \sim^o A \cup D_1$. Note that

$$\pi(A \cup D_1) = \eta(A) + \eta(D_1)$$

$$\pi(F \cup D_1) = \mu(F) + \eta(D_1)$$

$$1 - \pi([A \cup D_1]^c) = 1 - \pi(D_1^c) + \eta(A)$$

$$1 - \pi([F \cup D_1]^c) = 1 - \pi(D_1^c) + \mu(F)$$

Therefore, $F \cup D_1 \sim^o A \cup D_1$ implies that $\mu(F) = \eta(A)$. The same argument shows that $\mu(F) = \eta'(A)$. Thus, it follows that $\eta(A) = \eta'(A)$ for all $A \subset G_1$ such that $\mu_*(A) = 0$. For an arbitrary $A \subset G_1$, let $C_n \in \mathcal{C}$ such that $\eta(C_n) \leq 1/n$ and define $D_n = C_n \cap G_1$. By Lemma 4(ii), C_n exists for all n. Moreover, $\mu_*(D_n) = \mu_*(A \backslash D_n) = 0$ and, therefore, $\eta(A \backslash D_n) = \eta'(A \backslash D_n)$ and $\eta(D_n) = \eta'(D_n)$. Since,

$$\eta(A \backslash D_n) \leq \eta(A) \leq \eta(A \backslash D_n) + \eta(D_n)$$
$$\eta'(A \backslash D_n) \leq \eta'(A) \leq \eta'(A \backslash D_n) + \eta'(D_n)$$

it follows that $|\eta(A) - \eta'(A)| \leq \eta(D_n) \leq 1/n$. Since this inequality holds for all n, it follows that $\eta(A) = \eta'(A)$, for all $A \subset G_1$. Note that, by symmetry, the same argument shows that $\eta(A) = \eta'(A)$ for all $A \in G_2$. Since $\eta(A) = \eta(A \cap G_1) + \eta(A \cap G_2)$ and $\eta'(A) = \eta'(A \cap G_1) + \eta'(A \cap G_2)$ it follows that $\eta = \eta'$, as desired.

Let $(a, b) = (\pi(A), \bar{\pi}(A)) = (\pi'(A), \bar{\pi}'(A))$. Choose $E \in \mathcal{E} = \mathcal{E}'$ such that $A \sim^o E$. Hence, $\rho(a, b) = \pi(E) = \pi'(E) = \rho'(a, b)$ proving that $\rho = \rho'$. □

Proof of Proposition 4: Let (a, b) be such that $b > a$ and assume $L_\rho > L_{\hat{\rho}}$. Hence,

$$\alpha := \frac{L_\rho(a, b)}{1 + L_\rho(a, b)} > \frac{L_{\hat{\rho}}(a, b)}{1 + L_{\hat{\rho}}(a, b)} =: \hat{\alpha}$$

Let $\beta \in (\hat{\alpha}, \alpha)$. Let $a'(\epsilon) = a + (1 - \beta)\epsilon, b'(\epsilon) = b - \beta\epsilon$. Note that

$$\lim_{\epsilon \to 0} \frac{\rho(a, b) - \rho(a'(\epsilon), b'(\epsilon))}{\epsilon} = \rho_1(a, b)(1 - \beta) - \rho_2(a, b)\beta$$
$$\lim_{\epsilon \to 0} \frac{\hat{\rho}(a, b) - \hat{\rho}(a'(\epsilon), b'(\epsilon))}{\epsilon} = \hat{\rho}_1(a, b)(1 - \beta) - \hat{\rho}_2(a, b)\beta$$

Since $\beta > \hat{\alpha}$, $0 > \hat{\rho}_1(a, b)(1-\beta) - \hat{\rho}_2(a, b)\beta$ and since $\beta < \alpha$, $\rho_1(a, b)(1-\beta) - \rho_2(a, b)\beta > 0$. Therefore, $\hat{\rho}$ is not more ambiguity averse than ρ proving necessity of the condition for all (a, b) such that $b > a$.

To prove sufficiency, first assume that both ρ and $\hat{\rho}$ are strict. Let $a \leq a' \leq b' \leq b$ and $\rho(a', b') \geq \rho(a, b)$. By the monotonicity and continuity of ρ, we may choose $(a^*, b^*) \leq (a', b')$ such that $\rho(a^*, b^*) = \rho(a, b)$. For $c \in (a, a^*)$, let $f(c) \in [0, 1]$ be real number such that $\rho(c, f(c)) = \rho(a, b)$. Note that f is a continuously differentiable function and $f' = -\rho_1/\rho_2$. Since $\rho_2 > 0$, f' is well defined. Therefore,

$$\rho(a^*, b^*) - \rho(a, b) = \int_a^{a^*} [\rho_1(c, f(c)) + \rho_2(c, f(c)] f'(c) dc = 0$$

By assumption $\hat{\rho}_1(a, b)/\hat{\rho}_2(a, b) \geq \rho_1(a, b)/\rho_2(a, b)$ and, therefore,

$$\hat{\rho}(a^*, b^*) - \hat{\rho}(a, b) = \int_a^{a^*} [\hat{\rho}_1(c, f(c)) + \hat{\rho}_2(c, f(c)] f'(c) dc \geq 0$$

Then, monotonicity implies $\hat{\rho}(a', b') \geq \hat{\rho}(a, b)$, as desired.

If $\hat{\rho}$ is maximally ambiguity averse, then $\hat{\rho}(a', b') \geq \hat{\rho}(a, b)$ for all (a', b') such that $a' \geq a$ and hence there is nothing to prove. If ρ is maximally ambiguity averse, then $L_{\hat{\rho}} \geq L_\rho$ implies $\hat{\rho}$ is also maximally ambiguity averse and again we are done. Similarly, if ρ is maximally ambiguity loving, there is nothing to prove; if $\hat{\rho}$ is maximally ambiguity loving, then $L_{\hat{\rho}} \geq L_\rho$ implies ρ is also maximally ambiguity loving and again, we are done. $\square$

9. Appendix D: Proof of Theorem 2

Assume that $\succeq^*$ is a sophisticated extension of $\succeq$ (that is, $\succeq^*$ satisfies (X) and (WS)) and, in addition, satisfies (A), (S) and (P4). Let π be the uncertainty measure that represents $\succeq$. In particular, $\pi(E) = \mu(E)$ for every $\succeq$-unambiguous E. Let $\mathcal{E}$ be the set of $\succeq$-unambiguous events and let $\mathcal{F}_e$ be the set of all $\mathcal{E}$-measurable acts.

Step 1: (Monotonicity) If $y > x$, then $\pi(A) = 0$ implies $yAh \sim^* xAh$ and $\pi(A) > 0$ implies $yAh \succ^* xAh$.

Proof: If $\pi(A) = 0$, then, for $B = \emptyset$, we have $zBw(A \cup B)w = w \sim^* zAw = zAw(A \cup B)w$ since $\succeq^*$ is an extension of $\succeq$. Then, (WS) yields $xAh = yBx(A \cup B)h \sim^* yAx(A \cup B)h = yAh$. If $\pi(A) > 0$, we have $zBw(A \cup B)w \not\succeq^* zAw(A \cup B)w$, since $\succeq^*$ is an extension of $\succeq$ and $B = \emptyset$. Then, (WS) implies $yBx(A \cup B)h \not\succeq^* yAx(A \cup B)h$ and hence $xAh \not\succeq^* yAh$; that is, $yAh \succ^* xAh$. $\square$

Step 2: There exists a strictly increasing, continuous and onto utility index $u : X \to [0, 1]$ such that $u^{-1}(\pi(E)) \sim^* zEw$.

Proof: By (A) there exists E such that $x \sim^* zEw$. Let $u(x) = \pi(E)$. By (X), $zEw \sim^* zFw$ if and only if $\pi(E) = \pi(F)$. So, u is well-defined. Suppose $y > x$ and $y \sim^* zEw$, $x \sim^* zFw$. Then, by Step 1, $y \succ^* x$ and, therefore, $zEw \succ^* zFw$. It follows that $E \succ F$ and therefore $\pi(E) > \pi(F)$, proving that u is strictly increasing. For any $\alpha \in [0,1]$, we may choose E such that $\mu(E) = \alpha$. By (A), there is x such that $x \sim^* zEw$. Therefore, u is onto. To conclude the proof of Step 2, we note that a strictly increasing, onto function from a one compact interval to another must be continuous. $\square$

Step 3: For all $0 < a \leq b \leq 1$, there is $x(a,b) \in [0,1]$ such that $E \subset F$, $\pi(F) = b$, $\pi(E) = a$ implies $x(a,b)Fh \sim^* (zEw)Fh$ for all h.

Proof: Let $x(a,b) = x$ such that $xFw \sim^* zEw$. Let E', F' be two other unambiguous sets such that $\pi(E') = a$ and $\pi(F') = b$. By (X) and (P4), $xF'w \sim^* xFw$ and $zE'w \sim^* zEw$. Therefore, the x above only depends on a and b, not on the specific choice of E, F. Then, (S) yields $x(a,b)Fh \sim^* (zEw)Fh$ for all h and E, F such that $\pi(E) = a, \pi(F) = b$. $\square$

A real number b *dyadic* if $\pi(E) = k2^{-n}$ for some integers k, n.

Step 4: For all $0 < a \leq b \leq 1$, $x(a,b) = u^{-1}(a/b)$.

Proof: First, we will show that if $b = 2^{-n}$ for some integer n, $E \subset F$ and $\pi(E) = ab$, then $u^{-1}(a)Fh \sim^* (zEw)Fh$. If $n = 0$, then the result follows from Step 2. Next, suppose the result holds for n and assume $\pi(F) = b = 2^{-n-1}$. Let $F_1 = F$ and partition F^c into $2^n + 1$ sets, $F_2, \ldots, F_{2^{n+1}}$ such that $\pi(F_i) = 2^{-n-1}$. Since π is nonatomic, this can be done. Then, set $E_1 = E$ and for each $i = 2, \ldots, 2^{n+1}$ choose $E_i \subset F_i$ such that $\pi(E_i) = ab$. Let $h(s) = z$ if $s \in \bigcup E_i$ and $h(s) = w$ otherwise, Then, by Step 3,

$$h \sim^* x(ab,b)F_1h \sim^* x(ab,b)(F_1 \cup F_2)h$$

which, in turn, implies that $x(ab,b) = x(2ab, 2b)$. By the inductive hypothesis, $x(2ab, 2b) = u^{-1}(ab)$ and hence $x(ab,b) = u^{-1}(a)$ as desired. Extending the result to arbitrary dyadic b's is straightforward. To conclude the proof, we will show that the result holds for arbitrary b. Suppose $x(ab,b) < u^{-1}(a)$ and choose a dyadic number d such that

$$u(x(ab,b)) \cdot b + (b - a) < d < b$$

Then, choose $E' \subset E \subset F$ such that $\pi(E) = a$, $\pi(F) = b$ and $c := \pi(E') = d - b + a$. Let $F' = (F \backslash E) \cup E'$. Note that $\pi(F') = d$ and hence F' is dyadic and conclude that $x(c,d) = u^{-1}(c/d)$. Hence, we have

$$(x(c,d)F'z)Fw \sim^* x(a,b)Fw$$

Our choice of E', E, F' and F yields $u(x(c,d)) > u(x(a,b))$ and hence $x(c,d) > x(a,b)$ which means $(x(c,d)F'z)Fw$ yields a strictly larger prize at every state in F than $x(c,d)Fw$ which given monotonicity, contradicts the display equation above. A symmetric argument yields a contradiction for $x(a,b) > u^{-1}(a/b)$. $\square$

Let $E(uf) = \int u \circ f d\pi$ for all $f \in \mathcal{F}_e$.

Step 5: $f \succeq^* g$ if and only if $E(uf) \geq E(ug)$ for all $f, g \in \mathcal{F}_e$.

Proof: The proof is by induction on the cardinality of the set $\{x \in (w,z) \,|\, \pi(f^{-1}(x)) > 0\}$. Let $n(f)$ denote this cardinality. If $n(f) = 0$, then the result follows from Step 2 and the definition of u. Suppose the result holds for all f' such that $n(f') = n$ and assume $n(f) = n + 1$. Let $F = f^{-1}(y)$ and pick any $y \in \{x \in (w,z) \,|\, \pi(f^{-1}(x)) > 0\}$. By (A), there is $E \subset F$ such that $yFw \sim^* (zEw)Fw$ and, by (S), $f = yFf \sim^* (zEw)Ff := g$. By Step 4, $E(ug) = E(uf)$ and, by construction, $n(g) = n$. Then, the inductive hypothesis yields $f \sim^* g \sim^* zE'w$ for E' such that $\pi(E') = E(ug) = E(uf)$. For the converse, note that if $\pi(E^*) \neq \pi(E')$, then Step 2 ensures that zE^*w is not indifferent to $zE'w$ and hence not indifferent to f. $\square$

If $\mathcal{E} = \Sigma$, Step 5 ensures that $\succeq^*$ has an expected utility representation. Then $C^\pi = \{\pi\}$ and choosing any α establishes the desired representation. Henceforth, we assume that $\mathcal{E} \neq \Sigma$ and choose some G such that (G, G^c) is an essential partition of $\succeq$.

Step 6: For all $E \subset G^c$, $A \in \mathcal{C}_E$ and F' satisfying $\pi(E) > 0$, $E \cap F' = \emptyset$, there is a unique $\beta_E \in [0,1]$ such that for all $y > x$, $A \in \mathcal{C}_E$, $yA \cup F'x \succeq^* yFx$ if and only if $\eta(A) + \pi(F') + \beta_E(\pi(E) - \eta(E)) \geq \pi(F)$.

Proof: Let $A \in \mathcal{C}_E$. By (A) there exists F^* such that $zA \cup F'w \sim^* zF^*w$. From (X) it follows that $F^* \not\succ A \cup F'$ and $A \cup F' \not\succ F^*$. Since F^* is unambiguous, it follows that $\pi(F^*) \geq$

$\eta(A) + \pi(F')$ and $1 - \pi(F^*) \geq \pi((A \cup F')^c) = \eta(E \backslash A) + \pi((E \cup F')^c)$. Since $\eta(E\backslash A) = \eta(E) - \eta(A)$, the second inequality simplifies to $\pi(F^*) \leq \eta(A) + \pi(F') + \pi(E) - \eta(E)$. It follows that there is some $\beta_E \in [0,1]$ such that $\eta(A) + \pi(F') + \beta_E(\pi(E) - \eta(E)) = \pi(F^*)$. Since $\pi(E) > 0$, this β_E is unique. We have established that $z(A \cup F')w \sim^* zF^*w$ implies that $\eta(A) + \pi(F') + \beta_E(\pi(E) - \eta(E)) = \pi(F)$. Since $F^* \succeq F$ if and only if $\pi(F^*) \geq \pi(F)$, Step 6 now follows from (X) and (P4).

Step 7: $\beta_E = \beta_F$ for all $E, F \subset G^c$ such that $\pi(E) \cdot \pi(F) > 0$ and $\pi(E) - \eta(E) = \pi(F) - \eta(F)$.

Proof: Without loss of generality, assume $\pi(E) \geq \pi(F)$. Choose $F' \subset F^c$ such that $\pi(F') = \pi(E) - \pi(F)$ and $A \in \mathcal{C}_E, B \in \mathcal{C}_F$ such that $\eta(A) - \eta(B) = \pi(F')$. Since $\eta(E) - \eta(F) = \pi(E) - \pi(F)$ and η is nonatomic, this can be done. Then, note that

$$\pi(A) = \eta(A) = \eta(B) + \pi(F') = \pi(B \cup F')$$

$$\pi(A^c) = 1 - \pi(E) + \eta(E) - \eta(A) = 1 - \pi(F) - \pi(F') + \eta(F) - \eta(B) = \pi((B \cup F')^c)$$

Hence, $A \sim B \cup F'$ and, by (X), $zAw \sim^* z(B \cup F')w$. By (A) there is F^* such that $zAw \sim^* zF^*w$. By Step 6, $\eta(A) + \beta_E(\pi(E) - \eta(E)) = \pi(F^*)$. Since $zAw \sim^* z(B \cup F')w$, Step 6 implies $\eta(B) + \pi(F') + \beta_F(\pi(F) - \eta(F)) = \pi(F^*)$. The last two equations, and the fact that $\pi(E) - \eta(E) = \pi(F) - \eta(F) > 0$ yield $\beta_E = \beta_F$. □

Step 8: For all $E, F \subset G^c$ such that $\pi(E) \cdot \pi(F) > 0$ and $E \cap F = \emptyset$,

$$\beta_{E \cup F} = \frac{(\pi(E) - \eta(E))\beta_E + (\pi(F) - \eta(F))\beta_F}{\pi(E) + \pi(F) - \eta(E) - \eta(F)}$$

Proof: Choose $A \in \mathcal{C}_E$, $B \in \mathcal{C}_F$, $F_1 \subset E$ and $F_2 \subset F$ such that $\pi(F_1) = \eta(A) + \beta_E(\pi(E) - \eta(E))$ and $\pi(F_2) = \eta(B) + \beta_F(\pi(F) - \eta(F))$. Then, by Steps 6 and 7, $(zAw)Ew = zAw \sim^* zF_1w = (zF_1w)Ew$ and $(zBw)Fw = zBw \sim^* zF_2w = (zF_2w)Fw$. By (S), $(zAw)E(zBw) \sim^* (zF_1w)E(zBw) \sim^* (zF_1w)E(zF_2w) = z(F_1 \cup F_2)w$. Hence, $\eta(A \cup B) + \beta_{E \cup F}(\pi(E \cup F) - \eta(E \cup F)) = \pi(F_1 \cup F_2) = \pi(F_1) + \pi(F_2) = \eta(A) + \eta(B) + \eta(\pi(E) - \eta(E)) + \beta_F(\pi(F) - \eta(F))$. Since $\eta(A \cup B) = \eta(A) + \eta(B)$, the desired result follows. □

For all $E \subset G^c$ such that $\pi(E) > 0$, let $\nu(E) = \frac{\pi(E) - \eta(E)}{\pi(G^c) - \eta(G^c)}$. Clearly, ν is countably additive, nonnegative, and $\nu(G^c) = 1$. If follows that ν is a nonatomic probability measure on the σ-algebra $\mathcal{E}^* = \{E \subset G^c\}$.

Step 9: For all $E, F \subset G^c$ such that $\nu(E) \cdot \nu(F) > 0$ and $\nu(E), \nu(F)$ are both rational, $\beta_E = \beta_F$.

Proof: By assumption, there are k, m and n such that $\nu(E) = k/n$ and $\nu(F) = m/n$. Partition G^c into n sets $E_1, \dots, E_n$ such that $\nu(E_i) = 1/n$ for all i. By Step 7, $\beta_{E_i} = \beta_{E_j}$ for all i, j. Let $F_1 = \bigcup_{i=1}^k E_i$ and $F_2 = \bigcup_{i=1}^m E_i$. By Step 8, $\beta_{F_1} = \beta_{E_1} = \beta_{F_2}$. Again, by Step 7, $\beta_E = \beta_{F_1}$ and $\beta_F = \beta_{F_2}$ and therefore $\beta_E = \beta_F$. $\square$

Step 10: $\beta_E = \beta_F$ for all $E, F \subset G^c$ such that $\nu(E) \cdot \nu(F) > 0$.

Proof: Suppose $\beta_E > \beta_F$ for some $E, F \subset G^c$ such that $\nu(E) \cdot \nu(F) > 0$. Then choose real numbers $a, b \in (0, 1)$ such that

$$(\beta_E - (1 - a))/a > \beta_F/b$$

Since $\beta_E > \beta_F$ such a, b exist. Then choose $E' \subset E$ and $F' \subset F$ such that $\nu(E')$, $\nu(F')$ are both rational and $\nu(E') \geq a\nu(E)$, $\nu(F') \geq b\nu(F)$. By Step 8,

$$\beta_E = [\nu(E')\beta_{E'} + (\nu(E \backslash E')\beta_{E \backslash E'}]/\nu(E)$$
$$\beta_F = [\nu(F')\beta_{F'} + (\nu(F \backslash F')\beta_{F \backslash F'}]/\nu(F)$$

Since $\beta_{E'}, \beta_{F'} \in [0, 1]$, it follows that

$$(\beta_E - (1 - c))/c \geq (\beta_E - (1 - a))/a > \beta_F/b \geq \beta_F/d$$

where $c = \nu(E')/\nu(E)$ and $d = \nu(F')/\nu(F)$. Thus, we conclude that $\beta_{E'} > \beta_{F'}$, $\nu(E') \cdot \nu(F') > 0$ and both $\nu(E'), \nu(F')$ are rational numbers, a contradiction. Hence, $\beta_E \leq \beta_F$ and by symmetry $\beta_F \leq \beta_E$ completing the proof $\square$.

Let β the common β_E for all E. For any f and $E \subset G^c$ such that $\pi(E) > 0$, let

$$\mathcal{F}_E = \{f \mid f^{-1}(x) \cap E \neq \emptyset \text{ implies } f^{-1}(x) \in \mathcal{C}_E\}$$

Note that $f \in \mathcal{F}_E$ if and only if $fEh \in \mathcal{F}_E$ for all h. Let

$$V_E(f) = \sum_y u(y)\eta(E \cap f^{-1}(y)) + \beta(\pi(E) - \eta(E))u(\max f(E))$$
$$+ (1 - \beta)(\pi(E) - \eta(E))u(\min f(E))$$

Step 11: Let $E \subset G^c, \pi(E) > 0$, and $f \in \mathcal{F}_E$. Then, $fEh \sim_* xEh$ if and only if $V_E(fEh) = \pi(E)u(x)$

Proof: We will prove the assertion by induction over the cardinality of $f(E)$. By definition $f_E(E)$ has cardinality greater or equal to 2. Suppose $f(E) = \{x_1, x_2\}$, $x_1 > x_2$, and $fEh \sim^* xEh$. Then $fEh = (x_1 A x_2)Eh$ for some $A \in \mathcal{C}_E$. By (A), we may choose $F \subset E, x$ such that $(x_1 A x_2)Eh \sim^* (x_1 F x_2)Eh \sim^* xEh$. By (S), $x_1 A x_2 \sim^* x_1 F x_2 \sim^* xEx_2$. Steps 6 and 10 yield $a := \eta(A) + \beta(\pi(E) - \eta(E)) = \pi(F)$ which is equivalent to $au(x_1) + (1-a)u(x_2) = \pi(F)u(x_1) + (1 - \pi(F)u(x_2)$. Since $x_1 F x_2 \sim^* xEx_2$, Step 5 yields $\pi(F)u(x_1) + (1 - \pi(F)u(x_2) = \pi(E)u(x) + (1 - \pi(E)u(x_2)$; it follows that

$$V_E(fEh) = V_E((x_1 A x_2)Ex_2) = au(x_1) + (\pi(E) - a)u(x_2) = \pi(E)u(x)$$

The argument reverses to yield $V_E(fEh) = au(x_1) + (\pi(E) - a)u(x_2)$ implies $fEh \sim^* xEh$.

Assume that the result is true for all f' such that $f'(E) = n-1 \geq 2$. Let $\{x_1, \ldots, x_n\} = f(E)$ such that $x_1 > x_2 > \ldots > x_n$ and let $A_i = f^{-1}(x_i) \cap E$. We first prove the result for $\pi(E) \leq 1/2$. Note that $A_i \in \mathcal{C}_E$ for all i and $A_i \cong A_j$ for all i, j. Define $f_1 = (x_1 A_1 x_2)Ef$ and $f_2 = (x_1 A_1 \cup A_2 x_2)Ef$. Note that $V_E(f_2) - V_E(f_1) = \eta(A_2)(u(x_1) - u(x_2))$. By the inductive hypothesis, $y_i E x_n \sim^* f_i E x_n$ for $u(y_i)\pi(E) = V_E(f_i)$. Step 5 then implies that $y_1 E(x_1 F x_2) \sim^* y_2 E x_2$ for $F \subset E^c$, $\pi(F) = \eta(A_2)$. Such an F exists since $\pi(E) \leq 1/2$ and, therefore, $\eta(A_2) < 1/2$. Thus, by (S), $f_1 E(x_1 F x_2) \sim^* y_1 E(x_1 F x_2) \sim^* y_2 E x_2 \sim^* f_2 E x_2$. Note that $(A_1, (A_1 \cup A_2)^c)$ and $(A_2, (A_1 \cup A_2)^c)$ are both conforming pairs. Therefore, (WS) implies that, for $f^* = f_2(A_1 \cup A_2)f$, we have $(f_1 A \cup Bf)E(x_1 F x_2) = fE(x_1 F x_2) \sim^* (f_2 A \cup Bf)Ex_2 = f^* E x_2$. By the inductive hypothesis, $\hat{y} E x_2 \sim^* f^* E x_2$ for $\pi(E)u(\hat{y}) = V_E(f^*)$. Let y be such that $yEx_1 F x_2 \sim^* fE(x_1 F x_2)$. Then, $yEx_1 F x_2 \sim^* \hat{y} E x_2$. By Step 5, this implies that $\pi(E)u(y) = V_E(f^*) - \eta(A_2)(u(x_1) - u(x_2)) = V_E(f)$, as desired. Again, the argument reverses to yield the desired conclusion.

If $\pi(E) > 1/2$, then find set $E_1, E_2 \subset E$ such that $\pi(E_1) = \pi(E)/2 = \pi(E_2)$. Choose y_i such that $V_{E_i}(f) = y_i$ for $i = 1, 2$. Then, by the preceding argument $fE_i h \sim^* y_i E_i(fEh)$. such that $V_{E_i}(f) = \pi(E_i)u(y_i)$ for $i = 1, 2$ for some y_i. The additivity of η implies $V_E(f) = V_{E_i}(f) + V_{E_i}(f)$ and (S) implies $fEw \sim^* (y_1 E_1 y_2)Ew$. Then, Step

5 ensures $fEw \sim^* xEw$ for x such that $V_E(f) = \pi(E)u(x)$. Then, monotonicity yields $fEh \sim^* xEh$ if and only if $V_E(f) = \pi(E)u(x)$. $\square$

Proof of Theorem 2: Consider any $f \in \mathcal{F}$. Let $f(\Omega) = \{x_1, \ldots, x_n\}$ such that $x_i < x_{i+1}$, let $A_i = f^{-1}(x_i)$, let $\mathcal{N}$ be the non-empty subsets of $\{1, \ldots, n\}$ and let $\mathcal{N}^*$ be the subset of $\mathcal{N}$ that are not singleton sets. Let E_κ for $\kappa \in \mathcal{N}$ be an μ-split of $A_1, \ldots, A_n$. If (G, G^c) is the essential partition, then $\mu(E_\kappa \Delta G) = 0$ for all $\kappa \in \mathcal{N}^*$. Thus, we may assume that $E_\kappa \subset G$ implies $\kappa = \{i\}$ for some $i \in \{1, \ldots, n\}$ and, therefore, f is constant on E_κ. Define

$$V(f) = \sum_{i=1}^{n} \pi(E_i)u(f(E_i) + \sum_{\kappa \in \mathcal{N}^*} V_{E_\kappa}(f)$$

Steps 5, 11 and (S) imply that $f \succeq^* g$ if and only if $V(f) \geq V(g)$. Note that $V(f) = (1-\beta)V^1(f) + \beta V^2(f)$ where

$$\begin{aligned}
V^1(f) &= \sum_{i=1}^{n} \pi(E_i)u(x_i) + \sum_{\kappa \in \mathcal{N}^*} \left(\sum_i u(x_i)\eta(E_\kappa \cap A_i) + (\pi(E) - \eta(E))u(\min_{i \in \kappa} x_i) \right) \\
&= \int u dG_f^\pi \\
V^2(f) &= \sum_{i=1}^{n} \pi(E_i)u(x_i) + \sum_{\kappa \in \mathcal{N}^*} \left(\sum_i u(x_i)\eta(E_\kappa \cap A_i) + (\pi(E) - \eta(E))u(\max_{i \in \kappa} x_i) \right) \\
&= \int u dF_f^\pi
\end{aligned}$$

Thus, setting $\alpha = 1 - \beta$ proves the sufficiency part of the Theorem.

For the converse, suppose $\succeq^*$ has a DEU representation with uncertainty measure π and continuous utility index u. Let $\succeq$ be the QUA that π represents. Clearly $\succeq^*$ is an extension of $\succeq$. Given the continuity of u, verifying that $\succeq^*$ satisfies (A) is straightforward; (P4) is an immediate consequence of the additive separability of the representation; (S) and (WS) follow from the fact that π represents the QUA $\succeq$. $\square$

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