

Lecture notes about many-body localization, Princeton summer school, 5 Aug, 2009

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A very basic question in quantum statistical mechanics is: What systems of infinitely-many interacting quantum degrees of freedom can function as a heat bath and thus serve to equilibrate themselves and any smaller systems that they are in contact with? The first suggestion that certain interacting quantum many-body systems may generically fail to equilibrate in the thermodynamic limit, and thus fail to function as a heat bath, appears to be Anderson's original (1958) paper about localization [1].

Here we want to consider the behavior of interacting quantum many-body systems that are excited to energy densities above that of their ground state, so we are not focusing on the zero-temperature or ground-state limit that is often the focus in many discussions of quantum condensed-matter systems. We also want to consider only generic behavior that is not sensitive to adding perturbations or interactions that are short-range and thus local in real-space.

For an example, let us consider spin-1/2 spins (or "q-bits") \vec{S}_i located at sites i in a one- two- or three-dimensional space, with general single-spin and two-spin short-range interactions, with quenched-random Hamiltonian

$$H = \sum_i \vec{h}_i \cdot \vec{S}_i + \sum_{\langle ij \rangle} \vec{S}_i \mathbf{J}_{ij} \vec{S}_j, \quad (1)$$

where the second sum is over pairs of nearby sites, and the interaction \mathbf{J}_{ij} is in general a second rank tensor coupling the different components of the two spins (a particular case is when it is proportional to the unit matrix and thus produces the dot-product or Heisenberg interaction between the spins). Alternatively, we may consider tight-binding/Hubbard lattice models of quantum particles (fermions or bosons) with short-range hopping and interactions, on-site random potentials, at a nonzero density of particles. We will not couple these models to any other degrees of freedom (any "bath"), thus we ask if the isolated large interacting quantum system itself can serve as its own heat bath in order to relax to thermal equilibrium. The model must have the property that when we take the thermodynamic limit of an infinite system, the number of near-neighbors of a given site that interact with it (or that particles can hop to) remains finite. The standard case of this is lattices in d -dimensional Euclidean space with $1 \leq d < \infty$.

First Anderson [1], and more recently and explicitly Basko, *et al.* [2], have shown that for systems with strong enough quenched disorder, localization can prevent the system from transporting energy or particles, so that it fails to equilibrate and to be its own heat bath. However,

we also expect [3, 4] that for weaker or no quenched disorder, interacting quantum many-body systems do generically equilibrate. Thus there is a *many-body localization phase transition* that can occur as the Hamiltonian or energy density is varied *at nonzero [2] or even infinite [5] temperature*. Here I will mostly discuss the important differences between the two phases on either side of this phase transition.

Before proceeding, I should note that this idealization of a perfectly isolated many-body quantum system is only an approximation for any real physical system. There will always be nonzero couplings to the ever-present electromagnetic field, and in any material also couplings to acoustic and other phonons. At any nonzero temperature, these couplings to an "external" heat bath must ultimately disrupt the many-body localization. But in a case where the interactions included within our Hamiltonian H are strong compared to the interactions with phonons and/or photons that we neglect, the idealization of a perfectly isolated system is a useful first approximation. See Refs. [2, 6–8] for more discussion of this point.

This many-body localization transition is a quantum version of a glass transition. In the localized or glassy phase the system does not equilibrate or function as a heat bath and the fundamental ergodic hypothesis of statistical mechanics is false, while in the non-localized ergodic phase the system in the thermodynamic limit does thermally equilibrate and thus can serve as its own heat bath. What these statements mean in the context of an isolated quantum system undergoing the unitary time evolution dictated by its Hamiltonian is a little subtle:

The time dynamics of a general quantum state $|\psi(t)\rangle$ due to Hamiltonian H is discussed in every basic quantum mechanics textbook and is in some sense trivial. To describe it we need to first find all the many-body eigenstates $\{|n\rangle\}$ and eigenenergies $\{E_n\}$ of H :

$$H|n\rangle = E_n|n\rangle. \quad (2)$$

Then we can express our general state and its time-dependence in terms of the eigenstates as

$$|\psi(t)\rangle = \sum_n a_n e^{-iE_n t/\hbar} |n\rangle. \quad (3)$$

These statements remain true on either side of and at the many-body localization transition, and from this point of view we see that the system is never fully ergodic: Its motion in the full many-body Hilbert space of the entire system (which is spanned by the $\{|n\rangle\}$) is neither chaotic nor ergodic; instead it is quasiperiodic with a conserved quantity (namely $|a_n|^2$) for each eigenstate. As is discussed below, the distinction between the ergodic

and localized phases lies in the properties of the many-body eigenstates $|n\rangle$ of the Hamiltonian, not in the simple unitary time-evolution shown above.

So when we say a many-body quantum system is either ergodic on the one hand or localized on the other, what do we mean? The practical distinction is dynamical, namely: Does or does not the isolated system, undergoing the unitary time-evolution dictated by its Hamiltonian, equilibrate at long times? But what do we mean here by equilibrate? It means the long-time behavior is not dependent on the details of the initial conditions. The equilibrium state is determined by the globally conserved quantities: the total energy and any other conserved quantities of the system, such as total particle numbers, spin, momentum, angular momentum, etc. Although one may consider systems with conservation laws in addition to that of total energy, the model (1) generically does not have any other conservation laws, and the phenomenon of the many-body localization transition that we are examining does not require any. We will **not** consider integrable systems that have an infinite number of distinct conserved quantities in the thermodynamic limit, since as far we know such systems are not generic: they do not remain integrable under adding arbitrary, but small, short-range interactions to the Hamiltonian.

If we considered the full many-body quantum state of the *entire* system in asking whether the system has equilibrated, then the answer would be “no”, since if the system starts in a particular energy eigenstate $|n\rangle$, then it remains in that eigenstate for all times, so it does remember the “details” of the initial state. Instead, we ask whether any *finite* subsystem S of the entire system equilibrates, in the thermodynamic limit where the size of the entire system is taken to infinity while leaving S finite. As a diagnostic of this we consider the reduced density operator ρ_S of each finite subsystem S :

$$\rho_S(t) = \text{Trace}_{\bar{S}}\{|\psi(t)\rangle\langle\psi(t)|\}, \quad (4)$$

where this trace is only over \bar{S} , and \bar{S} consists of all degrees of freedom that are *not* in the finite subsystem S . To say that the system generically equilibrates, $\rho_S(t)$ must relax to its thermal equilibrium state at long time, independent of the details of the initial state. Since the phase relations in (3) between the amplitudes of different many-body eigenstates of different energies vary at long times, it appears that the contributions to $\rho_S(t)$ that are from two different many-body eigenstates will dephase and vanish in the limit of long times and large systems. For a given S , it appears that the only way the long time behavior of $\rho_S(t)$ can generically be independent of the details of the initial state is for it to be the same for all many-body eigenstates of a given energy density (and of a given density of all other globally conserved quantities) in the thermodynamic limit.

This is the idea of *eigenstate thermalization* [3]. It says that for a generic interacting quantum many-body system that does thermally equilibrate in the thermodynamic limit at a given energy density, each individual

many-body eigenstate of the entire system at that energy is itself thermal. This means that in the thermodynamic limit, for any such many-body eigenstate $|n\rangle$ of the entire system, the reduced density matrix ρ_S of any finite subsystem S is simply the equilibrium thermal (Boltzmann-Gibbs) distribution of S at the temperature, chemical potentials, etc. dictated by the total energy, particle numbers, etc. of the eigenstate.

The many-body ergodic phase is the regime of parameters and Hamiltonians where this *eigenstate thermalization hypothesis* is true, while the many-body localized phase is the regime where it is false. This eigenstate thermalization hypothesis plays the role for quantum statistical mechanics that the usual ergodic hypothesis plays for classical statistical mechanics [3, 4]. It means that the microcanonical ensemble for such a many-body quantum system that does equilibrate can be chosen to be a single many-body eigenstate of the entire system. Note that the ergodicity is not of the full system. It is only for subsystems that the dynamics is fully ergodic (meaning the long-time $\rho_S(t)$ generically has nonzero probability for any state of S).

Since the system in the ergodic phase does thermally equilibrate, it must transport energy and any other globally-conserved quantities. If any conserved quantity has an initial distribution in space that differs from equilibrium, then it must be moved to an equilibrium distribution at long times. Thus the DC thermal and other conductivities are nonzero. Also, the system is fully *decoherent* at long times, since the reduced density operator ρ_S of any finite subsystem S within which one might try to store quantum (or classical) information goes to equilibrium at long times: At long times any quantum or other information about the initial state is dispersed over the entire system in this ergodic phase. Also, this means that, given any initial *pure* state of the entire system, the long-time entanglement is extensive, since the von Neumann entropy of entanglement, $\text{Trace}\{\rho_S \log \rho_S\}$, of any finite subsystem S with the remainder of the system \bar{S} is simply the equilibrium thermal entropy of that subsystem, which has a nonzero entropy density provided the equilibrium energy density is larger than that of the ground state (so the temperature is nonzero).

To what extent has this eigenstate thermalization hypothesis been tested? Since it is about the properties of the exact eigenstates of the many-body system, it appears that exact diagonalization of H may be the only route to directly testing it numerically. So this of course limits the sizes of the systems that can be studied this way. Rigol, *et al.* [4] have recently confirmed the expected behavior for an example of 5 hard-core bosons hopping on a certain cluster of 21 sites (which is equivalent to a spin-1/2 model of the form (1)). Pal and I [9] have also confirmed it for interacting one-dimensional hard-core bosons in a weak random potential (or equivalently for a random-field spin chain). We look at all eigenstates of systems of various lengths L (with periodic boundary conditions) and find that the deviation of

the correlations within each many-body eigenstate from those at equilibrium decrease with increasing L as expected due to the exponential growth of the dimension of the Hilbert space. In this quantum-chaotic many-body ergodic phase, the many-body energy-levels should have the statistics of the Gaussian orthogonal ensemble (GOE) of random matrices, for these simple time-reversal-invariant Hamiltonians, and Oganessian and I [5] did indeed see this for a model of interacting one-dimensional fermions in a weak enough random potential.

To get a first understanding of the many-body *localized* phase, on the other hand, let's start with an extreme (and admittedly nongeneric) example, namely our spin Hamiltonian (1) in the limit of no interactions:

$$H = \sum_i \vec{h}_i \cdot \vec{S}_i, \quad (5)$$

with, say, Gaussian-distributed quenched random fields $\{\vec{h}_i\}$ on each spin. Here the many-body eigenstates are simply product states, with each spin aligned either parallel or anti-parallel to its random field, and thus no entanglement between spins. For N spins, there are 2^N such many-body eigenstates, since each spin has two local eigenstates and any (outer) product of these is a many-body eigenstate. This set of all many-body eigenstates of (5) is, of course, a complete set of states spanning the full many-body Hilbert space and can be represented as the corners of an N -dimensional hypercube, which I will call ‘‘Fock space’’. In the absence of interactions, the many-body eigenstates of Hamiltonian (5) are each fully localized at one corner of this hypercube.

This simple noninteracting many-body system (5) does not equilibrate. Nor does it transport energy. Nor does it relax the quantum state of any individual spin. All that happens dynamically for any initial quantum state of the system is that each spin \vec{S}_i precesses around the axis defined by its local random field \vec{h}_i at its Larmor frequency $h_i \equiv |\vec{h}_i|$.

Now turn on weak short-range (in real-space) interactions $\{\mathbf{J}_{ij}\}$ as in (1). In the simple basis of the many-body eigenstates of (5), these interactions generate off-diagonal matrix elements in the Hamiltonian (1) that connect many-body basis states that differ in the orientations of at most two neighboring (interacting) spins. The resulting eigenstates of (1) are each thus an admixture of eigenstates of the noninteracting system (5). The amplitude of this mixing (of the local ‘‘quantum fluctuations’’ as viewed in this basis) will be at most of order $|\mathbf{J}_{ij}|/h_i$ due to the terms (if present) that simply flip spin i and of order $|\mathbf{J}_{ij}|/(h_i - h_j)$ due to the terms that flip both spins i and j . For weak interactions these fluctuations will be weak for almost all spins; they will only be large for the few spins that have very small h_i or very nearly equal (nearly resonant) h_i and h_j . As a result, for weak short-range spin-spin interactions the many-body eigenstates of (1) appear to remain localized and non-

thermal, with reduced density matrices ρ_S for finite subsystems S that differ from the equilibrium thermal distribution. This stability of many-body localization to weak but nonzero short-range interactions has been argued in detail by Basko, *et al.* [2]; although they consider interacting electrons, their arguments are of more general applicability and cover the spin model I am discussing, among other models.

In the many-body localized phase, the *eigenstates* of H have only short-range quantum fluctuations and thus short-range (in real space) entanglement. This limits the transport of energy, so the system has strictly zero DC thermal conductivity in the thermodynamic limit. The distance range (in real space) over which energy and correlations are transported is the localization length, ξ , which presumably diverges continuously as the phase transition to the ergodic phase is approached.

In this many-body localized phase, I expect that one can produce states with localized quantum information that remains coherent for all times, by, for example, starting with a many-body eigenstate and then doing some local coherent operations (e.g., spin rotations) to make local superpositions at one location. This localized coherence will also be confined on a length scale of order ξ , the localization length. However, a recent proof by Linden, *et al.* [10] appears to imply that this protection against decoherence is not very general, even in the localized phase. They prove that a generic initial state dephases, and this appears to imply that one cannot store a nonzero density of quantum information in this system, since q-bits at any finite distance from one another will interact via virtual excitations, thus forming a ‘‘spin bath’’, and this will generically produce decoherence (but not necessarily relaxation to thermal equilibrium) at long times.

The limited entanglement in the many-body localized phase makes simulation with ‘‘tDMRG’’ or matrix-product states techniques a good approximation, as recently demonstrated by Znidaric, *et al.* [11], who study a random-field spin chain and conclude that for strong enough random fields all many-body eigenstates appear to be localized. This is in good agreement with the arguments of Basko, *et al.* [2], and Pal and I [9] also have confirmed this in exact diagonalization studies of a similar spin chain.

The nature of the many-body localization transition between the ergodic and localized phases is still very much an open question. In one published attempt to study it, Oganessian and I [5] looked at the many-body eigenenergy-level statistics, seeing the expected crossover from GOE statistics in the ergodic phase to Poisson statistics in the localized phase. But, since the localization transition is occurring in the infinite-dimensional many-body Fock space, it appears that the level statistics are also Poisson *at* the transition, which apparently causes the simple one-parameter finite-size scaling analysis we used [5] to fail to capture any other useful information about the phase transition. Now, in work that is still in progress, Pal and I [9] are studying the finite-

size scaling of the correlations within each eigenstate of a random-field spin chain, hoping to learn more about this many-body localization transition.

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- [1] P. W. Anderson, “*Absence of diffusion in certain random lattices*”, Phys. Rev. **109**, 1492 (1958).
- [2] D. M. Basko, I. L. Aleiner and B. L. Altshuler, “*Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states*”, Annals of Physics **321**, 1126 (2006); “*On the problem of many-body localization*”, cond-mat/0602510 .
- [3] Mark Srednicki, “*Chaos and quantum thermalization*”, Phys. Rev. E **50**, 888 (1994); J. M. Deutsch, “*Quantum statistical mechanics in a closed system*”, Phys. Rev. A **43**, 2046 (1991).
- [4] Marcos Rigol, Vanja Dunjko and Maxim Olshanii, “*Thermalization and its mechanism for generic isolated quantum systems*”, Nature **452**, 854 (2008).
- [5] Vadim Oganesyan and David A. Huse, “*Localization of interacting fermions at high temperature*”, Phys. Rev. B **75**, 155111 (2007).
- [6] D. M. Basko, I. L. Aleiner and B. L. Altshuler, “*Possible experimental manifestations of the many-body localization*”, Phys. Rev. B **76**, 052203 (2007).
- [7] Boris L. Altshuler, Vladimir E. Kravtsov, Igor V. Lerner and Igor L. Aleiner, “*Jumps in current-voltage characteristics in disordered films*”, Phys. Rev. Lett. **102**, 176803 (2009).
- [8] M. Ovadia, B. Sacépé and D. Shahar, “*Electron-phonon decoupling in disordered insulators*”, Phys. Rev. Lett. **102**, 176802 (2009).
- [9] Arijeet Pal and David A. Huse, *work in progress*.
- [10] Noah Linden, Sandu Popescu, Anthony J. Short and Andreas Winter, “*Quantum mechanical evolution towards thermal equilibrium*”, arXiv:0812.2385 .
- [11] Marko Znidaric, Tomaz Prosen and Peter Prelovsek, “*Many-body localization in the Heisenberg XXZ magnet in a random field*”, Phys. Rev. B **77**, 064426 (2008).