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# HISTORY VERSUS EXPECTATIONS: A COMMENT

KYOJI FUKAO AND ROLAND BENABOU

## I. INTRODUCTION

Recently in this *Journal* Krugman [1991] developed a model of an economy with external economies and adjustment costs to show how these effects of the economy determine the relative importance of history and expectations in shaping equilibrium. If the characteristic roots of the dynamic system are complex and the labor force initially employed in a sector with Marshallian externality lies inside a certain range, which is referred to as the "overlap," there will be multiple self-fulfilling equilibrium paths. In this note we show that the terminal condition used in the paper is incorrect. Since the true equilibrium paths are different from Krugman's, his analysis of the width of the "overlap" also needs to be revised.

## II. KRUGMAN'S MODEL

Krugman's model can be summarized as follows. Consider a one-factor ( $L$ ), two-goods ( $C$  and  $X$ ) open economy. The economy's total labor supply is  $\bar{L}$ .  $C$  is produced with constant returns, and the wage rate in the  $C$  sector is normalized to unity. The  $X$  sector is characterized by Marshallian externalities; the larger the labor force  $L_X$  it employs, the higher its average productivity  $\pi(L_X)$ , which is also equal to the wage rate in that sector. We assume that  $\pi(L_X)$  is continuously differentiable with  $\pi' > 0$ ,  $0 < \pi(0) < 1$ ,  $1 < \pi(\bar{L}) < +\infty$ . Let  $L_X^*$  denote the critical value that satisfies  $\pi(L_X^*) = 1$ .

The economy faces constant world prices for the two goods and for the rate of interest  $r > 0$ . When the two sectors' discounted present values of future wage income differ, workers in the lower wage sector want to move to the other sector. Workers who move, however, incur a moving cost. The economy's total moving cost is assumed to be quadratic, reflecting congestion; the moving cost for each worker is  $|L_X|/\gamma > 0$ .

The economy is described by the dynamic system,

$$\begin{aligned} (1) \quad & \dot{L}_X = \gamma q, \\ (2) \quad & \dot{q} = r q - \pi(L_X) + 1, \end{aligned}$$

where  $q$  is defined by Krugman [p. 659] as "the shadow price placed on the asset of having a unit of labor in the  $X$  sector rather than the

C sector." Equation (1) implies that labor moves at a rate determined by the equality of marginal moving costs and the shadow price. Equation (2) is an arbitrage condition; the rate of return on the "asset" must equal the world interest rate.

Figure I shows the phase diagram of the system. The unique stationary point is  $(L_X = L_X^*, q = 0)$ , and at that point the characteristic roots of the linearized system are

$$\rho = [r \pm (r^2 - 4\gamma\pi'(L_X^*))^{1/2}]/2.$$

As Krugman showed, equilibrium paths can take two possible shapes in the vicinity of the stationary point. If  $r^2 > 4\gamma\pi'(L_X^*)$ , both roots are real and positive, so the linearized system steadily diverges from the stationary point. If  $r^2 < 4\gamma\pi'(L_X^*)$ , there are two complex roots with a positive real part, and the linearized system diverges from the stationary point in expanding oscillations. In both cases divergence makes it necessary to supplement Krugman's *local* analysis by a study of the *global* behavior of the system.

PROPOSITION 1. For any initial values  $(L_X(0), q(0))$  that satisfy  $0 < L_X(0) < 1$ , except for the stationary point  $(L_X(0) = L_X^*, q(0) = 0)$ , the solution to the dynamic system (1) and (2) will hit a boundary  $L_X = 0$  or  $L_X = 1$  in finite time.

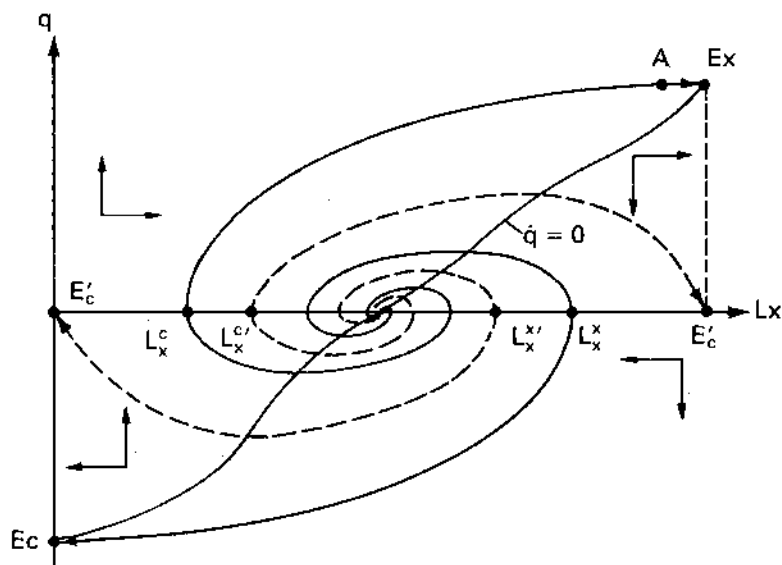


FIGURE I

*Proof.* First, we show that there is no closed orbit. On any solution path of equations (1) and (2), we have  $\{\pi(L_X) - 1\}dL_X + \gamma q dq = r q dL_X$ . Suppose that there exists a closed orbit. By integrating along it, we get

$$\oint \{\pi(L_X) - 1\} dL_X + \gamma \oint q dq = r \oint q dL_X.$$

From Green's Theorem the left-hand side of the equation equals zero, and the right-hand integral is the area lying within the limit cycle—a contradiction. We also note that once  $q$  becomes greater than  $\{\pi(\bar{L}) - 1\}/r$  or smaller than  $\{\pi(0) - 1\}/r$ , the economy will hit a boundary  $L_X = 0$  or  $L_X = 1$  in finite time. Finally, by the Poincaré-Bendixson Theorem (see Hirsch and Smale [1974]) these two characteristics, plus the fact that the unique stationary point is a source, imply the Proposition.

Q.E.D.

Since  $q$  is a "jumping" variable, a terminal condition is required in order to select an economically meaningful equilibrium path from the infinite number of trajectories. Krugman asserts that the economy approaches one of two terminal points,  $(L_X = \bar{L}, q = \{\pi(\bar{L}) - 1\}/r)$  or  $(L_X = 0, q = \{\pi(0) - 1\}/r)$ . In Figure I these points are denoted by  $E_X$  and  $E_C$ , and Krugman's "equilibrium" paths drawn in solid curves. Mathematically, Krugman defines  $q(t)$  by his equation (6):

$$(3) \quad q(t) = \int_t^{\infty} (\pi(L_X(\tau)) - 1) e^{-r(\tau-t)} d\tau.$$

Our point is that Krugman's presumption about terminal conditions is not correct. The true equilibrium paths are those that lead to points  $E'_X (L_X = \bar{L}, q = 0)$  or  $E'_C (L_X = 0, q = 0)$ ; they are represented by dotted curves in Figure I. Accordingly, the correct formula for  $q(t)$  is

$$(4) \quad q(t) = \int_t^T (\pi(L_X(\tau)) - 1) e^{-r(\tau-t)} d\tau,$$

where  $T < +\infty$  is the time at which one of the boundaries is reached. The interpretation of (1) is then that the adjustment cost  $|L_X|/\gamma > 0$  is what a worker is willing to pay to move at  $t < T$  rather than at  $T$ , when moving becomes costless. We first present the intuition behind (4), which is very simple. The rigorous proof will be given in the next section.

Suppose that the economy is located at point  $A$  in Figure I, which is very close to the "terminal" point  $E_X$  and is located on

Krugman's "equilibrium" path. The economy is about to hit the "terminal" point, so  $q$  is close to  $(\pi(\bar{L}) - 1)/r$ . At this point, a worker who still works in the  $C$  sector can gain by deviating from the "equilibrium" behavior. By waiting until all other workers have moved, he can avoid all congestion and move at zero cost. Since this yields a capital gain of  $|\dot{L}_X|/\gamma = q = \{\pi(\bar{L}) - 1\}/r$ , point  $A$  cannot be an equilibrium. The only paths that exclude this kind of arbitrage behavior are *soft landing* ones, along which  $q$  approaches zero. We show below that the true equilibrium paths are those that lead to point  $(L_X = \bar{L}, q = 0)$  or  $(L_X = 0, q = 0)$ , and that after the economy hits one of these terminal points, it remains there forever. Note that the arbitrage condition (2) does not hold anymore at these points, but this is consistent with market equilibrium, since no labor is left in the lower wage sector.

### III. THE TRUE EQUILIBRIUM PATH

We now derive the true global dynamics of the system. Let  $v_X(t)$  and  $v_C(t)$  denote the expected discounted sum of future wage income minus moving costs of a worker *currently* working in the  $X$  and the  $C$  sectors, respectively. Define  $T$  as the first time the economy hits the boundary  $L_X = 0$  or  $\bar{L}$ . Since some workers choose to remain in each sector until  $T$ ,  $v_X(t)$  and  $v_C(t)$  must satisfy

$$(5) \quad v_X(t) = \int_t^T \pi(L_X(\tau)) e^{-r(\tau-t)} d\tau + v_X(T^-) e^{-r(T-t)}$$

and

$$(6) \quad v_C(t) = \int_t^T e^{-r(\tau-t)} d\tau + v_C(T^-) e^{-r(T-t)} \quad \text{for all } t \in [0, T),$$

where  $v_i(T^-)$  denotes the limit from the left, which must be determined by a terminal condition of the model. In particular, if all the workers have left sector  $i$  by time  $T$ , there is no reason to presume that  $v_i(T^-)$  is equal to the discounted sum of future wages in sector  $i$ .

Note that at any  $t < T$ ,  $v_X(t)$  and  $v_C(t)$  are continuously differentiable. Moreover, since workers in each sector have the option to move to the other sector by paying the marginal adjustment cost  $|\dot{L}_X|/\gamma$ ,

$$(7) \quad v_X(t) \geq v_C(t) - \frac{|\dot{L}_X(t)|}{\gamma}, \quad \text{with equality if } \dot{L}_X(t) < 0$$

and

$$(8) \quad v_C(t) \geq v_X(t) - \frac{|\dot{L}_X(t)|}{\gamma}, \quad \text{with equality if } \dot{L}_X(t) > 0.$$

Let us therefore replace Krugman's definition of  $q(t)$ , given by equation (3), with the appropriate shadow price on which workers base their moving decisions:

$$q(t) \equiv v_X(t) - v_C(t).$$

The system's laws of motion remain unchanged; from equations (5) and (6) we get equation (2); and from equations (7) and (8) we get equation (1). These equations hold for any  $t \in [0, T]$  and more generally wherever  $0 < L_X(t) < 1$  (as one can always take  $(L_X(t), q(t))$  as the initial conditions).

Finally, we determine the system's terminal conditions.

**PROPOSITION 2.** The true equilibrium paths are those that lead to point  $(L_X = \bar{L}, q = 0)$  or  $(L_X = 0, q = 0)$ . After reaching such a point, the economy stays there forever, except for nongeneric cases.

For the proof we shall use the following lemmas.

**LEMMA 1.** Suppose that after the economy hits the boundary at time  $T$  it stays there at least for a certain period,  $[T, T']$ , where  $T < T'$ . Then the equilibrium path must satisfy

$$q(T^-) \equiv v_X(T^-) - v_C(T^-) = 0.$$

*Proof.* First, we study the case in which the economy hits the right boundary,  $L_X = 1$ . Then necessarily  $q(T^-) \geq 0$ , as shown on the phase diagram (Figure I). Suppose that  $q(T^-) > 0$ . Then for  $\epsilon > 0$  small enough,  $\dot{L}_X(t)/\gamma = q(t) > 0$ , and  $0 < L_X(t) < 1$  on  $[T - \epsilon, T)$ . During this interval workers in the  $C$  sector are indifferent between staying and moving. Moreover, once in  $X$ , staying there until  $T + \epsilon$  is an optimal strategy since  $L_X(T) > 0$  on  $[T - \epsilon, T + \epsilon)$ . Hence,

$$\begin{aligned} v_C(T - \epsilon) &= -(\dot{L}_X(T - \epsilon))/\gamma + v_X(T - \epsilon) \\ &= -q(T - \epsilon) + 2\epsilon\pi(\bar{L}) + v_X(T + \epsilon)e^{-2r\epsilon} + o(\epsilon), \end{aligned}$$

where  $o(\epsilon)$  represents terms that are negligible compared with  $\epsilon$ . But if a worker chose to wait until  $T + \epsilon$  to move from  $C$  to  $X$ , he would get

$$\hat{v}_C(T - \epsilon) = 2\epsilon + v_X(T + \epsilon)e^{-2r\epsilon} + o(\epsilon).$$

Since  $\hat{v}_C(T - \epsilon) - v_C(T - \epsilon)$  tends to  $q(T^-)$  as  $\epsilon$  tends to zero, this deviation from equilibrium behavior is strictly profitable—a contradiction. A similar proof applies to the case where the economy hits the left boundary  $L_X = 0$ .

Q.E.D.

LEMMA 2. Suppose that after the economy stays at the boundary,  $L_X = 0$  or 1 for a certain period,  $[T, T']$ , where  $T < T'$ , it returns to inner equilibrium,  $0 < L_X < \bar{L}$  at time  $T'$ . Then the equilibrium path must satisfy

$$q(T'^+) \equiv v_X(T'^+) - v_C(T'^+) = 0.$$

*Proof.* First, we study the case in which the economy starts from the right boundary,  $L_X = 1$ . As the phase diagram shows, we must have  $q(T'^+) \leq 0$ . If  $-q(T'^+) > 0 = q(T'^-)$ , one of the  $-\dot{L}_X(T'^+) > 0$  workers who move from  $X$  to  $C$  just after  $T'$  could avoid this adjustment cost by instead moving just before  $T'$ . Again, the effect on his wage stream would be infinitesimal, but the cost savings would be  $-q(T'^+)$ , making the deviation profitable. A similar proof by contradiction applies when  $L_X = 0$  on  $[T, T']$ .

Q.E.D.

LEMMA 3. Suppose that after the economy hits the boundary  $L_X = 0$  or  $L_X = 1$  at time  $T$ , it immediately returns to inner equilibrium,  $0 < L_X < \bar{L}$ . Then the equilibrium path must satisfy

$$q(T^-) = q(T^+) = 0.$$

*Proof.* Again, we focus without loss of generality on the case in which the economy rebounds from the right boundary,  $L_X = 1$ . As the phase diagram shows, we must have  $q(T^-) \geq 0$  and  $q(T^+) \leq 0$ . By assumption, just before  $T$  some agents are moving from  $C$  to  $X$ , while just after  $T$  some are moving from  $X$  to  $C$ . Since all agents are identical, it must be an optimal strategy to move to  $X$  just before  $T$ , remain until just after  $T$ , and switch back to  $C$ . But if instead an agent remained in  $C$  throughout this infinitesimal interval of time, he would save the two-way moving costs  $q(T^-) - q(T^+)$ , with only an infinitesimal effect on his wage stream. Equilibrium then requires  $q(T^-) - q(T^+) \leq 0$ , hence the result.

Q.E.D.

We now turn to the proof of Proposition 2.

*Proof of Proposition 2.* Lemmas 1 and 3 imply that the true equilibrium paths are those that lead to either point  $(L_X = \bar{L}$ ,

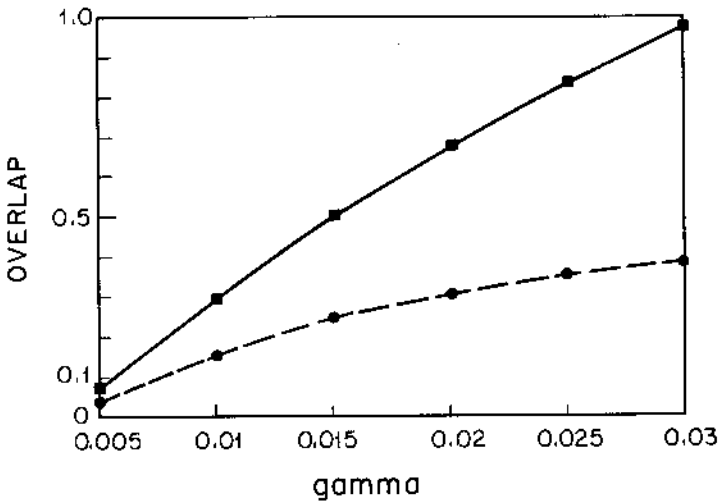


FIGURE II

$q = 0$ ) or  $(L_X = 0, q = 0)$ . Lemmas 2 and 3 imply that if the economy returns to inner equilibrium,  $0 < L_X < \bar{L}$  from the boundary,  $L_X = 0$  or  $1$ , the equilibrium path will start at either point  $(L_X = \bar{L}, q = 0)$  or  $(L_X = 0, q = 0)$ . But then the new path will hit a boundary again with nonzero  $q$  in finite time and violate equilibrium conditions, except for the nongeneric case in which an equilibrium path links one equilibrium boundary point to the opposite one. So generically, once the system reaches a boundary equilibrium point it remains there forever. In the nongeneric case nonexistence of closed orbits implies that the economy will remain at the new terminal point forever.

Q.E.D.

#### IV. ON THE WIDTH OF THE "OVERLAP"

Following Krugman, let us linearize the model by specifying  $\pi(L_X) = 1 + \beta(L_X - L_X^*)$ . If  $r^2 > 4\gamma\beta$ , the system will steadily diverge from  $(L_X = L_X^*, q = 0)$ . In this case, as Krugman argued, history (i.e., the initial allocation of labor) uniquely determines the equilibrium path. If  $r^2 < 4\gamma\beta$ , the system diverges from the stationary point in expanding oscillations. In this case, if  $L_X$  lies inside a certain range, referred to as the "overlap," there will be multiple self-fulfilling equilibrium paths.



Our revision on the terminal point does not invalidate Krugman's criterion for self-fulfilling prophecies. But his analysis of the width of the "overlap" needs revision. In Figure I, Krugman's "overlap" is the range  $(L_X^C, L_X^X)$ . The true "overlap" is the range  $(L_X^{C'}, L_X^{X'})$ . In Section V of his paper Krugman explicitly calculates how the adjustment cost parameter  $\gamma$  affects the width of the overlap, using a numerical example:  $L_X^* = 0.5$ ,  $\beta = 1$ , and  $r = 0.1$ . The solid curve in Figure II plots Krugman's calculated width. It corresponds to his Figure V. The true relationship between  $\gamma$  and the "overlap" is given by the dotted curve.<sup>1</sup> The true "overlap" is much narrower than Krugman argued.

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1. The width of the "overlap" is derived as follows. Let us define  $t = 0$  as the time when the economy reaches either one of the terminal points. We assume that  $L = 1$ . From equations (1), (2), and our terminal conditions, we can derive explicit solutions. In the complex roots case, the solutions are

$$(9) \quad L_X(t) = \pm \frac{1}{2 \sin \theta} e^{(\gamma t/2)} \sin \left( \sqrt{\beta\gamma - \frac{r^2}{4}} t - \theta \right) + \frac{1}{2},$$

$$(10) \quad q(t) = \pm \frac{\sqrt{\beta\gamma}}{2\gamma \sin \theta} e^{(\gamma t/2)} \sin \left( \sqrt{\beta\gamma - \frac{r^2}{4}} t \right).$$

The solution in which the first term in equation (9) takes a positive value is paired with the solution in which the first term of equation (10) takes a positive value and vice versa. The parameter  $\theta$  is defined by  $\cos \theta = r/(2(\beta\gamma)^{0.5})$ . Equation (10) implies that the economy reaches  $L_X^C$  or  $L_X^X$  of Figure I at  $t = -\pi/(\beta\gamma - r^2/4)^{0.5}$ . Equation (9) implies that the width of the "overlap" is  $\exp(-r\pi/(4\beta\gamma - r^2)^{0.5})$ .