Online Appendix for "Aggregation and Estimation of Constant Elasticity of Substitution (CES) Preferences" (Not for Publication)

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A.1 Introduction

This online appendix contains technical derivations and the proofs of propositions. Section A.2 derives the forward, backward and geometric mean differences of the unit expenditure function. Section A.3 shows that all of our results in the paper are invariant with respect the choice of units in which to measure appeal (the cardinalization of utility). Section A.4 proves our main aggregation result for CES preferences. Section A.5 provides condition under which the forward and backward appeal indexes are equal to one another. Section A.6 shows that the forward and backward appeal indexes are equal to one another up to a first-order approximation. Section A.7 characterizes the asymptotic properties of the OLS demand system estimates under the assumption of CES preferences and log normally distributed price and appeal shocks. Section A.8 shows that the forward-backward (FB) and reverse-weighting (RW) estimators are consistent estimators of the elasticity of substitution as appeal shocks become small for each good. Section A.9 characterizes the properties of the FB and RW estimators for the case in which price and appeal shocks are joint log normally distributed and correlated with one another. Section A.10 shows that our results for CES preferences also hold for logit preferences, because of the well known result that idiosyncratic discrete choice decisions of individual consumers aggregate across these individual consumers to yield CES preferences at the aggregate level.

A.2 Derivation of Differenced Unit Expenditure Functions

Forward Difference We start with the forward difference for the change in the unit expenditure function from period $t-1$ to t from equation (3) in the paper:

$$\frac{P_t}{P_{t-1}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}} \left[\frac{\sum_{k\in\Omega_t^*}\left(\frac{p_{kt}}{\varphi_{kt}}\right)^{1-\sigma}}{\sum_{k\in\Omega_t^*}\left(\frac{p_{kt-1}}{\varphi_{kt-1}}\right)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}}, \tag{A.42}$$

which can be re-written as follows:

$$\frac{P_t}{P_{t-1}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}} \left[\sum_{k\in\Omega_t^*}\frac{\left(\frac{p_{kt}}{\varphi_{kt}}\right)^{1-\sigma}}{\frac{1}{s_{kt-1}^*}\left(\frac{p_{kt-1}}{\varphi_{kt-1}}\right)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}}, \tag{A.43}$$

$$= \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}} \left[\sum_{k\in\Omega_t^*} s_{kt-1}^*\left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{A.44}$$

which corresponds to the forward difference of the unit expenditure function using initial-period expenditure share weights in equation (8) in the paper.

Backward Difference We next turn to the backward difference for the change in the unit expenditure function from period t to $t-1$, which from equation (3) in the paper can be written as:

$$\frac{P_{t-1}}{P_t} = \left(\frac{\lambda_{t-1}}{\lambda_t}\right)^{\frac{1}{\sigma-1}} \left[\frac{\sum_{k\in\Omega_t^*}\left(\frac{p_{kt-1}}{\varphi_{kt-1}}\right)^{1-\sigma}}{\sum_{k\in\Omega_t^*}\left(\frac{p_{kt}}{\varphi_{kt}}\right)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}}, \tag{A.45}$$

which can be re-written as follows:

$$\frac{P_{t-1}}{P_t} = \left(\frac{\lambda_{t-1}}{\lambda_t}\right)^{\frac{1}{\sigma-1}} \left[\sum_{k\in\Omega_t^*}\frac{\left(\frac{p_{kt-1}}{\varphi_{kt-1}}\right)^{1-\sigma}}{\frac{1}{s_{kt}^*}\left(\frac{p_{kt}}{\varphi_{kt}}\right)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}}, \tag{A.46}$$

$$= \left(\frac{\lambda_{t-1}}{\lambda_t}\right)^{\frac{1}{\sigma-1}} \left[\sum_{k\in\Omega_t^*} s_{kt}^*\left(\frac{p_{kt}/\varphi_{kt}}{p_{kt-1}/\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{\frac{1}{1-\sigma}}, \tag{A.47}$$

Re-arranging this relationship, we obtain the backward difference of the unit expenditure function using end-period expenditure share weights in equation (9) in the paper:

$$\frac{P_t}{P_{t-1}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}} \left[\sum_{k\in\Omega_t^*} s_{kt}^*\left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}. \tag{A.48}$$

Geometric Mean Difference Re-arranging the common goods expenditure share at time t from equation (6), we obtain the following expression for the unit expenditure function for common goods at time t (P_t^*):

$$P_t^* = \frac{p_{kt}}{\varphi_{kt}} \left(s_{kt}^*\right)^{\frac{1}{\sigma-1}}. \tag{A.49}$$

Taking the geometric mean across common goods $k \in \Omega_t^*$, we obtain:

$$P_t^* = \frac{\widetilde{p}_t}{\widetilde{\varphi}_t} \widetilde{s}_t^{\frac{1}{\sigma-1}}, \tag{A.50}$$

where recall that a tilde above a variable denotes a geometric mean across common goods, such that $\tilde{x}_t = \left(\prod_{k\in\Omega_t^*} x_{kt}\right)^{1/N_t^*}$, where $N_t^* = |\Omega_t^*|$ is the number of common goods.

Re-arranging the common goods expenditure share at time t from lagged equation (6) on period, we obtain the following analogous expression for the unit expenditure function for common goods at time $t-1$ (P_{t-1}^*):

$$P_{t-1}^* = \frac{p_{kt-1}}{\varphi_{kt-1}} \left(s_{kt-1}^*\right)^{\frac{1}{\sigma-1}}. \tag{A.51}$$

Taking the geometric mean across common goods $k \in \Omega_t^*$, we obtain:

$$P_{t-1}^* = \frac{\widetilde{p}_{t-1}}{\widetilde{\varphi}_{t-1}} \widetilde{s}_{t-1}^{\frac{1}{\sigma-1}}. \tag{A.52}$$

Taking the ratio of equations (A.50) and (A.52), we obtain the change in the unit expenditure function for common goods:

$$\frac{P_t^*}{P_{t-1}^*} = \frac{\widetilde{p}_t/\widetilde{p}_{t-1}}{\widetilde{\varphi}_t/\widetilde{\varphi}_{t-1}} \left(\frac{\widetilde{s}_t}{\widetilde{s}_{t-1}}\right)^{\frac{1}{\sigma-1}}. \tag{A.53}$$

Substituting equation (A.53) into the overall change in the unit expenditure function in equation (3) in the paper, we obtain the geometric mean difference of the unit expenditure function in equation (11) in the paper:

$$\Phi_t^G \left(\left\{ \frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}} \right\}_{k\in\Omega_t\cup\Omega_{t-1}} \right) = \frac{P_t}{P_{t-1}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}} \frac{\tilde{p}_t/\tilde{p}_{t-1}}{\tilde{\varphi}_t/\tilde{\varphi}_{t-1}} \left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}. \tag{A.54}$$

A.3 Invariance With Respect to Units for Appeal

In this section of the online appendix, we show that appeal for each good can be recovered from observed prices and expenditure shares up to a multiplicative constant. We show that all of our results in the paper are invariant with respect to the choice of this this multiplicative constant, because we express appeal for each good as a ratio to its geometric mean ($\varphi_{kt}/\widetilde{\varphi}_t$), such that this multiplicative constant cancels from the numerator and denominator of this ratio.

From equation (6) in the paper, the share of individual common good in all expenditure on common goods (s_{kt}^*) is:

$$s_{kt}^* = \frac{\left(p_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{\ell\in\Omega_t^*} \left(p_{\ell t}/\varphi_{\ell t}\right)^{1-\sigma}}. \tag{A.55}$$

Prices and expenditure shares are observed in the data, but the appeal shifters are not observed. Noting that the CES expenditure share system is homogenous of degree zero in prices and appeal, the appeal shifters can be recovered (up to a multiplicative choice of units) from solving the following fixed point system:

$$s_{kt} - \frac{\left(p_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{\ell\in\Omega^*}\left(p_{\ell t}/\varphi_{k\ell t}\right)^{1-\sigma}} = 0, \tag{A.56}$$

which provides a system of equations for the N goods in the N unobserved appeal shifters.

This system of equations (A.56) only determines the appeal shifters up to a multiplicative constant, because this multiplicative constant cancels from the numerator and denominator of the fraction on the left-hand side. This multiplicative constant corresponds to a particular choice of units in which to measure appeal and hence a particular cardinalization of utility. Noting that the denominator of the fraction on the left-hand side of equation (A.56) is a power function of the common goods unit expenditure function ($(P_t^*)^{1-\sigma}$), we can write appeal as:

$$\varphi_{kt} = \frac{p_{kt}}{P_t^*}\left(s_{kt}^*\right)^{\frac{1}{\sigma-1}}\xi_t, \tag{A.57}$$

where ξ_t depends on the choice of units in which to measure appeal and hence the cardinalization of utility. Taking the geometric mean across goods in equation (A.57), we also have:

$$\widetilde{\varphi}_t = \frac{\widetilde{p}_t}{P_t^*}\left(\widetilde{s}_t^*\right)^{\frac{1}{\sigma-1}}\xi_t, \tag{A.58}$$

where a tilde above a variable denotes a geometric mean across common goods, such that $\widetilde{x}_t = \left(\prod_{k=1}^{N_t^*} x_{kt}\right)^{1/N_t^*}$.

Using equation (A.58) to substitute for ξ_t in equation (A.57), we can equivalently write appeal for each common good:

$$\frac{\varphi_{kt}}{\widetilde{\varphi}_t} = \frac{p_{kt}}{\widetilde{p}_t}\left(\frac{s_{kt}^*}{\widetilde{s}_t^*}\right)^{\frac{1}{\sigma-1}}, \tag{A.59}$$

which holds for any multiplicative choice of units in which to measure appeal (and hence for any cardinalization of utility).

A.4 Proof of Proposition 1 (CES Aggregation)

Forward Difference of the Unit Expenditure Function We start with the forward difference of the unit expenditure function from equation (8) in the paper, which can be written as:

$$\frac{P_t}{P_{t-1}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\left[\sum_{k\in\Omega_t^*} s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}, \tag{A.60}$$

which can be re-written as follows:

$$\frac{P_t^*}{P_{t-1}^*} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\Theta_t^F\left[\sum_{k\in\Omega_t^*} s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{A.61}$$

where Θ_t^F is a forward aggregate demand shifter that is defined as:

$$\Theta_t^F \equiv \left[\frac{\sum_{k\in\Omega_t^*} s_{kt-1}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}}{\sum_{k\in\Omega_t^*} s_{kt-1}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}}, \tag{A.62}$$

Note that the common goods expenditure share in equation (6) in the paper implies:

$$\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1} = \frac{s_{kt}^*}{s_{kt-1}^*} \left(\frac{P_t^*}{P_{t-1}^*}\right)^{1-\sigma}. \tag{A.63}$$

Using this result, we can re-write the forward aggregate demand shifter in equation (A.62) as follows:

$$\Theta_t^F \equiv \left[\frac{\sum_{k\in\Omega_t^*} s_{kt-1}^* \frac{s_{kt}^*}{s_{kt-1}^*} \left(\frac{P_t^*}{P_{t-1}^*}\right)^{1-\sigma}}{\sum_{k\in\Omega_t^*} s_{kt-1}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}}. \tag{A.64}$$

Note that the common goods expenditure share in equation (6) in the paper also implies:

$$\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma} = \frac{s_{kt}^*}{s_{kt-1}^*} \left(\frac{P_t^*}{P_{t-1}^*}\right)^{1-\sigma} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{1-\sigma}, \tag{A.65}$$

Using this result, we can re-write the forward aggregate demand shifter in equation (A.64) as follows:

$$\Theta_t^F \equiv \left[\frac{\sum_{k\in\Omega_t^*} s_{kt-1}^* \frac{s_{kt}^*}{s_{kt-1}^*} \left(\frac{P_t^*}{P_{t-1}^*}\right)^{1-\sigma}}{\sum_{k\in\Omega_t^*} s_{kt-1}^* \frac{s_{kt}^*}{s_{kt-1}^*} \left(\frac{P_t^*}{P_{t-1}^*}\right)^{1-\sigma} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}}, \tag{A.66}$$

which simplifies to:

$$\Theta_t^F \equiv \left[\frac{1}{\sum_{k\in\Omega_t^*} s_{kt}^* \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}} = \left[\sum_{k\in\Omega_t^*} s_{kt}^* \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(\sigma-1)} \right]^{-\frac{1}{1-\sigma}}. \tag{A.67}$$

Using this result in equation (A.61), we obtain the expression for the forward difference of the unit expenditure function in Proposition 1 in the paper:

$$\Phi_t^F \left(\left\{ \frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}} \right\}_{k\in\Omega_t\cup\Omega_{t-1}} \right) = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}} \frac{\Phi_t^{F*}\left(\left\{ \frac{p_{kt}}{p_{kt-1}} \right\}_{k\in\Omega_t^*} \right)}{\Phi_t^{B*}\left(\left\{ \frac{\varphi_{kt}}{\varphi_{kt-1}} \right\}_{k\in\Omega_t^*} \right)},$$

$$\Phi_t^{F*}\left(\left\{ \frac{p_{kt}}{p_{kt-1}} \right\}_{k\in\Omega_t^*} \right) = \left[\sum_{k\in\Omega_t^*} s_{kt-1}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}},$$

$$\Phi_t^{B*}\left(\left\{ \frac{\varphi_{kt}}{\varphi_{kt-1}} \right\}_{k\in\Omega_t^*} \right) = \left[\sum_{k\in\Omega_t^*} s_{kt}^* \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}.$$

Backward Difference of the Unit Expenditure Function We next turn to the backward difference of the unit expenditure function from equation (9) in the paper, which can be written as:

$$\frac{P_t}{P_{t-1}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}} \left[\sum_{k\in\Omega_t^*} s_{kt}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(\sigma-1)}\right]^{-\frac{1}{1-\sigma}}, \tag{A.68}$$

which can be further re-written as follows:

$$\frac{P_t}{P_{t-1}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}} \Theta_t^B \left[\sum_{k\in\Omega_t^*} s_{kt}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}, \tag{A.69}$$

where Θ_t^B is a backward aggregate demand shifter that is defined as:

$$\Theta_t^B \equiv \left[\frac{\sum_{k\in\Omega_t^*} s_{kt}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(\sigma-1)}}{\sum_{k\in\Omega_t^*} s_{kt}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}}\right]^{-\frac{1}{1-\sigma}}. \tag{A.70}$$

Note that the common goods expenditure share in equation (6) in the paper implies:

$$\left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(\sigma-1)} = \frac{s_{kt-1}^*}{s_{kt}^*}\left(\frac{P_{t-1}^*}{P_t^*}\right)^{1-\sigma}. \tag{A.71}$$

Using this result, we can re-write the backward aggregate demand shifter in equation (A.70) as:

$$\Theta_t^B \equiv \left[\frac{\sum_{k\in\Omega_t^*} s_{kt}^* \frac{s_{kt-1}^*}{s_{kt}^*}\left(\frac{P_{t-1}^*}{P_t^*}\right)^{1-\sigma}}{\sum_{k\in\Omega_t^*} s_{kt}^* \left(\frac{p_{kt-1}}{p_{kt}}\right)^{1-\sigma}}\right]^{-\frac{1}{1-\sigma}}. \tag{A.72}$$

Note that the common goods expenditure share in equation (6) in the paper also implies:

$$\left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)} = \frac{s_{kt-1}^*}{s_{kt}^*}\left(\frac{P_{t-1}^*}{P_t^*}\right)^{1-\sigma}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(1-\sigma)}. \tag{A.73}$$

Using this result, we can re-write the backward aggregate demand shifter in equation (A.72) as:

$$\Theta_t^B \equiv \left[\frac{\sum_{k\in\Omega_t^*} s_{kt}^* \frac{s_{kt-1}^*}{s_{kt}^*}\left(\frac{P_{t-1}^*}{P_t^*}\right)^{1-\sigma}}{\sum_{k\in\Omega_t^*} s_{kt}^* \frac{s_{kt-1}^*}{s_{kt}^*}\left(\frac{P_{t-1}^*}{P_t^*}\right)^{1-\sigma}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(1-\sigma)}}\right]^{-\frac{1}{1-\sigma}}, \tag{A.74}$$

which simplifies to:

$$\Theta_t^B \equiv \left[\frac{1}{\sum_{k\in\Omega_t^*} s_{kt-1}^* \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(1-\sigma)}}\right]^{-\frac{1}{1-\sigma}} = \left[\sum_{k\in\Omega_t^*} s_{kt-1}^* \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}. \tag{A.75}$$

Using this result in equation (A.69), we obtain the expression for the forward difference of the unit expenditure function in Proposition 1 in the paper:

$$\Phi_t^B\left(\left\{\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right\}_{k\in\Omega_t\cup\Omega_{t-1}}\right) = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}} \frac{\Phi_t^{B*}\left(\left\{\frac{p_{kt}}{p_{kt-1}}\right\}_{k\in\Omega_t^*}\right)}{\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)},$$

$$\Phi_t^{B*}\left(\left\{\frac{p_{kt}}{p_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\left[\sum_{k\in\Omega_t^*}s_{kt}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}},$$

$$\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\left[\sum_{k\in\Omega_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}.$$

A.5 Proof of Proposition 2 (Appeal Indexes)

Assume that price and appeal shocks are independently distributed across varieties and uncorrelated with one another and initial-period expenditure shares, such that the following moment condition holds:

$$\text{plim}_{N_t^*\to\infty}\left\{\left[\frac{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}\right]}{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\right]\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}\right]}\right]^{\frac{1}{1-\sigma}}\right\}=1. \tag{A.76}$$

Note that the common goods expenditure share in equation (6) in the paper implies:

$$s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}=s_{kt}^*\left(\frac{P_t^*}{P_{t-1}^*}\right)^{1-\sigma}. \tag{A.77}$$

Using this result, we can re-write equation (A.76) as follows:

$$\text{plim}_{N_t^*\to\infty}\left\{\frac{\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{P_t^*}{P_{t-1}^*}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}}\right\}=1. \tag{A.78}$$

Note that the common goods expenditure share in equation (6) in the paper also implies:

$$s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}=s_{kt}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(\sigma-1)}\left(\frac{P_t^*}{P_{t-1}^*}\right)^{1-\sigma}. \tag{A.79}$$

Using this result, we can re-write equation (A.78) as follows:

$$\text{plim}_{N_t^*\to\infty}\left\{\frac{\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{P_t^*}{P_{t-1}^*}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(\sigma-1)}\left(\frac{P_t^*}{P_{t-1}^*}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}}\right\}=1, \tag{A.80}$$

which simplifies to:

$$\text{plim}_{N_t^*\to\infty}\left\{\frac{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}\right\}=1. \tag{A.81}$$

Using the definitions of $\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)$ and $\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)$ from Proposition 1 in the paper, we have established Proposition 2 in the paper:

$$\lim_{N_t^*\to\infty}\left\{\frac{\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k=1}^{N_t^*}\right)}{\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}^{N_t^*}\right)}\right\}=1.$$

A.6 Proof of Proposition 3 (First-Order Approximation)

Backward Appeal Index From the proof of Proposition 1, the backward appeal index can be written in the following two equivalent forms:

$$\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}=\left[\frac{\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}}{\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}}\right]^{-\frac{1}{1-\sigma}}. \tag{A.82}$$

Taking logarithms in the second of these expressions, we have:

$$\log\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=-\frac{1}{1-\sigma}\log\left[\frac{\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}}{\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}}\right]. \tag{A.83}$$

Taking a Taylor-series expansion of $\log\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)$ around $(p_{kt}/p_{kt-1})=1$ and $(\varphi_{kt}/\varphi_{kt-1})=1$, we obtain:

$$\begin{aligned}\log\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=&-\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}-1\right)+\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right)\\&+\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}-1\right)+O_F^2\left(\mathbf{s},\mathbf{p}\right),\end{aligned} \tag{A.84}$$

which simplifies to:

$$\log\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right)+O_B^2\left(\mathbf{s_{t-1}},\mathbf{p_t},\mathbf{p_{t-1}}\right), \tag{A.85}$$

where bold math font denotes a vector; and $O_B^2\left(\mathbf{s_{t-1}},\mathbf{p_t},\mathbf{p_{t-1}}\right)$ denotes the second-order and higher terms, such that:

$$\begin{aligned}O_B^2\left(\mathbf{s_{t-1}},\mathbf{p_t},\mathbf{p_{t-1}}\right)=&-(2-\sigma)\sum_{k=1}^{N_t^*}s_{kt-1}^*\left[\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right]\left[\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right]\\&+2(1-\sigma)\sum_{k=1}^{N_t^*}s_{kt-1}^*\left[\frac{p_{kt}}{p_{kt-1}}-1\right]\left[\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right]\\&+(1-\sigma)\sum_{k=1}^{N_t^*}\sum_{\ell=1}^{N_t^*}s_{kt-1}^*s_{\ell t-1}^*\left[\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right]\left[\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right]\\&-2(1-\sigma)\sum_{k=1}^{N_t^*}\sum_{\ell=1}^{N_t^*}s_{kt-1}^*s_{\ell t-1}^*\left[\frac{p_{kt}}{p_{kt-1}}-1\right]\left[\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right]+O_B^3\left(\mathbf{s_{t-1}},\mathbf{p_t},\mathbf{p_{t-1}}\right),\end{aligned} \tag{A.86}$$

where these second-order terms depend on the covariance of price and appeal shocks; and $O_B^3\left(\mathbf{s_{t-1}},\mathbf{p_t},\mathbf{p_{t-1}}\right)$ denotes the third-order and higher terms.

Forward Appeal Index From the proof of Proposition 1, the forward appeal index can be written in the following two equivalent forms:

$$\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}=\left[\frac{\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(\sigma-1)}}{\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}}\right]^{\frac{1}{1-\sigma}}. \tag{A.87}$$

Taking logarithms in the first of these expressions, we have:

$$\log\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\frac{1}{\sigma-1}\log\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}\right] \tag{A.88}$$

Taking a Taylor-series expansion of $\log\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)$ around $(p_{kt}/p_{kt-1})=1$ and $(\varphi_{kt}/\varphi_{kt-1})=1$, we obtain:

$$\log\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right)+O_F^2\left(\mathbf{s_{t-1}},\mathbf{p_{t-1}},\mathbf{p_t}\right), \tag{A.89}$$

where $O_F^2\left(\mathbf{s_{t-1}},\mathbf{p_{t-1}},\mathbf{p_t}\right)$ denotes the second-order and higher terms, such that:

$$\begin{aligned}O_F^2\left(\mathbf{s_{t-1}},\mathbf{p_{t-1}},\mathbf{p_t}\right)=&(2-\sigma)\sum_{k=1}^{N_t^*}s_{kt-1}^*\left[\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right]\left[\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right]\\&-(1-\sigma)\sum_{k=1}^{N_t^*}\sum_{\ell=1}^{N_t^*}s_{kt-1}^*s_{\ell t-1}^*\left[\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right]\left[\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right]+O_F^3\left(\mathbf{s_{t-1}},\mathbf{p_{t-1}},\mathbf{p_t}\right).\end{aligned} \tag{A.90}$$

where $O_F^3\left(\mathbf{s_{t-1}},\mathbf{p_{t-1}},\mathbf{p_t}\right)$ denotes the third-order and higher terms. Therefore, from equations (A.85) and (A.89), the forward and backward appeal indexes are equal to one another up to a first-order approximation:

$$\log\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)\approx\log\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)\approx\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}-1\right). \tag{A.91}$$

A.7 Proof of Proposition 4 (Consistency OLS Demand Systems)

From CES demand in equation (12) in the paper, we have:

$$\log\left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}^*}\right)=(1-\sigma)\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)+(\sigma-1)\log\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right), \tag{A.92}$$

where, in general, $\log\left(\left(\varphi_{kt}/\tilde{\varphi}_t\right)/\left(\varphi_{kt-1}/\tilde{\varphi}_{t-1}\right)\right)$ and $\log\left(\left(p_{kt}/\tilde{p}_t\right)/\left(p_{kt-1}/\tilde{p}_{t-1}\right)\right)$ are correlated. From the joint log normality of appeal and price shocks in equation (15) in the paper, we have:

$$\log\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)=\gamma\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)+\delta\log\left(\frac{v_{kt}}{v_{kt-1}}\right), \tag{A.93}$$

where $\log\left(v_{kt}/v_{kt-1}\right)$ is orthogonal to $\log\left(\left(p_{kt}/\tilde{p}_t\right)/\left(p_{kt-1}/\tilde{p}_{t-1}\right)\right)$ by construction. Using equation (A.93) to substitute for appeal shocks in equation (A.92), we obtain:

$$\log\left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}^*}\right)=(1-\sigma)(1-\gamma)\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)+\delta(\sigma-1)\log\left(\frac{v_{kt}}{v_{kt-1}}\right). \tag{A.94}$$

We can represent equation (A.94) as the following reduced-form regression that can be estimated using OLS:

$$\log\left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}^*}\right)=\beta\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)+\log\left(\frac{u_{kt}}{u_{kt}}\right), \tag{A.95}$$

where $\beta = (1-\sigma)(1-\gamma)$ and $\log(u_{kt}/u_{kt}) = \delta(\sigma-1)\log\left(\frac{v_{kt}}{v_{kt-1}}\right)$ is orthogonal to $\log\left((p_{kt}/\tilde{p}_t)/(p_{kt-1}/\tilde{p}_{t-1})\right)$ by construction. From the normal equations, the OLS estimated slope coefficient is:

$$\hat{\beta}^{OLS} = \frac{\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\log\left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}^*}\right)\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)}{\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^2}. \tag{A.96}$$

As the number of common goods becomes large ($N_t^* \to \infty$), the sample covariance of log sales and prices in the numerator and the sample variance of log prices in the denominator converge to their population counterparts. Using these consistency properties of OLS and equation (A.94), we obtain:

$$\text{plim}_{N_t^*\to\infty}\left\{\hat{\beta}^{OLS}\right\} = (1-\sigma)(1-\gamma), \tag{A.97}$$

which establishes the first part of the proposition. Using this result (A.97) and equations (A.94) and (A.95), as the number of common goods becomes large, the estimated OLS residual converges to its population counterpart:

$$\begin{aligned}\text{plim}_{N_t^*\to\infty}\left\{\log\left(\frac{\widehat{u}_{kt}^{OLS}}{\widehat{u}_{kt-1}^{OLS}}\right)\right\} &= \text{plim}_{N_t^*\to\infty}\left\{\log\left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}^*}\right) - \hat{\beta}\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)\right\}, \\ &= \left[\log\left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}^*}\right) - (1-\sigma)(1-\gamma)\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)\right] \\ &= \delta(\sigma-1)\log\left(\frac{v_{kt}}{v_{kt-1}}\right).\end{aligned} \tag{A.98}$$

Using this result (A.98) and noting that $\log(v_{kt}/v_{kt-1})$ has an independent standard normal distribution, $\delta(\sigma-1)$ can be consistently estimated from the standard deviation of the OLS residuals:

$$\text{plim}_{N_t^*\to\infty}\left\{\sqrt{\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\log\left(\frac{\widehat{u}_{kt}^{OLS}}{\widehat{u}_{kt-1}^{OLS}}\right)^2}\right\} = \delta(\sigma-1), \tag{A.99}$$

and the marginal appeal shocks $\log(v_{kt}/v_{kt-1})$ can be consistently estimated from the standardized OLS residuals:

$$\text{plim}_{N_t^*\to\infty}\left\{\frac{\log\left(\frac{\widehat{u}_{kt}^{OLS}}{\widehat{u}_{kt-1}^{OLS}}\right)}{\sqrt{\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\log\left(\frac{\widehat{u}_{kt}^{OLS}}{\widehat{u}_{kt-1}^{OLS}}\right)^2}}\right\} = \log\left(\frac{v_{kt}}{v_{kt-1}}\right), \tag{A.100}$$

which completes the proof of the proposition.

A.8 Proof Proposition 5 (Consistency Small Appeal Shocks)

Forward-Backward Estimator From equations (8) and (9) in the paper, we have the following equality between the forward and backward differences of the unit expenditure function:

$$\left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} = \left[\sum_{k=1}^{N_t^*} s_{kt}^*\left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}},$$

where the variety correction terms have cancelled from both sides. We can rewrite this equality as:

$$\frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}\left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} = \frac{\left[\sum_{k=1}^{N_t^*} s_{kt}^*\left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*} s_{kt}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}}\left[\sum_{k=1}^{N_t^*} s_{kt}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}.$$

Using the common goods expenditure share (6), we can further re-write this equality as:

$$\Theta_t^F \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} = \Theta_t^B \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}. \tag{A.101}$$

$$\Theta_t^F \equiv \frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}} = \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(\sigma-1)}\right]^{-\frac{1}{1-\sigma}},$$

$$\Theta_t^B \equiv \frac{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(\sigma-1)}\right]^{-\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}} = \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}.$$

We thus have:

$$\begin{array}{c} \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}}\right)^{-(\sigma-1)}\right]^{-\frac{1}{1-\sigma}} \\ \times \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \end{array} = \begin{array}{c} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}} \\ \times \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}} \end{array}, \tag{A.102}$$

where we have divided both sides of equation (A.102) by $\widetilde{\varphi}_t/\widetilde{\varphi}_{t-1}$ and $\widetilde{p}_t/\widetilde{p}_{t-1}$ to make clear that this relationship is invariant to the units in which prices and appeal are measured. Taking the limit as appeal shocks become small for each good ($\varphi_{kt}/\varphi_{kt-1} \to 1$ for each k), we have:

$$\lim_{\{\varphi_{kt}/\varphi_{kt-1}\to 1\}_{k=1}^{N_t^*}} \left\{ \log\left\{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}}\right)^{-(\sigma-1)}\right]^{-\frac{1}{1-\sigma}}\right\} - \log\left\{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}\right\}\right\} = 0.$$

Using this result in equation (A.102), we obtain the forward-backward moment condition:

$$\lim_{\{\varphi_{kt}/\varphi_{kt-1}\to 1\}_{k=1}^{N_t^*}} \left\{ \log\left\{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\right\} - \log\left\{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}\right\}\right\} = 0, \tag{A.103}$$

Therefore, as appeal shocks become small for each good ($\varphi_{kt}/\varphi_{kt-1} \to 1$ for each k), the forward-backward estimator consistently estimates the elasticity of substitution (σ).

Reverse-Weighting Estimator From equations (8), (9) and (11) in the paper, we have the following equalities between (i) the forward difference and geometric mean difference of the unit expenditure function and (ii) the backward difference and geometric mean difference of the unit expenditure function:

$$\frac{\tilde{p}_t/\tilde{p}_{t-1}}{\tilde{\varphi}_t/\tilde{\varphi}_{t-1}} \left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}},$$

$$\frac{\tilde{p}_t/\tilde{p}_{t-1}}{\tilde{\varphi}_t/\tilde{\varphi}_{t-1}} \left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}},$$

where the variety correction terms have cancelled from both sides of the equality. We can rewrite these equalities as:

$$\frac{\tilde{p}_t/\tilde{p}_{t-1}}{\tilde{\varphi}_t/\tilde{\varphi}_{t-1}}\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}=\frac{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}},$$

$$\frac{\tilde{p}_t/\tilde{p}_{t-1}}{\tilde{\varphi}_t/\tilde{\varphi}_{t-1}}\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}=\frac{\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}}\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}},$$

Using the common goods expenditure share (6), we can further re-write these equalities as:

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}=\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{-(\sigma-1)}\right]^{-\frac{1}{1-\sigma}}\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{A.104}$$

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}=\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}},$$

where prices and appeal in each period are both scaled by their geometric means in each period, and hence this relationship is invariant to the units in which prices and appeal are measured. Taking the limit as appeal shocks become small for each good ($\varphi_{kt}/\varphi_{kt-1}\rightarrow 1$ for each k), we have:

$$\lim_{\{\varphi_{kt}/\varphi_{kt-1}\rightarrow 1\}_{k=1}^{N_t^*}}\left\{\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{-(\sigma-1)}\right]^{-\frac{1}{1-\sigma}}\right\}=1,$$

$$\lim_{\{\varphi_{kt}/\varphi_{kt-1}\rightarrow 1\}_{k=1}^{N_t^*}}\left\{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}\right\}=1.$$

Using these results in equation (A.104), we obtain the reverse-weighting moment conditions:

$$\lim_{\{\varphi_{kt}/\varphi_{kt-1}\rightarrow 1\}}\left\{\log\left\{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\right\}-\log\left\{\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right\}\right\}=0, \tag{A.105}$$

$$\lim_{\{\varphi_{kt}/\varphi_{kt-1}\rightarrow 1\}}\left\{\log\left\{\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}\right\}-\log\left\{\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right\}\right\}=0.$$

Therefore, as appeal shocks become small for each good ($\varphi_{kt}/\varphi_{kt-1}\rightarrow 1$ for each k), the reverse-weighting estimator consistently estimates the elasticity of substitution (σ).

A.9 Joint Log Normal and Correlated Price and Appeal Shocks

Forward-Backward Estimator From equations (8) and (9) in the paper, we have the following equality between equivalent ways of writing the change in the unit expenditure function:

$$\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma}\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}=\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)}\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{-(\sigma-1)}\right]^{-\frac{1}{1-\sigma}}. \tag{A.106}$$

Using the assumption of joint log normally distributed price and appeals shocks from equation (15) in the paper, we can further re-write this equality as:

$$\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]^{\frac{1}{1-\sigma}} = \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{-\delta(\sigma-1)}\right]^{-\frac{1}{1-\sigma}}. \tag{A.107}$$

We can re-write this equality as:

$$\begin{aligned}&\frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]^{\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{1-\sigma}}} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{1-\sigma}} \\ &= \frac{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{-\delta(\sigma-1)}\right]^{-\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{1-\sigma}}} \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{1-\sigma}}.\end{aligned} \tag{A.108}$$

From the common goods expenditure share in equation (6) in the paper, we have:

$$\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)} = \left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}}\right)^{-1} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}. \tag{A.109}$$

Using this relationship, we can re-write equation (A.108) as:

$$\begin{aligned}&\frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]^{\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{1-\sigma}}} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{1-\sigma}} \\ &= \frac{1}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]^{-\frac{1}{1-\sigma}}} \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{1-\sigma}},\end{aligned} \tag{A.110}$$

and hence:

$$\begin{aligned}&\left[\frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}} \\ &= \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{(1-\sigma)(1-\gamma)}}.\end{aligned} \tag{A.111}$$

Since $\log\left(\frac{v_{kt}}{v_{kt-1}}\right)$ is an independent standard normal random variable, the expectation of the product of price shocks and this independent standard normal random variable is equal to the product of their expectations:

$$\text{plim}_{N_t^* \to \infty}\left\{\left[\frac{\left[\frac{1}{N_t^*}\sum_{k=1}^{N_t^*} \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]}{\left[\frac{1}{N_t^*}\sum_{k=1}^{N_t^*} \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]\left[\frac{1}{N_t^*}\sum_{k=1}^{N_t^*} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\right\} = 1. \tag{A.112}$$

Noting that common goods expenditure shares at time $t-1$ (s_{kt-1}^*) are pre-determined at time t, we can replace the unweighted expectation in equation (A.112) by the weighted expectation using initial expenditure share weights to obtain:

$$\text{plim}_{N_t^* \to \infty} \left\{ \left[\frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}} \right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}} \right)^{\delta(\sigma-1)} \right]}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}} \right)^{(1-\sigma)(1-\gamma)} \right] \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{v_{kt}}{v_{kt-1}} \right)^{\delta(\sigma-1)} \right]} \right]^{\frac{1}{(1-\sigma)(1-\gamma)}} \right\} = 1. \tag{A.113}$$

Using this result in equation (A.111), we have:

$$\text{plim}_{N_t^* \to \infty} \left\{ \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}} \right)^{(1-\sigma)(1-\gamma)} \right]^{\frac{1}{(1-\sigma)(1-\gamma)}} = \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}} \right)^{-(1-\sigma)(1-\gamma)} \right]^{-\frac{1}{(1-\sigma)(1-\gamma)}} \right\}. \tag{A.114}$$

Taking logarithms, we obtain:

$$\text{plim}_{N_t^* \to \infty} \left\{ \log \left\{ \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}} \right)^{(1-\sigma)(1-\gamma)} \right]^{\frac{1}{(1-\sigma)(1-\gamma)}} \right\} - \log \left\{ \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}} \right)^{-(1-\sigma)(1-\gamma)} \right]^{-\frac{1}{(1-\sigma)(1-\gamma)}} \right\} \right\} = 0. \tag{A.115}$$

Comparing equation (A.115) to the moment condition for the FB estimator in equation (19), we see that when price and appeal shocks are joint log normally distributed and correlated, the FB estimator estimates $(1-\sigma)(1-\gamma)$ rather than $(1-\sigma)$. Noting that $\widehat{\beta}^{OLS} = (1-\sigma)(1-\gamma)$, it follows that when price and appeal shocks are joint log normally distributed and correlated, the FB estimator exhibits similar properties as the OLS estimator.

Reverse-Weighting Estimator From equations (8), (9) and (11) in the paper, we have the following two equalities between equivalent ways of writing the change in the unit expenditure function:

$$\frac{\tilde{p}_t/\tilde{p}_{t-1}}{\tilde{\varphi}_t/\tilde{\varphi}_{t-1}} \left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*} \right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}, \tag{A.116}$$

$$\frac{\tilde{p}_t/\tilde{p}_{t-1}}{\tilde{\varphi}_t/\tilde{\varphi}_{t-1}} \left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*} \right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}} \right)^{-(1-\sigma)} \right]^{-\frac{1}{1-\sigma}},$$

which can be written as:

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*} \right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}} \right)^{1-\sigma} \left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}} \right)^{\sigma-1} \right]^{\frac{1}{1-\sigma}}, \tag{A.117}$$

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*} \right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}} \right)^{-(1-\sigma)} \left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}} \right)^{-(\sigma-1)} \right]^{-\frac{1}{1-\sigma}}.$$

Using the assumption of joint log normally distributed price and appeals shocks from equation (15) in the paper, we can further re-write these equalities as:

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*} \right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}} \right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}} \right)^{\delta(\sigma-1)} \right]^{\frac{1}{1-\sigma}}, \tag{A.118}$$

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*} \right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}} \right)^{-(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}} \right)^{-\delta(\sigma-1)} \right]^{-\frac{1}{1-\sigma}}.$$

We can now re-write these expressions as:

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}} = \frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]^{\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{1-\sigma}}} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{1-\sigma}}, \tag{A.119}$$

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}} = \frac{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{-\delta(\sigma-1)}\right]^{-\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{1-\sigma}}} \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{1-\sigma}}.$$

From the common goods expenditure share in equation (6) in the paper, we have:

$$\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)} = \left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}}\right)^{-1} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}. \tag{A.120}$$

Using this relationship, we can re-write equation (A.119) as:

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{(\sigma-1)(1-\gamma)}} = \left[\frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}, \tag{A.121}$$

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{(\sigma-1)(1-\gamma)}} = \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}} \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{(1-\sigma)(1-\gamma)}},$$

Note that $(1-\sigma)(1-\gamma)\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)$ and $\delta(\sigma-1)\log\left(\frac{v_{kt}}{v_{kt-1}}\right)$ are jointly normally distributed and uncorrelated with one another. Therefore $\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}$ and $\left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}$ are joint log normally distributed and uncorrelated:

$$\begin{pmatrix} \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \\ \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)} \end{pmatrix} \sim \text{LogNormal}\left(\begin{bmatrix} \mu_{e\gamma} \\ \mu_{ev} \end{bmatrix}, \begin{bmatrix} \Xi_{\gamma\gamma} & \Xi_{\gamma v} \\ \Xi_{\gamma v} & \Xi_{vv} \end{bmatrix}\right), \tag{A.122}$$

$$\mu_{e\gamma} = \exp\left\{\frac{1}{2}(\sigma-1)^2(1-\gamma)^2\chi_p^2\right\}, \qquad \mu_{ev} = \exp\left\{\frac{1}{2}\delta^2(\sigma-1)^2\right\}, \qquad \Xi_{\gamma\varphi} = 0,$$

$$\Xi_{\gamma\gamma} = \Lambda \times \left[\exp\left\{(\sigma-1)^2(1-\gamma)^2\chi_p^2\right\} - 1\right], \qquad \Xi_{vv} = \Lambda \times \left[\exp\left\{\delta^2(\sigma-1)^2\right\} - 1\right].$$

Recall that the sum of normally distributed random variables is also normally distributed. Therefore, it follows that log adjusted price shocks ($\log\left[\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]$) are also normally distributed, and the level of adjusted price shocks is log normally distributed:

$$\left(\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right) \sim \text{LogNormal}\left(\mu_{er}, \Xi_{er}\right), \tag{A.123}$$

$$\mu_{er} = \exp\left\{\frac{1}{2}(\sigma-1)^2\left((1-\gamma)^2\chi_p^2 + \delta^2\right)\right\},$$

$$\Xi_{er} = \exp\left\{(\sigma-1)^2\left((1-\gamma)^2\chi_p^2+\delta^2\right)\right\}\left[\exp\left\{(\sigma-1)^2\left((1-\gamma)^2\chi_p^2+\delta^2\right)\right\}-1\right].$$

Using these results for a joint log normal distribution with zero correlation, we obtain:

$$\text{plim}_{N_t^*\to\infty}\left\{\left[\frac{\left[\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]}{\left[\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\right\}=\left[\exp\left\{\frac{1}{2}(\sigma-1)^2\delta^2\right\}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}. \tag{A.124}$$

$$\text{plim}_{N_t^*\to\infty}\left\{\left[\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\right\}=\left[\exp\left\{\frac{1}{2}(\sigma-1)^2\delta^2\right\}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}.$$

Noting that common goods expenditure shares at time $t-1$ (s_{kt-1}^*) are pre-determined at time t, we can replace the unweighted expectation in equation (A.124) by the weighted expectation using initial expenditure share weights to obtain:

$$\text{plim}_{N_t^*\to\infty}\left\{\left[\frac{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]}{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\right\}=\left[\exp\left\{\frac{1}{2}(\sigma-1)^2\delta^2\right\}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}} \tag{A.125}$$

$$\text{plim}_{N_t^*\to\infty}\left\{\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{v_{kt}}{v_{kt-1}}\right)^{\delta(\sigma-1)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\right\}=\left[\exp\left\{\frac{1}{2}(\sigma-1)^2\delta^2\right\}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}.$$

Using these results in equation (A.121), we obtain:

$$\text{plim}_{N_t^*\to\infty}\left\{\log\left\{\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{(\sigma-1)(1-\gamma)}}\right\}-\log\left\{\left[\exp\left\{\frac{1}{2}(\sigma-1)^2\delta^2\right\}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\left[\sum_{k=1}^{N_t^*}s_{kt-1}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\right\}\right\}=0, \tag{A.126}$$

$$\text{plim}_{N_t^*\to\infty}\left\{\log\left\{\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{(\sigma-1)(1-\gamma)}}\right\}-\log\left\{\left[\exp\left\{\frac{1}{2}(\sigma-1)^2\delta^2\right\}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\left[\sum_{k=1}^{N_t^*}s_{kt}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{(1-\sigma)(1-\gamma)}}\right\}\right\}=0$$

Comparing equation (A.126) with the reverse-weighting moment condition (22), we see that when price and appeal shocks are joint log normally distributed and correlated, the RW moment condition is in general not satisfied. Furthermore, the RW estimator in general differs from the FB estimator, because it is affected by standard deviation of appeal shocks (χ_φ) in equation (A.126), whereas the FB estimator in equation (A.115) is unaffected by the standard deviation of appeal shocks.

A.10 Logit

In this section of the web appendix, we show that our family of estimators for CES preferences also can be applied for logit preferences. Following McFadden (1974), we suppose that the utility of an individual consumer i who consumes c_{ik} units of product k at time t is given by:

$$U_{it} = u_{kt} + z_{ikt}, \qquad u_{kt} \equiv \log\varphi_{kt} + \log c_{ikt} \tag{A.127}$$

where φ_{kt} captures product appeal that is common across consumers; z_{ikt} captures idiosyncratic consumer tastes for each product that are drawn from an independent Type-I Extreme Value distribution:

$$G(z) = e^{-e^{(-z/\nu-\kappa)}}, \tag{A.128}$$

where ν is the scale parameter of the extreme value distribution and $\kappa \approx 0.577$ is the Euler-Mascheroni constant.

Each consumer has the same expenditure E_t and chooses their preferred product given the observed realizations for idiosyncratic tastes. Therefore the consumer's budget constraint implies:

$$c_{ikt} = \frac{E_t}{p_{ikt}}. \tag{A.129}$$

The probability that individual i chooses product k at time t is:

$$\begin{aligned} x_{ikt} &= \text{Prob}\left(u_{ikt} + z_{ikt} > u_{i\ell t} + z_{i\ell t}, \forall \ell \neq k\right), \\ &= \text{Prob}\left(z_{i\ell t} < z_{ikt} + v_{ikt} - v_{i\ell t}, \forall \ell \neq k\right). \end{aligned}$$

Therefore, using the distribution of idiosyncratic tastes (A.128), we have:

$$x_{ikt}|z_{ikt} = \prod_{\ell \neq k} e^{-e^{-\left(z_{ikt}+u_{ikt}-u_{i\ell}\right)/\nu+\kappa}}.$$

Integrating across the probability density function for z_{ikt}, we have:

$$x_{ikt} = \int_{-\infty}^{\infty} \left(\prod_{\ell \neq k} e^{-e^{-\left(y+u_{ikt}-u_{i\ell t}\right)/\nu+\kappa}} \right) \frac{1}{\nu} e^{-y/\nu+\kappa} e^{-e^{-(-y\nu+\kappa)}} dy.$$

Noting that $u_{ikt} - u_{ikt} = 0$, this expression can be re-written as:

$$x_{ikt} = \int_{-\infty}^{\infty} \left(\prod_{\ell \in \Omega_t} e^{-e^{-\left(y+u_{ikt}-u_{i\ell t}\right)/\nu+\kappa}} \right) \frac{1}{\nu} e^{-y/\nu+\kappa} dy,$$

which can be in turn re-written as:

$$x_{ikt} = \int_{-\infty}^{\infty} \exp\left(-\sum_{\ell \in \Omega_t} e^{-\left(y+u_{ikt}-u_{i\ell t}\right)/\nu+\kappa} \right) \frac{1}{\nu} e^{-y/\nu+\kappa} dy,$$

and hence:

$$x_{ikt} = \int_{-\infty}^{\infty} \exp\left(-e^{-y/\nu+\kappa} \sum_{\ell \in \Omega_t} e^{-\left(u_{ikt}-u_{i\ell t}\right)/\nu} \right) \frac{1}{\nu} e^{-y/\nu+\kappa} dy.$$

Now define the following change of variable:

$$h = \exp\left(-y/\nu + \kappa\right),$$

where

$$-\frac{1}{\nu} \exp\left(-y/\nu + \kappa\right) dy = dh.$$

As $y \to \infty$, we have $h \to 0$. As $y \to -\infty$, we have $h \to \infty$. Using this change of variable, we have:

$$x_{ikt} = \int_{\infty}^{0} \exp\left(-h \sum_{\ell \in \Omega_t} e^{-(u_{ikt} - u_{i\ell t})/\nu}\right) - dh,$$

or equivalently:

$$x_{ikt} = \int_{0}^{\infty} \exp\left(-h \sum_{\ell \in \Omega_t} e^{-(u_{ikt} - u_{i\ell t})/\nu}\right) dh,$$

which yields:

$$x_{ikt} = \left[\frac{\exp\left(-h \sum_{\ell \in \Omega_t} e^{-(u_{ikt} - u_{i\ell t})/\nu}\right)}{-\sum_{\ell \in \Omega_t} e^{-(u_{ikt} - u_{i\ell t})/\nu}}\right]_0^{\infty},$$

and hence:

$$x_{ikt} = \frac{1}{\sum_{\ell \in \Omega_t} e^{-(u_{ikt} - u_{i\ell t})/\nu}}.$$

The probability that individual i chooses product k at time t is therefore:

$$x_{ikt} = \frac{e^{u_{ikt}/\nu}}{\sum_{\ell \in \Omega_t} e^{u_{i\ell t}/\nu}},$$

which from the definition of u_{ikt} in (A.127) and the consumer's budget constraint in (A.129) becomes:

$$s_{ikt} = s_{kt} = \frac{\left(p_{kt}/\varphi_{kt}\right)^{-1/\nu}}{\sum_{\ell \in \Omega_t} \left(p_{\ell t}/\varphi_{\ell t}\right)^{-1/\nu}}. \tag{A.130}$$

As shown in Anderson, De Palma and Thisse (1992) and Train (2009), the expected utility of consumer i at time t is:

$$\mathbb{E}\left[U_{it}\right] = \mathbb{E}\left[\max\left\{u_{i1t} + z_{i1t}, \ldots, u_{iNt} + z_{iNt}\right\}\right] = \nu \log\left[\sum_{\ell \in \Omega_t} \exp\left(\frac{u_{i\ell t}}{\nu}\right)\right]. \tag{A.131}$$

Using the definition of u_{ikt} in (A.127) and the consumers budget constraint in (A.129), expected utility can be written as:

$$\mathbb{E}\left[U_{it}\right] = \log\left[\frac{E_t}{P_t}\right] \tag{A.132}$$

where P_t is the unit expenditure function:

$$P_t = \left[\sum_{k \in \Omega_t} \left(p_{kt}/\varphi_{kt}\right)^{-1/\nu}\right]^{-\nu}. \tag{A.133}$$

Total expenditure on product k across all consumers i at time t is:

$$E_t = \sum_i E_{ikt} = \sum_i s_{kt} E_{it} = s_{kt} E_t, \tag{A.134}$$

where we have used the fact that each consumer has the same expenditure E_t. Combining equations (A.130) and (A.134), total expenditure on product k at time t can be written as:

$$E_{kt} = \left(p_{kt}/\varphi_{kt}\right)^{-1/\nu} P_t^{1/\nu} E_t, \tag{A.135}$$

where P_t is again the unit expenditure function (A.133).

Comparing equations (A.130) and (A.133) for logit preferences to equations (2) and (1) in the paper for CES preferences, the predictions of logit demand for aggregate expenditure shares and the unit expenditure function are the same as those of CES demand, where $1/\nu = \sigma - 1$. Therefore, our FB and RW estimators can be applied to estimate the dispersion of idiosyncratic tastes (ν) in the logit model, using this relationship that $1/\nu = \sigma - 1$.

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