Aggregation and Estimation of Constant Elasticity of Substitution (CES) Preferences*

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Abstract

We provide new aggregation and estimation results for constant elasticity of substitution (CES) preferences. We show that CES preferences have a fractal-like property: The change in the appeal-adjusted price of an individual variety can be written as the ratio of price changes to appeal changes, and the change in the aggregate unit expenditure function can be written as the ratio of price indexes to appeal indexes. We show that this property holds regardless of the correlation between price changes and appeal changes and regardless of the cardinality of utility. We characterize the properties of these appeal indexes and use these properties to develop two new estimators of the elasticity of substitution: the forward-backward and reverse-weighting estimators. We provide conditions under which these estimators consistently estimate the elasticity of substitution. Even when these conditions are not satisfied, we show that a comparison of the forward-backward and OLS estimators provides a specification check on the combined assumption of CES preferences and joint log normally distributed price changes and appeal changes.

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1 Introduction

One of the most commonly-used preferences structures in international trade and macroeconomics is the constant elasticity of substitution (CES) preference structure. The assumption of a constant elasticity substitution brings substantial gains in tractability and allows this specification easily to be embedded in general equilibrium. A close relative is the logit preference structure in discrete choice settings, which aggregates across individual consumers to yield a constant elasticity preference structure at the aggregate level. More broadly, the CES preference structure provides a tractable approach to introducing variable substitution elasticities through the assumption of multiple types of consumers with heterogeneous (constant) elasticities of substitution, as in the mixed CES and mixed logit specifications, or multiple nests of utility, as in the nested CES or nested logit specifications. For example, if there are multiple types of consumers with different elasticities of substitution, and the shares of these types in the population varies across markets, one obtains variable elasticity of substitution at the market level that depends on the composition of types.

In this paper, we provide new aggregation and estimation results for CES preferences. We develop our main results for a single CES nest, where utility is defined over the consumption of horiziontally-differentiated varieties. We allow each variety to differ in terms of its price and appeal, where appeal corresponds to the CES weighting parameter for each variety, which captures both vertical differences in product quality and horizontal differences in product characteristics. We being by developing our main aggregation result. We show that CES preferences have a fractal-like property: The change in the appeal-adjusted price of an individual variety can be written as the ratio of price changes to appeal changes, and the change in the aggregate unit expenditure function can be written as the ratio of price indexes to appeal indexes. We show that this result holds regardless of the correlation between price changes and appeal changes and regardless of the cardinality of utility (the choice of units in which to measure appeal). In the presence of variety entry and exit, we show that this aggregation result holds for the subset of surviving goods, which we refer to as common goods, in the sense that they are common to a pair of time periods. For nested CES, this aggregation result holds for each nest of utility. For mixed CES, it holds for each type of consumers.

We next establish some properties of these appeal indexes underlying our aggregation result. We show that they are expenditure share weighted averages of the appeal changes for each variety. Furthermore, there are two alternative representations of these appeal indexes, one using expenditure shares in the initial period, and one using expenditure shares in the end period. If price changes and appeal changes are independently distributed and uncorrelated across varieties, we show that these two appeal indexes are equal to one another. More generally, when price changes and appeal changes are correlated across varieties, we show that this equality holds up to a first-order approximation.

We use these properties of the appeal indexes to develop two new estimators of the elasticity of substitution between varieties: the forward-backward and reverse-weighting estimators. We show that both are consistent estimators of the elasticity of substitution as appeal shocks become small for each good. More generally, for large appeal shocks, these two estimators differ from one another. Therefore, the comparison between them provides a metric for assessing the empirical relevance of appeal shocks in the estimation of demand systems for CES preferences.

In the case in which preferences are CES and price changes and appeal changes are correlated and joint log normally distributed, we show that the forward-backward estimator converges to the ordinary least squares (OLS) slope coefficient from regressing log changes in expenditure shares on log changes in prices. In this case in which price changes and appeal changes are correlated, neither the forward-backward estimator nor the OLS estimator are consistent. But the comparison of the forward-backward estimator and the OLS slope coefficient provides a specification check on the combined assumptions of CES preferences and a joint log normal distribution of price changes and appeal changes. Although this result requires a functional form assumption for both preferences and the distribution of price and appeal shocks, this assumption of a joint log normal distribution can be rationalized through a central limit theorem argument as the limiting distribution of correlated price and appeal shocks as the number of varieties becomes large, if these shocks are independently distributed across varieties.

Throughout the paper, we illustrate each of these analytical results using a Monte Carlo simulation assuming CES preferences and joint log normally distributed price and appeal shocks. Under the assumption of CES preferences and monopolistic competition, markups are constant for each variety, and this joint log normal distribution for price and appeal shocks follows from the assumption that marginal cost and appeal shocks are joint log normally distributed.

Our research is related to the large literature on CES preferences in international trade and macroeconomics, including Dixit and Stiglitz (1977), Anderson and van Wincoop (2003), Krugman (1980), Krugman (1991), Feenstra (1994), Melitz (2003), Broda and Weinstein (2006), Bernard et al. (2007), Hsieh and Klenow (2009), Hottman et al. (2016), and Jaravel (2019), among many others. Our results also speak to the wide applied micro literature using logit preferences for discrete choice problems. In such specifications, individual consumers make different discrete decisions based on idiosyncratic preference shocks. However, integrating across these individuals, aggregate consumer behavior is the same as under CES preferences, as shown in Anderson et al. (1992) and Train (2009).

Our paper is most closely connected to existing research on CES price indexes. Sato (1976) and Vartia (1976) derive an exact price index for the change in the unit expenditure function between a pair of time periods under the assumption of CES preferences, a constant set of goods, and constant appeal for each good. This price index depends only on observed prices and expenditure shares and not on the elasticity of substitution. More generally, Feenstra (1994) derives the exact price index for CES preferences in the presence of entry and exit under the assumption of constant appeal for each good. This Feenstra price index combines the Sato-Vartia index for the subset of common goods and a variety correction term for entry and exit. This variety correction term depends on the share of common goods in total expenditure in each time period and the elasticity of substitution between varieties. Finally, Redding and Weinstein (2020) develop an exact price index for CES preferences in the presence of entry and exit and changes in appeal for each variety. This price index assumes a choice of units for appeal (a normalization), such that the mean log change in appeal is equal to zero across common goods. Relative to this literature, we derive a new aggregation result for CES preferences, and use it to derive estimators of the elasticity of substitution in the presence of entry and exit and changes in appeal for each variety. Our aggregation result and our estimators continue to hold regardless of the choice of units for appeal (regardless of the cardinality of utility).

Our work also makes connections to the broader literature on price indexes. The Laspeyres (1871) price index computes changes in the cost of living using initial-period expenditure shares. In contrast, the Paasche (1875) price index evaluates changes in the cost of living using end-period expenditure shares. A classic theoretical result in this price index literature is that these two price indexes bound the change in the cost of living for a general preference structure with a constant set of goods and constant appeal for each good. For CES preferences under these same assumptions of a constant set of goods and constant appeal for each good, Moulton (1996) shows that the change in unit expenditure function can be written in a share-relative form, which uses either initial-period or end-period expenditure shares and the elasticity of substitution. Balk (1999) uses these share-relative price indexes and the Feenstra (1994) variety correction to examine the respective importance of substitution bias and new goods for the measurement of changes in the cost of living. We extend the share-relative price indexes in Moulton (1996) to incorporate changes in appeal for each good. Our main aggregation result starts with these generalized price indexes and shows that the change in the CES unit expenditure function can be written as the ratio of a price index to an appeal index, where both of these indexes are expressed in share-relative form.

Our analysis also builds on the traditional literature on demand systems estimation, dating back to Wright (1928, 1934) and Theil (1953), as reviewed in Pollak and Wales (1992). CES preferences imply a log linear relationship between changes in expenditure shares, prices and appeal. If appeal changes are unobserved and correlated with price changes, the OLS slope coefficient in a regression of log changes in expenditure shares on log changes in prices is biased and inconsistent, because of conventional omitted variables bias. In the case where changes in prices and appeal are joint log normally distribution, the OLS slope coefficient converges asymptotically to one minus the elasticity of substitution multiplied by one minus the projection coefficient of log changes in prices on log changes in appeal. We combine this well-known result for OLS with our new forward-backward estimator to develop a specification check for the combined assumptions of CES preferences and joint log normally distributed price and appeal shocks. Under these assumptions, the forward-backward estimator, which uses the full non-linear structure of CES preferences, converges asymptotically to the OLS slope coefficient.

Existing research on demand estimation for CES preferences takes one of two main approaches to overcoming the inconsistency of the OLS estimator. The first approach is instrumental variables estimation, which requires finding supply-side instruments that are powerful predictors of price changes, and satisfy the exclusion restriction of only affecting expenditure shares through price changes, and not through appeal changes. Although these instruments are sometimes available for some sectors, it can be challenging to find valid instruments for the many sectors considered in applications in international trade and macroeconomics.

The second approach uses the assumption of heteroskedasticity in price and appeal changes following Feenstra (1994), as used in Broda and Weinstein (2006) and Soderbury (2015). In a classic demand-supply identification problem, a researcher observes price and quantity changes for each good. Given any assumed supply elasticity, the demand elasticity can be estimated from the observed price and quantity changes. Therefore, for different assumed supply elasticities, one obtains a rectangular hyperbola in demand elasticity - supply elasticity space that provides partial identification, as in Leamer (1981). If price changes and quantity changes are heteroskedastic, the intersection of these rectangular hyperbolas for different goods can be used to separately identify the demand and supply elasticities, as shown in Feenstra (1994). In contrast, our estimators do not use heteroskedasticity for identification, and instead exploit properties of CES preferences and joint log normally distributed demand and supply shocks.

The remainder of the paper is structured as follows. Section 2 briefly summarizes properties of CES preferences. Section 3 derives our main aggregation result and characterizes analytically the properties of the price and appeal indexes underlying this aggregation result. Section 4 uses these properties to develop two new estimators of the elasticity of substitution and provides conditions under which these estimators are consistent. Even when these conditions are not satisfied, we show how our our aggregation result can be used to derive our specification check for the combined assumptions of CES preferences and joint log normally distributed price and appeal shocks. Section 5 shows that all our results for CES preferences also hold for logit preferences. Section 6 concludes.

2 CES Preferences

We assume that the representative consumer's utility is defined over horizontally-differentiated varieties. Preferences are characterized by a constant elasticity of substitution. Varieties can differ in price and appeal. We allow the set of varieties available to change over time through entry and exit.

Although we focus for simplicity on a representative consumer with a single CES nest, it is straightforward to extend the analysis to incorporate multiple consumer types, or multiple nests of utility, in which case our results hold for each consumer type or nest.

Our main aggregation result uses only properties of CES preferences. Therefore, this result holds regardless of supply-side assumptions for the determinants of prices, appeal and entry and exit. When we characterize the properties of some of our estimators below, we invoke the additional assumption that price and appeal shocks are joint log normally distributed.

2.1 Unit Expenditure Function

Under the assumption of CES preferences, the unit expenditure function for the cost of obtaining a unit of utility (P_t) depends on the price (p_{kt}) and appeal (φ_{kt}) for each good k at time t as follows:

$$P_t = \left[\sum_{k \in \Omega_t} \left(\frac{p_{kt}}{\varphi_{kt}} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}, \qquad \sigma > 1, \tag{1}$$

where σ is the constant elasticity of substitution between goods; appeal can capture both vertical differences in product quality and horizontal differences in product characteristics; we derive all our results for any cardinalization of utility and hence any choice of units for appeal; we assume that goods are substitutes ($\sigma > 1$); and Ω_t is the set of goods available at time t.

Applying Shepherd's Lemma to this unit expenditure function, we obtain the demand system, in which the expenditure share (s_{kt}) for each good k and time period t is:

$$s_{kt} \equiv \frac{p_{kt}c_{kt}}{\sum_{\ell \in \Omega_t} p_{\ell t}c_{\ell t}} = \frac{\left(p_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{\ell \in \Omega_t} \left(p_{\ell t}/\varphi_{\ell t}\right)^{1-\sigma}} = \frac{\left(p_{kt}/\varphi_{kt}\right)^{1-\sigma}}{P_t^{1-\sigma}}, \qquad k \in \Omega_t, \tag{2}$$

where c_{kt} denotes consumption of good k at time t.

2.2 Entry and Exit

CES preferences provide a tractable approach for taking account of the impact of entry and exit on the evolution of the cost of living over time. The change in the unit expenditure function between a pair of time periods $t-1$ and t can be exactly decomposed into two components: (i) a variety correction term that controls for the impact of the entry and exit of goods ($(\lambda_t/\lambda_{t-1})^{\frac{1}{\sigma-1}}$); (ii) the change in the unit expenditure function for common goods that are supplied in both time periods (P_t^*/P_{t-1}^*):

$$\Phi_t = \frac{P_t}{P_{t-1}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}} \frac{P_t^*}{P_{t-1}^*}, \tag{3}$$

where we use Φ_t to denote a ratio of unit expenditure functions, and we use asterisks to denote the value of variables for common goods.

The variety correction term depends on the ratio of the aggregate shares of expenditure on common goods in total expenditure in the two time periods (λ_t/λ_{t-1}). From the CES expenditure share in equation (2), these aggregate common goods expenditure shares are given by:

$$\lambda_t \equiv \frac{\sum_{k \in \Omega_t^*} \left(\frac{p_{kt}}{\varphi_{kt}}\right)^{1-\sigma}}{\sum_{k \in \Omega_t} \left(\frac{p_{kt}}{\varphi_{kt}}\right)^{1-\sigma}}, \qquad \lambda_{t-1} \equiv \frac{\sum_{k \in \Omega_t^*} \left(\frac{p_{kt-1}}{\varphi_{kt-1}}\right)^{1-\sigma}}{\sum_{k \in \Omega_{t-1}} \left(\frac{p_{kt-1}}{\varphi_{kt-1}}\right)^{1-\sigma}}, \tag{4}$$

where Ω_t^* is the set of common goods, such that $\Omega_t^* = \Omega_t \cap \Omega_{t-1}$, and we use $N_t^* = |\Omega_t^*|$ to denote the number of common goods.

The change in the unit expenditure function for common goods (P_t^*/P_{t-1}^*) takes the conventional CES form:

$$\Phi_t^* = \frac{P_t^*}{P_{t-1}^*} = \left[\frac{\sum_{k \in \Omega_t^*} \left(\frac{p_{kt}}{\varphi_{kt}}\right)^{1-\sigma}}{\sum_{k \in \Omega_t^*} \left(\frac{p_{kt-1}}{\varphi_{kt-1}}\right)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}}, \tag{5}$$

Intuitively, if common goods account for a smaller share of expenditure in period t than in period $t-1$ ($\lambda_t/\lambda_{t-1} < 1$), and varieties are substitutes ($\sigma > 1$), entering goods must be relatively more attractive than exiting goods, which reduces the cost of living ($(\lambda_t/\lambda_{t-1})^{\frac{1}{\sigma-1}} < 1$) in equation (3).

Within the set of common goods, we can define the share of expenditure of an individual common good in all expenditure on common goods as:

$$s_{kt}^* \equiv \frac{p_{kt}c_{kt}}{\sum_{\ell \in \Omega_t^*} p_{\ell t}c_{\ell t}} = \frac{\left(p_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{\ell \in \Omega_t^*} \left(p_{\ell t}/\varphi_{\ell t}\right)^{1-\sigma}} = \frac{\left(p_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\left(P_t^*\right)^{1-\sigma}}, \qquad k \in \Omega_t^*, \tag{6}$$

which takes the same form as in equation (2), except that the summation in the denominator of equation (6) is only over the set of common goods (Ω_t^*), instead of over all goods available in period t (Ω_t).

3 CES Aggregation

We now develop our main aggregation result for the fractal-like property of CES preferences: The change in the aggregate unit expenditure function can be written as the ratio of price indexes to appeal indexes, just as the change in the appeal-adjusted price of an individual variety can be written as the ratio of price changes to appeal changes.

We begin by defining this change in the appeal-adjusted price of an individual common variety k that continues to be supplied between a pair of periods $t-1$ and t:

$$\phi_{kt}^{*} = \frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}. \tag{7}$$

We next rewrite the change in the unit expenditure function for common goods (P_t^*/P_{t-1}^*) in equations (3) and (5) using the share of an individual common good in all expenditure on common goods at time $t-1$ (using s_{kt-1}^* from lagging equation (6) one period). We thus obtain the following forward difference of the unit expenditure function using initial-period expenditure share weights:

$$\Phi_t^F\left(\left\{\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right\}_{k\in\Omega_t\cup\Omega_{t-1}}\right) = \frac{P_t}{P_{t-1}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\left[\sum_{k\in\Omega_t^*} s_{kt-1}^*\left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{8}$$

as shown in Section A.2 of the online appendix; the superscript F indicates a forward difference using initial-period expenditure share weights; and the use of the parentheses ($\Phi_t^F(\cdot)$) indicates that this forward difference is a function of the set of appeal-adjusted prices for each variety ($\left\{\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right\}_{k\in\Omega_t\cup\Omega_{t-1}}$). Equation (8) generalizes the share-relative expression in Moulton (1996) using initial-period expenditure shares to allow for the entry and exit of varieties and changes in the appeal of each common variety.

Equivalently, we can rewrite the change in the unit expenditure function for common goods (P_t^*/P_{t-1}^*) in equations (3) and (5) using the share of an individual common good in all expenditure on common goods at time t (using s_{kt}^* from equation (6)). We thus obtain the following backward difference of the unit expenditure function using end-period expenditure share weights:

$$\Phi_t^B\left(\left\{\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right\}_{k\in\Omega_t\cup\Omega_{t-1}}\right) = \frac{P_t}{P_{t-1}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\left[\sum_{k\in\Omega_t^*} s_{kt}^*\left(\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}, \tag{9}$$

as shown in Section A.2 of the online appendix; the superscript B indicates a backward difference using end-period expenditure share weights; and the use of the parentheses ($\Phi_t^B(\cdot)$) indicates that this backward difference is a function of the set of appeal-adjusted prices for each variety ($\left\{\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right\}_{k\in\Omega_t\cup\Omega_{t-1}}$). Equation (9) generalizes the share-relative expression in Moulton (1996) using end-period expenditure shares to allow for the entry and exit of varieties and changes in the appeal of each common variety.

By construction, the forward and backward differences of the unit expenditure function in equations (8) and (9) are equal to one another, since they are both derivation from equations (3) and (5). Furthermore, they are derived for any cardinalization of utility, and hence hold regardless of the units chosen in which to measure prices and appeal. By combining these forward and backward differences in equations (8) and (9) with the shares of individual common goods in all expenditure on common goods (s_{kt}^* and s_{kt-1}^* from equation (6)), we now derive our main aggregation result.

Proposition 1. *Assume CES preferences with entry and exit of varieties and changes in the appeal of each common variety. The change in the unit expenditure function can be written as the variety correction term multiplied by the ratio of price indexes to appeal indexes for common goods:*

$$\Phi_t^F\left(\left\{\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right\}_{k\in\Omega_t\cup\Omega_{t-1}}\right)=\left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\frac{\Phi_t^{F*}\left(\left\{\frac{p_{kt}}{p_{kt-1}}\right\}_{k\in\Omega_t^*}\right)}{\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)},$$

$$\Phi_t^B\left(\left\{\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right\}_{k\in\Omega_t\cup\Omega_{t-1}}\right)=\left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\frac{\Phi_t^{B*}\left(\left\{\frac{p_{kt}}{p_{kt-1}}\right\}_{k\in\Omega_t^*}\right)}{\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)},$$

where the price and appeal indexes for common goods are forward and backward differences that take the same form as those for appeal-adjusted prices in equations (8) and (9):

$$\Phi_t^{F*}\left(\left\{\frac{p_{kt}}{p_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\left[\sum_{k\in\Omega_t^*}s_{kt-1}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}},$$

$$\Phi_t^{B*}\left(\left\{\frac{p_{kt}}{p_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\left[\sum_{k\in\Omega_t^*}s_{kt}^*\left(\frac{p_{kt}}{p_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}},$$

$$\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\left[\sum_{k\in\Omega_t^*}s_{kt-1}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}},$$

$$\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k\in\Omega_t^*}\right)=\left[\sum_{k\in\Omega_t^*}s_{kt}^*\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}.$$

Proof. See Section A.4 of the online appendix. □

Proposition 1 is a remarkable property of CES preferences. Recall that the change in the unit expenditure function depends on an expenditure share weighted average of the ratio of price changes to appeal changes (as shown in equations (8) and (9)). Furthermore, these price and appeal changes can be correlated with one another across varieties. Nevertheless, the change in the unit expenditure function can be exactly decomposed into either the ratio of a forward price index to a backward appeal index, or equivalently the ratio of a backward price index to a forward appeal index. We show in the proof of Proposition 1 that this result is an implication of the independence of irrelevant alternatives (IIA) property of CES.

The forward price index ($\Phi_t^{F*}\left(\left\{\frac{p_{kt}}{p_{kt-1}}\right\}_{k\in\Omega_t^*}\right)$) evaluates the impact of price shocks on the cost of living using initial-period expenditure shares, while the backward price index ($\Phi_t^{B*}\left(\left\{\frac{p_{kt}}{p_{kt-1}}\right\}_{k\in\Omega_t^*}\right)$) assesses this impact of price shocks using end-period expenditure shares. The forward and backward appeal indexes are defined analogously, using initial-period and end-period expenditure shares, respectively. Despite the potential correlation between price and appeal shocks across varieties, the change in the unit expenditure function is log additively separable into two indexes for price shocks and appeal shocks. Therefore, changes in the cost of living can be exactly decomposed into these two separate components of price and appeal shocks.

We now show that the forward appeal index converges asymptotically to the same value as the backward appeal index as the number of common goods becomes large, if price and appeal shocks are independently distributed across varieties and uncorrelated with one another and initial-period expenditure shares.

Proposition 2. *Assume that price and appeal shocks are independently distributed across varieties and uncorrelated with one another and initial-period expenditure shares, such that:*

$$plim_{N_t^* \to \infty} \left\{ \left[\frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}}{p_{kt-1}} \right)^{1-\sigma} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}} \right)^{\sigma-1} \right]}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}}{p_{kt-1}} \right)^{1-\sigma} \right] \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{\varphi_{kt}}{\varphi_{kt-1}} \right)^{\sigma-1} \right]} \right]^{\frac{1}{1-\sigma}} \right\} = 1. \tag{10}$$

As the number of common goods becomes large ($N_t^* \to \infty$), the forward appeal index converges asymptotically to the same value as the backward appeal index:

$$plim_{N_t^* \to \infty} \left\{ \frac{\Phi_t^{F*} \left(\left\{ \frac{\varphi_{kt}}{\varphi_{kt-1}} \right\}_{k=1}^{N_t^*} \right)}{\Phi_t^{B*} \left(\left\{ \frac{\varphi_{kt}}{\varphi_{kt-1}} \right\}_{k=1}^{N_t^*} \right)} \right\} = 1.$$

Proof. See Section A.5 of the Online Appendix. □

Intuitively, if price and appeal shocks are uncorrelated across varieties, their combined impact on the cost of living is equal to the product of their separate impacts on the cost of living. More generally, if price and appeal shocks are correlated with one another across varieties, the forward and backward appeal indexes are not necessarily equal to one another. Nevertheless, we now show that the forward appeal index is equal to the backward appeal index up to a first-order approximation.

Proposition 3. *The forward appeal index is equal to the backward appeal index up to a first-order approximation:*

$$\log \Phi_t^{F*} \left(\left\{ \frac{\varphi_{kt}}{\varphi_{kt-1}} \right\}_{k \in \Omega_t^*} \right) = \log \Phi_t^{B*} \left(\left\{ \frac{\varphi_{kt}}{\varphi_{kt-1}} \right\}_{k \in \Omega_t^*} \right) + O^2,$$

where O^2 denotes the second-order and higher terms in Taylor-series expansions of $\log \Phi_t^{F*} \left(\left\{ \frac{\varphi_{kt}}{\varphi_{kt-1}} \right\}_{k \in \Omega_t^*} \right)$ *and* $\log \Phi_t^{B*} \left(\left\{ \frac{\varphi_{kt}}{\varphi_{kt-1}} \right\}_{k \in \Omega_t^*} \right)$ *around p_{kt}/p_{kt-1} and $\varphi_{kt}/\varphi_{kt-1} = 1$ for each variety k.*

Proof. See Section A.6 of the Online Appendix. □

Therefore, any difference between the forward appeal index and the backward appeal index is driven by the second-order and higher terms. We show in the online appendix that these second-order and higher terms depend on the covariance between price and appeal shocks. As appeal shocks become small, these second-order and higher terms converge to zero, and the forward appeal index converges to the backward appeal index. In the limit in which there are no appeal shocks ($\varphi_{kt}/\varphi_{kt-1} = 1$ for all $k \in \Omega^*$), the forward and backward appeal indexes are both equal to one, and hence equal to one another.

Before closing this section, we derive another expression for the change in the unit expenditure function, which is neither a forward nor a backward difference. Dividing the share of an individual common good in all expenditure on common goods in equation (6) by its geometric mean across common goods, and rearranging terms, we obtain the following expression for the change in the unit expenditure function in terms of geometric means:

$$\Phi_t^G\left(\left\{\frac{p_{kt}/p_{kt-1}}{\varphi_{kt}/\varphi_{kt-1}}\right\}_{k\in\Omega_t\cup\Omega_{t-1}}\right)=\frac{P_t}{P_{t-1}}=\left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\frac{\tilde{p}_t/\tilde{p}_{t-1}}{\tilde{\varphi}_t/\tilde{\varphi}_{t-1}}\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}, \tag{11}$$

as shown in Section A.2 of the online appendix; the subscript G indicates a geometric mean difference of the unit expenditure function; a tilde above a variable denotes a geometric mean across common goods, such that $\tilde{x}_t=\left(\prod_{k\in\Omega_t^*}x_{kt}\right)^{1/N_t^*}$, where $N_t^*=|\Omega_t^*|$ is the number of common goods.

By construction, the geometric mean difference of the unit expenditure function in equation (11) is equal to the forward and backward differences of the unit expenditure function in equations (8) and (9) above, since all of these expressions can be derived from equations (3) and (5). Furthermore, this geometric mean difference is derived for any cardinalization of utility, and hence again holds regardless of the units chosen in which to measure prices and appeal.

4 CES Estimation

We now use our aggregation result for changes in the unit expenditure function to derive two new estimators of the elasticity of substitution. We provide an analytical characterization of the conditions under which these estimators are consistent. Alongside these analytical results, we illustrate the finite sample performance of these estimators using a Monte Carlo simulation, in which we assume CES preferences and joint log normally distributed price and appeal shocks.

4.1 Demand Systems Estimation

Before developing our two new estimators, we first review demand systems estimation of the elasticity of substitution. We begin by dividing the common goods expenditure share in period t (s_{kt}^* in equation (6)) by its geometric mean. We next divide the common goods expenditure share in period $t-1$ (s_{kt-1}^* from lagging equation (6)) by its geometric mean. Finally, we take the log ratio of these relative common good expenditure shares in the two time periods to obtain the following structural demand system relating log changes in expenditure shares to log changes in prices and appeal:

$$\log\left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}^*}\right)=(1-\sigma)\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)+(\sigma-1)\log\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right). \tag{12}$$

Appeal shocks are typically unobserved and absorbed into a residual to obtain the following reduced-form demand system relating log changes in expenditure shares to log changes in prices:

$$\log\left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}^*}\right)=\beta\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)+\log\left(\frac{u_{kt}}{u_{kt-1}}\right), \tag{13}$$

where $\log\left(u_{kt}/u_{kt-1}\right)$ is the regression error; the normalization of the expenditure share and price ratios by their geometric means ensures that both variables have a mean of zero in logs; hence the regression constant is equal to zero.

Estimating this regression using ordinary least squares (OLS), the estimated slope coefficient (β) will provide a biased and inconsistent estimate of the structural parameter $(1-\sigma)$ if price and appeal shocks are correlated with one another, because the right-hand side variable ($\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)$) in equation (13) is correlated with the error term ($\log\left(\frac{u_{kt}}{u_{kt-1}}\right)$), resulting in conventional omitted variables bias. The standard approach to this omitted variables bias is to search for supply-side instruments that are powerful predictors of price changes, and satisfy the exclusion restriction of only affecting expenditure shares through price changes, and not through appeal changes. Although these instruments are sometimes available for some sectors, it can be challenging to find valid instruments for the many sectors considered in applications in macroeconomics and international trade.

Supposing that valid instruments are not available, further intuition for the inconsistency of the OLS estimator can be obtained by considering the case in which price and appeal shocks are joint log normally distributed with a general variance-covariance matrix:[1]

$$\left(\begin{array}{c} \log\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right) \\ \log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right) \end{array}\right) \sim \mathcal{N}\left(\left[\begin{array}{c} 0 \\ 0 \end{array}\right], \left[\begin{array}{cc} \chi_\varphi^2 & \rho\chi_\varphi\chi_p \\ \rho\chi_\varphi\chi_p & \chi_p^2 \end{array}\right]\right), \tag{14}$$

where the scaling of the appeal and price ratios by their geometric means ensures that both variables are mean zero in logs; χ_φ and χ_p are the standard deviations of appeal and price shocks respectively; and ρ is the correlation of appeal and price shocks. Under this distributional assumption, we can partition appeal shocks into a component that is correlated with price shocks and an orthogonal component:

$$\log\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right) = \gamma \log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right) + \delta \log\left(\frac{v_{kt}}{v_{kt-1}}\right), \tag{15}$$

where $\gamma = \rho\chi_\varphi/\chi_p$ is the projection coefficient of log changes in appeal on log changes in prices, $\delta = \chi_\varphi\sqrt{(1-\rho^2)}$, and $\log\left(v_{kt}/v_{kt-1}\right)$ has an independent standard normal distribution. Using this distributional assumption (15) in our demand system (12), we obtain:

$$\log\left(\frac{s_{kt}^*/\tilde{s}_t^*}{s_{kt-1}^*/\tilde{s}_{t-1}^*}\right) = (1-\sigma)(1-\gamma)\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right) + \delta(\sigma-1)\log\left(\frac{v_{kt}}{v_{kt-1}}\right), \tag{16}$$

where $\operatorname{cov}\left(\log\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right), \log\left(\frac{v_{kt}}{v_{kt-1}}\right)\right) = 0$ by construction.

From equations (13) and (16), and invoking the conventional properties of the OLS estimator, we obtain the following results under our assumptions of CES preferences and joint log normally distributed price and appeal shocks.

[1] Although, for simplicity, we directly assume that appeal and price shocks are joint normally distributed, one can instead assume CES demand and monopolistic competition, and derive this property from the primitive assumption that appeal and marginal cost shocks are joint normally distributed.

Proposition 4. *(Demand System) Assume CES preferences and joint log normally distributed price and appeal shocks. As the number of common goods becomes large ($N_t^* \to \infty$), the OLS estimates of the reduced-form demand system (13) have the following asymptotic properties:*

$$plim_{N_t^*\to\infty}\left\{\hat{\beta}^{OLS}\right\} = (1-\sigma)\,(1-\gamma)\,,$$

$$plim_{N_t^*\to\infty}\left\{\sqrt{\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\ln\left(\frac{\widehat{u}_{kt}^{OLS}}{\widehat{u}_{kt-1}^{OLS}}\right)^2}\right\} = \delta\,(\sigma-1)\,,$$

$$plim_{N_t^*\to\infty}\left\{\frac{\ln\left(\hat{u}_{kt}^{OLS}/\hat{u}_{kt-1}^{OLS}\right)}{\sqrt{\frac{1}{N_t^*}\ln\left(\widehat{\frac{u_{kt}}{u_{kt-1}}}\right)^2}}\right\} = \ln\left(v_{kt}/v_{kt-1}\right).$$

Proof. See Section A.7 of the online appendix. □

Therefore, the estimated OLS slope coefficient ($\widehat{\beta}^{OLS}$) is an inconsistent estimate of the structural parameter $(1-\sigma)$, because of conventional omitted variables bias. Recalling that $(1-\sigma) < 0$, we find that the estimated OLS slope coefficient ($\widehat{\beta}^{OLS}$) is too small for $\gamma < 0$ (too negative), and too large for $\gamma > 0$ (not negative enough). Rather than recovering the structural parameter $(1-\sigma)$, the estimated OLS slope coefficient ($\widehat{\beta}^{OLS}$) is instead a consistent estimator of $(1-\sigma)\,(1-\gamma)$. Additionally, the standard deviation of the regression residuals is a consistent estimator of $\delta\,(\sigma-1)$, and the residuals divided by their standard deviation are a consistent estimate of the independent standard normal marginal appeal shock (log (v_{kt}/v_{kt-1})). We use these results below when we develop a specification check on the combined assumptions of CES preferences and joint log normally distributed price and appeal shocks.

4.2 Unit Expenditure Function Estimation

We now use our main aggregation result to develop two new estimators of the elasticity of substitution from changes in the unit expenditure function.

Forward-Backward Estimator Our forward-backward (FB) estimator uses the equality between the forward and backward differences of the unit expenditure function from equations (8) and (9). Equating these two expression, the variety correction term cancels from the two sides of the equality to yield:

$$\left[\begin{array}{c}\left[\sum_{k=1}^{N_t^*} s_{kt}^*\left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}}\right)^{-(\sigma-1)}\right]^{-\frac{1}{1-\sigma}} \\ \times\left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\end{array}\right] = \left[\begin{array}{c}\left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}} \\ \times\left[\sum_{k=1}^{N_t^*} s_{kt}^*\left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}\end{array}\right], \qquad (17)$$

as shown in Section A.8 of the online appendix; and we have divided both sides of the equation by $\widetilde{\varphi}_t/\widetilde{\varphi}_{t-1}$ and $\widetilde{p}_t/\widetilde{p}_{t-1}$ to make clear that this relationship is invariant to the choice of units in which appeal and prices are measured.[2]

We derive our FB estimator from this equality by imposing the identifying assumption that the forward appeal index ($\Phi_t^{F*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k=1}^{N_t^*}\right)$) defined in Proposition 1 is equal to the backward appeal index ($\Phi_t^{B*}\left(\left\{\frac{\varphi_{kt}}{\varphi_{kt-1}}\right\}_{k=1}^{N_t^*}\right)$):

$$\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}} = \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_t}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}. \tag{18}$$

Using this identifying assumption in equation (17), we obtain the following moment condition that can be used to estimate the elasticity of substitution (σ) by the generalized method of moments (GMM):

$$m^{FB}(\sigma) = \log\left\{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\right\} - \log\left\{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}\right\} = 0, \tag{19}$$

$$\widehat{\sigma}^{FB} = \arg\min_{\sigma} m^{FB}(\sigma)^2.$$

Intuitively, the FB estimator chooses the elasticity of substitution to minimize the sum of squared deviations between the forward difference of the unit expenditure function, including only price shocks, and the backward difference of the unit expenditure function, including only price shocks.

Reverse-Weighting Estimator Our reverse-weighting (RW) estimator uses two sets of equalities between (i) the forward and geometric mean differences of the unit expenditure function and (ii) the backward and geometric mean difference of the unit expenditure function, as implied by equations (8), (9) and (11). Equating these expressions, the variety correction term cancels from the two sides of these equalities to yield:

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{-(\sigma-1)}\right]^{-\frac{1}{1-\sigma}} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{20}$$

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}} \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}},$$

where the scaling of prices and appeal by their geometric means in each period implies that these expressions are invariant to the choice of units in which appeal is measured.

We derive our RW estimator from this equality by imposing the joint identifying assumptions that (i) the ratio of the forward appeal index and the geometric mean appeal index is equal to one and (ii) the ratio of the

[2] In Section A.3 of the Online Appendix, we show that unobserved appeal (φ_{kt}) can be recovered from observed common goods expenditure shares (s_{kt}^*) and prices (p_{kt}) for any elasticity of substitution (σ) up to a multiplicative scalar that corresponds to a choice of units in which to measure appeal. This multiplicative scalar cancels from the numerator and denominator of the fractions on both sides of equation (17).

backward appeal index and the geometric mean appeal index is equal to one:

$$\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}} = 1, \tag{21}$$

$$\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}\widetilde{\varphi}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} = 1.$$

Using this identifying assumption in equation (20), we obtain the following moment condition that can be used to estimate the elasticity of substitution (σ) by the generalized method of moments (GMM):

$$m^{RW}(\sigma) = \begin{pmatrix} \log\left\{\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right\} - \log\left\{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\right\} \\ \log\left\{\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right\} - \log\left\{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}\right\} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{22}$$

$$\widehat{\sigma}^{RW} = \arg\min_{\sigma} \left\{ m^{RW}(\sigma)' \times \mathbb{I} \times m^{RW}(\sigma) \right\},$$

where $\mathbb{I}$ is the identity matrix.

Intuitively, the RW estimator chooses the elasticity of substitution to minimize the sum of squared deviations between the forward difference of the price index including only price shocks, the backward difference of the price index including only price shocks, and the geometric mean difference of the price index including only price shocks.

4.3 Consistency for Small Appeal Shocks

We now show that the FB and RW estimators are consistent estimators of the elasticity of substitution as appeal shocks become small.

Proposition 5. *As appeal shocks become small ($\varphi_{kt}/\varphi_{kt-1} \to 1$ for all k), the forward-backward estimator ($\widehat{\sigma}^{FB}$) and reverse-weighting ($\widehat{\sigma}^{RW}$) estimators consistently estimate the elasticity of substitution:*

$$plim_{\left\{\{\varphi_{kt}/\varphi_{kt-1}\}_{k=1}^{N_t^*}\to 1\right\}} \left\{\widehat{\sigma}^{FB}\right\} = \sigma,$$

$$plim_{\left\{\{\varphi_{kt}/\varphi_{kt-1}\}_{k=1}^{N_t^*}\to 1\right\}} \left\{\widehat{\sigma}^{RW}\right\} = \sigma.$$

Proof. See Section A.8 of the online appendix. □

Intuitively, as shown in Proposition 3 above, the forward and backward appeal indices are equal to one another up to a first-order approximation. Therefore, as appeal shocks become small for each good, these forward and backward appeal indexes converge to the same common value, which implies that the identifying assumption (18) of the FB estimator is satisfied, such that this estimator consistently estimates the elasticity of substitution (σ). Additionally, as appeal shocks become small for each good, both appeal indexes converge to the same common value of one, which implies that the identifying assumption (21) of the RW estimator is satisfied, such that this estimator consistently estimates the elasticity of substitution (σ).

4.4 CES Preferences and Log Normal Price and Appeal Shocks

We now examine the performance of the FB and RW estimator for large shocks for the case in which price and appeal shocks are joint log normally distributed.

4.4.1 Log Normally Distributed Price and Appeal Shocks

Under the assumption that log appeal and log price shocks are joint normally distribution according to equation (14), the level of appeal and price shocks have the following joint log normal distribution:

$$\begin{pmatrix} \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)} \\ \left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{(\sigma-1)} \end{pmatrix} \sim \text{LogNormal}\left(\begin{bmatrix} \mu_{ep} \\ \mu_{e\varphi} \end{bmatrix}, \begin{bmatrix} \Xi_{pp} & \Xi_{p\varphi} \\ \Xi_{p\varphi} & \Xi_{\varphi\varphi} \end{bmatrix}\right), \tag{23}$$

$$\mu_{ep} = \exp\left\{\frac{1}{2}(\sigma-1)^2\chi_p^2\right\}, \qquad \mu_{e\varphi} = \exp\left\{\frac{1}{2}(\sigma-1)^2\chi_\varphi^2\right\},$$

$$\Xi_{pp} = \Lambda \times \left[\exp\left\{(\sigma-1)^2\chi_p^2\right\} - 1\right], \qquad \Xi_{\varphi\varphi} = \Lambda \times \left[\exp\left\{(\sigma-1)^2\chi_\varphi^2\right\} - 1\right],$$

$$\Xi_{p\varphi} = \Lambda \times \left[\exp\left\{-(\sigma-1)^2\rho\chi_\varphi\chi_p\right\} - 1\right], \qquad \Lambda = \exp\left\{\frac{1}{2}\left[(\sigma-1)^2\chi_p^2 + (\sigma-1)^2\chi_\varphi^2\right]\right\}.$$

Recall that the difference of normally distributed random variables is also normally distributed. Therefore, it follows that log appeal-adjusted price shocks ($\log\left[\left(\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right) / \left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)\right)^{1-\sigma}\right]$) are also normally distributed, and the level of appeal-adjusted price shocks is log normally distributed:

$$\left(\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right) / \left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)\right)^{(1-\sigma)} \sim \text{LogNormal}\left(\mu_{er}, \Xi_{er}\right), \tag{24}$$

$$\mu_{er} = \exp\left\{\frac{1}{2}(\sigma-1)^2\left(\chi_p^2 + \chi_\varphi^2 - 2\rho\chi_\varphi\chi_p\right)\right\},$$

$$\Xi_{er} = \exp\left\{(\sigma-1)^2\left(\chi_p^2 + \chi_\varphi^2 - 2\rho\chi_\varphi\chi_p\right)\right\}\left[\exp\left\{(\sigma-1)^2\left(\chi_p^2 + \chi_\varphi^2 - 2\rho\chi_\varphi\chi_p\right)\right\} - 1\right].$$

Using these properties of log normal distributions, we obtain the following expression for the ratio of the expectation of the product of the price and appeal shocks to the product of the expectation of these price and appeal shocks:

$$\text{plim}_{N_t^*\to\infty}\left\{\left[\frac{\left[\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma}\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]}{\left[\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma}\right]\left[\frac{1}{N_t^*}\sum_{k=1}^{N_t^*}\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]}\right]^{\frac{1}{1-\sigma}}\right\} = \left[\exp\left\{-(\sigma-1)^2\left(\rho\chi_\varphi\chi_p\right)\right\}\right]^{\frac{1}{1-\sigma}}. \tag{25}$$

4.4.2 Orthogonal Appeal Shocks

We now use these results to characterize the properties of the FB and RW estimators when price and appeal shocks are joint log normally distributed according to equation (14). We begin in this subsection by considering the case in which price and appeal shocks are orthogonal ($\rho = 0$), before considering the case in which price and appeal shocks are correlated ($\rho \neq 0$) in the next subsection.

Forward-Backward Estimator Using the orthogonality of price and appeal shocks, and noting that common goods expenditure shares at time $t-1$ (s^*_{kt-1}) are pre-determined at time t, we can replace the unweighted expectation of the left-hand side of equation (25) by the weighted expectation using initial expenditure share weights to obtain:

$$\operatorname{plim}_{N_t^*\to\infty}\left\{\left[\frac{\left[\sum_{k=1}^{N_t^*} s^*_{kt-1}\left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{1-\sigma}\left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]}{\left[\sum_{k=1}^{N_t^*} s^*_{kt-1}\left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{1-\sigma}\right]\left[\sum_{k=1}^{N_t^*} s^*_{kt-1}\left(\frac{\varphi_{kt}/\widetilde{\varphi}_t}{\varphi_{kt-1}/\widetilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]}\right]^{\frac{1}{1-\sigma}}\right\}=1, \tag{26}$$

which is the condition (10) in Proposition 2 for the forward appeal shifter to converge to the same value as the backward appeal shifter.

Using the common goods expenditure share (6) in equation (26), and simplifying terms, we obtain the moment condition (19) for the forward-backward estimator to consistently estimate the elasticity of substitution:

$$\operatorname{plim}_{N_t^*\to\infty}\left\{\log\left\{\left[\sum_{k=1}^{N_t^*} s^*_{kt-1}\left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\right\}-\log\left\{\left[\sum_{k=1}^{N_t^*} s^*_{kt}\left(\frac{p_{kt}/\widetilde{p}_t}{p_{kt-1}/\widetilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}\right\}\right\}=0. \tag{27}$$

Intuitively, if price and appeal shocks are independently distributed across varieties and uncorrelated with one another and initial-period expenditure shares, the weighted expectation of the product of price and appeal shocks is equal to the product of the weighted expectations of price shocks and appeal shocks, where these weighted expectations use initial-period expenditure shares as weights. In the proof of Proposition 2, we show that this property implies that the forward appeal shifter converges asymptotically in the number of common goods to the same value as the backward appeal shifter, which ensures that that the moment condition for the forward-backward estimator in equation (19) is satisfied. Therefore, if price and appeal shocks are independently distributed across varieties and uncorrelated with one another and initial-period expenditure shares, the forward-backward estimate ($\widehat{\sigma}^{FB}$) converges asymptotically to the true elasticity of substitution (σ) as the number of common goods becomes large ($N_t^*\to\infty$).

Under this same identifying assumption that price and appeal shocks are independently distributed across varieties, the OLS estimator is also consistent. Although at first sight, this reduces the usefulness of the forward-backward estimator, we show below that this similarity of identifying assumptions to the OLS estimator can be used to develop a specification check for the joint assumptions of CES demand and joint log normally distributed price and appeal shocks. Whereas the OLS estimator is a linear estimator, the forward-backward estimator uses the full non-linear structure of CES preferences. Therefore, if preferences are characterized by a different functional form, there is no reason for these two estimators to have similar asymptotic properties, as examined in further detail below.

Reverse-Weighting Estimator The two equalities in equation (20) from which we derive the RW estimator can be written as follows:

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}} = \left[\frac{\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma} \left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{\sigma-1}}{\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{28}$$

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}} = \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right)^{\sigma-1}\right]^{\frac{1}{1-\sigma}} \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}.$$

Using the orthogonality of price and appeal shocks, and noting that common goods expenditure shares at time $t-1$ (s_{kt-1}^*) are pre-determined at time t, we can replace the weighted expectations in the first terms on the right-hand side of equation (28) with unweighted expectations. Using the properties of the log normal distribution in equation (23) and (24), and taking the limit as the number of common goods becomes large ($N_t^* \to \infty$), we obtain the following result:

$$\operatorname{plim}_{N_t^* \to \infty} \left\{ \log\left\{ \left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}} \right\} - \log\left\{ \left[\exp\left\{\frac{1}{2}(\sigma-1)^2 \chi_\varphi^2\right\}\right]^{\frac{1}{1-\sigma}} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \right\} \right\} = 0, \tag{29}$$

$$\operatorname{plim}_{N_t^* \to \infty} \left\{ \log\left\{ \left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}} \right\} - \log\left\{ \left[\exp\left\{\frac{1}{2}(\sigma-1)^2 \chi_\varphi^2\right\}\right]^{\frac{1}{1-\sigma}} \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}} \right\} \right\} = 0.$$

Even if price and appeal shocks are orthogonal, there is no reason in general that the terms in the standard deviation of appeal shocks (χ_φ) in equation (29) are equal to zero, as required for the moment condition of the RW estimator in equation (22) to hold. Therefore, with large and orthogonal price and appeal shocks, the identifying assumption of the RW estimator (21) need not be satisfied, in which case the RW and FB estimators will differ systematically from one another.

Furthermore, whereas the moment condition for the FB estimator holds regardless of the standard deviation of appeal shocks (χ_φ), the degree to which the moment condition for the RW estimator is violated (comparing equations (22) and (29)) depends on the dispersion of appeal shocks (χ_φ). Therefore, the difference between the FB and RW estimators is informative about the dispersion of appeal shocks.

4.4.3 Correlated Appeal Shocks

We now characterize the properties of the FB and RW estimators when price and appeal shocks are joint log normally distributed according to equation (14) and correlated.

Forward-Backward Estimator When log price shocks and log appeal shocks are joint normally distributed and correlated with one another, we can partition log appeal shocks into a price component and an orthogonal component, as in equation (15). Combining this result with the independence of irrelevant alternatives (IIA) properties of CES preferences, we can rewrite the equality between the forward difference (8) and backward difference (9) of the unit expenditure function as follows:

$$\left[\frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{\epsilon_{kt}}{\epsilon_{kt-1}}\right)^{\delta(\sigma-1)}\right]}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{\epsilon_{kt}}{\epsilon_{kt-1}}\right)^{\delta(\sigma-1)}\right]}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}$$

$$= \left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{(1-\sigma)(1-\gamma)}}, \quad (30)$$

as shown in Section A.9 of the online appendix.

Noting that that $\ln\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)$ and $\ln\left(\frac{\epsilon_{kt}}{\epsilon_{kt-1}}\right)$ are normally distributed and independent of one another and initial expenditure shares (s_{kt-1}^*), the first term on the left-hand side of equation (30) converges to one as the number of common goods becomes large ($N_t^* \to \infty$):

$$\text{plim}_{N_t^*\to\infty}\left\{\left[\frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)} \left(\frac{\epsilon_{kt}}{\epsilon_{kt-1}}\right)^{\delta(\sigma-1)}\right]}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{\epsilon_{kt}}{\epsilon_{kt-1}}\right)^{\delta(\sigma-1)}\right]}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\right\} = 1.$$

as also shown in Section A.9 of the online appendix.

Using this property in our equality (30) between the forward and backward difference of the unit expenditure function, we obtain the following result:

$$\text{plim}_{N_t^*\to\infty}\left\{\log\left\{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\right\} - \log\left\{\left[\sum_{k=1}^{N_t^*} s_{kt}^* \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{(1-\sigma)(1-\gamma)}}\right\}\right\} = 0. \quad (31)$$

Comparing equation (31) to the moment condition for the FB estimator in equation (19), we see that when price and appeal shocks are joint log normally distributed and correlated, the FB estimator estimates $(1-\sigma)(1-\gamma)$ rather than $(1-\sigma)$. Noting that $\widehat{\beta}^{OLS} = (1-\sigma)(1-\gamma)$ in the demand systems estimation in Section 4.1, it follows that when price and appeal shocks are joint log normally distributed and correlated, the FB estimator exhibits similar properties as the OLS estimator.

We thus obtain a specification check for the joint assumptions of CES demand and joint log normally distributed price and appeal shocks. Under these assumptions, and regardless of the correlation between price and appeal shocks, the FB and OLS estimators converge to the same value. Whereas the OLS estimator is a linear estimator, the forward-backward estimator uses the full non-linear structure of CES preferences. Therefore, under the alternative hypothesis that preferences are non-CES and/or the distribution of price and appeal shocks differs from a joint log normal distribution, there is no reason for the FB and OLS estimators to lie close together.

Reverse-Weighting Estimator Using the joint log normality of price and appeal shocks, we again can partition log appeal shocks into a price component and an orthogonal component, as in equation (15). Combining this result with the independence of irrelevant alternatives (IIA) properties of CES preferences, we can rewrite our two equalities between the three equivalent expressions for the change in unit expenditure function from equations (8), (9) and (11) as follows:

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{(\sigma-1)(1-\gamma)}} = \left[\frac{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\left(\frac{\epsilon_{kt}}{\epsilon_{kt-1}}\right)^{\delta(\sigma-1)}\right]}{\left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}} \left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}, \tag{32}$$

$$\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{(\sigma-1)(1-\gamma)}} = \left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{\epsilon_{kt}}{\epsilon_{kt-1}}\right)^{\delta(\sigma-1)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}} \left[\sum_{k=1}^{N_t^*} s_{kt}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{(1-\sigma)(1-\gamma)}},$$

as shown in Section A.9 of the Online Appendix.

Note that $\ln\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)$ and $\ln\left(\frac{\epsilon_{kt}}{\epsilon_{kt-1}}\right)$ are normally distributed and independent of both one another and initial expenditure shares (s_{kt-1}^*) in the first terms on the right-hand side of equation (32). Using this property, and taking the limit as the number of common goods becomes large ($N_t^* \to \infty$), we obtain the following result:

$$\operatorname{plim}_{N_t^*\to\infty}\left\{\log\left\{\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{(\sigma-1)(1-\gamma)}}\right\} - \log\left\{\left[\exp\left\{\frac{1}{2}(\sigma-1)^2\delta^2\chi_\varphi^2\right\}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\left[\sum_{k=1}^{N_t^*} s_{kt-1}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{(1-\sigma)(1-\gamma)}\right]^{\frac{1}{(1-\sigma)(1-\gamma)}}\right\}\right\} = 0, \tag{33}$$

$$\operatorname{plim}_{N_t^*\to\infty}\left\{\log\left\{\left(\frac{\tilde{s}_t^*}{\tilde{s}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right\} - \log\left\{\left[\exp\left\{\frac{1}{2}(\sigma-1)^2\delta^2\chi_\varphi^2\right\}\right]^{\frac{1}{1-\sigma}}\left[\sum_{k=1}^{N_t^*} s_{kt}^*\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{-(1-\sigma)(1-\gamma)}\right]^{-\frac{1}{(1-\sigma)(1-\gamma)}}\right\}\right\} = 0,$$

as also shown in Section A.9 of the Online Appendix.

Comparing equation (33) to the moment condition for the RW estimator in equation (22), we see that when price and appeal shocks are joint log normally distributed and correlated, the RW moment condition is in general not satisfied. Furthermore, the RW estimator in general differs from the FB estimator, because it is affected by standard deviation of appeal shocks (χ_φ) in equation (33), whereas the FB estimator in equation (31) is unaffected by the standard deviation of appeal shocks. Therefore, the difference between the FB and RW estimators is again informative about the dispersion of appeal shocks, when price and appeal shocks are correlated with one another.

4.4.4 Monte Carlo Simulation

We now illustrate our analytical results using a Monte Carlo simulation. Recall that our estimators use only the subset of common goods, because the variety correction term cancels from equivalent expressions for the change in the unit expenditure function. Therefore, we focus on this subset of common goods, and are not required to make assumptions about entering and exiting goods. We assume 10,000 common goods, which is not unusual in micro data. We assume CES preferences with an elasticity of substitution between varieties of $\sigma = 4$, which is central value in the empirical literature using international trade and barcode data.

We begin by drawing initial values for appeal (φ_{kt-1}) and prices (p_{kt-1}) in period $t-1$ from a joint log log normal distribution:

$$\begin{pmatrix} \ln(\varphi_{kt-1}/\tilde{\varphi}_{t-1}) \\ \ln(p_{kt-1}/\tilde{p}_{t-1}) \end{pmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \chi_\varphi^2 & \rho\chi_\varphi\chi_p \\ \rho\chi_\varphi\chi_p & \chi_p^2 \end{bmatrix}\right). \tag{34}$$

Since initial appeal (φ_{kt-1}) and initial prices (p_{kt-1}) are expressed relative to their geometric means ($\tilde{\varphi}_{t-1}$ and $\tilde{p}_{t-1}$, respectively), they are mean zero in logs by construction. We set the standard deviation for initial prices to one ($\chi_p = 1$); we consider three different values for the correlation between initial prices and appeal ($\rho \in \{-0.5, 0, 0.5\}$); and we examine five different values for the standard deviation for initial appeal ($\chi_\varphi \in \{0.001, 0.01, 0.1, 0.5, 1\}$).

We use these initial realizations for appeal (φ_{kt-1}) and prices (p_{kt-1}) to solve for initial equilibrium expenditure shares (s^*_{kt-1}) in period $t-1$:

$$s^*_{kt-1} = \frac{\left(p_{kt-1}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{\ell \in \Omega^*} \left(p_{\ell t-1}/\varphi_{\ell t-1}\right)^{1-\sigma}}. \tag{35}$$

We next draw appeal shocks ($\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}$) and price shocks ($\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}$) from period $t-1$ to period t from the same joint log log normal distribution:

$$\begin{pmatrix} \ln\left(\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}\right) \\ \ln\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right) \end{pmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \chi^2_\varphi & \rho\chi_\varphi\chi_p \\ \rho\chi_\varphi\chi_p & \chi^2_p \end{bmatrix}\right), \tag{36}$$

where appeal shocks ($\frac{\varphi_{kt}/\tilde{\varphi}_t}{\varphi_{kt-1}/\tilde{\varphi}_{t-1}}$) and price shocks ($\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}$) are expressed relative to their geometric means ($\tilde{\varphi}_t/\tilde{\varphi}_{t-1}$ and $\tilde{p}_t/\tilde{p}_{t-1}$, respectively) and are hence mean zero in logs by construction. We use these realizations for appeal and price shocks to solve for prices (p_{kt}) and expenditure shares (s^*_{kt}) in period t:

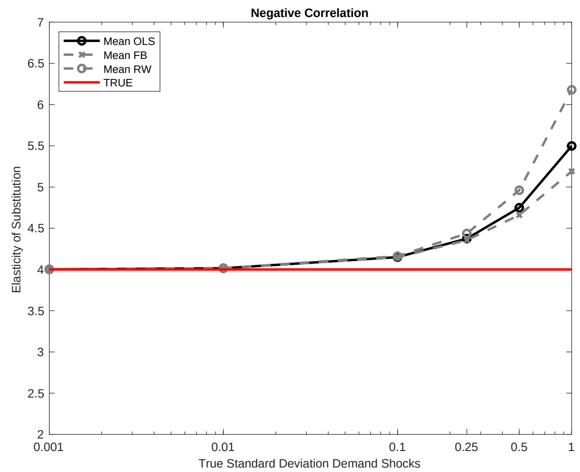
$$p_{kt} = \left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right) \times p_{ut-1}, \tag{37}$$

$$s^*_{kt} = \frac{\left(\frac{p_{kt}/\tilde{p}_t}{p_{kt-1}/\tilde{p}_{t-1}}\right)^{1-\sigma} \times \left(\frac{\varphi_{ut}/\tilde{\varphi}_t}{\varphi_{ut-1}/\tilde{\varphi}_{t-1}}\right)^{\sigma-1} \times s^*_{kt-1}}{\sum_{\ell\in\Omega^*}\left(\frac{p_{\ell t}/\tilde{p}_t}{p_{\ell t-1}/\tilde{p}_{t-1}}\right)^{1-\sigma} \times \left(\frac{\varphi_{\ell t}/\tilde{\varphi}_t}{\varphi_{\ell t-1}/\tilde{\varphi}_{t-1}}\right)^{\sigma-1} \times s^*_{\ell t-1}}. \tag{38}$$

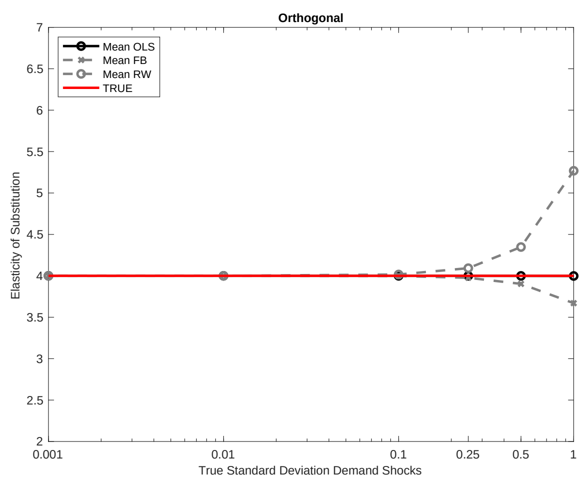
Given the prices and expenditure shares for periods $t-1$ and t (p_{kt-1}, p_{kt}, s^*_{kt-1}, s^*_{kt}), we estimate the elasticity of substitution using OLS (using $\widehat{\sigma}^{OLS} = 1 - \widehat{\beta}^{OLS}$), our FB estimator ($\widehat{\sigma}^{FB}$) and the RW estimator ($\widehat{\sigma}^{RW}$). In Figure 1, we display the mean estimated elasticity of substitution across 250 replications for each of these estimators. In each panel, the vertical axis displays the elasticity of substitution (σ), and the horizontal axis shows the standard deviation of appeal shocks (χ_φ) using a log scale. The top panel reports results for a negative correlation of price and appeal shocks ($\rho = -0.5$); the middle panel gives results for orthogonal price and appeal shocks ($\rho = 0$); and the bottom panel presents results for a positive correlation between price and appeal shocks ($\rho = 0.5$).

Figure 1: Mean Estimated Elasticities of Substitution ($\widehat{\sigma}^{OLS}$, $\widehat{\sigma}^{FB}$, $\widehat{\sigma}^{RW}$) for Different Correlations Between Price and Appeal Shocks

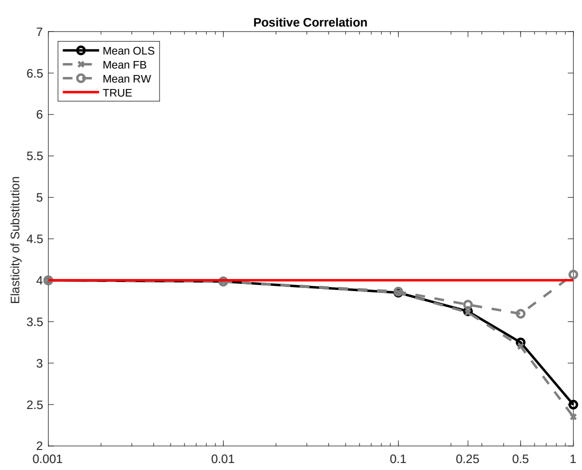
(a) Negative Correlation ($\rho = -0.5$)



(b) Orthogonal ($\rho = 0$)



(c) Positive Correlation ($\rho = 0.5$)



Note: 1,000 Monte Carlo simulations; 10,000 varieties; joint log normal distribution of price and appeal shocks; we set the standard deviation for initial prices to one ($\chi_p = 1$); we consider three different values for the correlation between initial prices and appeal ($\rho \in \{-0.5, 0, 0.5\}$); we consider five different values for the standard deviation of appeal shocks ($\chi_\varphi \in \{0.001, 0.01, 0.1, 0.5, 1\}$).

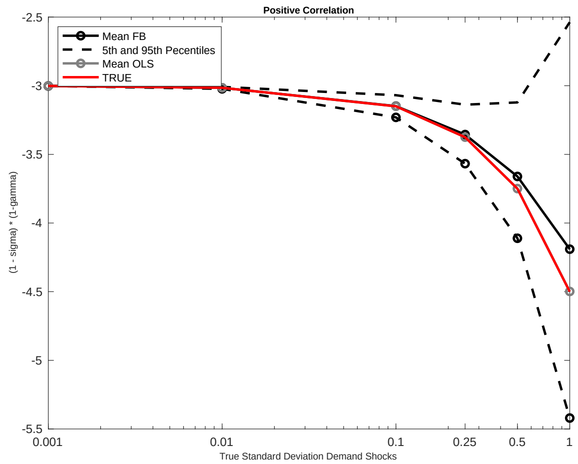
As appeal shocks become small, we find that all three estimates converge to the true elasticity of substitution, regardless of the correlation between price and appeal shocks (consistent with Proposition 5). In the middle panel where price and appeal shocks are orthogonal, the mean OLS estimate ($\widehat{\sigma}^{OLS}$) is centered on the true parameter value (consistent with Proposition 4). Furthermore, the FB estimate lies close to the OLS estimate (consistent with our results in Section 4.4.2). As the standard deviation of appeal shocks becomes large, the FB estimate begins to depart from the OLS estimates. The reason is that this Monte Carlo imposes that the population unweighted expectation of price and appeal shocks is zero, which ensure consistency of the OLS estimate. In contrast, the weighted expectation of price and appeal shocks using initial expenditure share weights can diverge from zero in any finite sample, in which case the FB estimator departs from the true parameter value (from Proposition 2).

In the top panel where price and appeal shocks are negatively correlated, the mean OLS estimate ($\widehat{\sigma}^{OLS}$) lies above the true elasticity of substitution, which is consistent with conventional omitted variables bias. The OLS elasticity is substitution is negatively related to the OLS slope coefficient in the demand system regression ($\widehat{\sigma}^{OLS} = 1 - \widehat{\beta}^{OLS}$). High prices reduce expenditure shares, whereas high appeal increases expenditure shares, in this demand system regression. Therefore, a negative correlation between prices and appeal implies that the negative impact of high prices on sales is magnified by low appeal, resulting in a too negative estimated OLS slope coefficient on prices (too negative $\widehat{\beta}^{OLS}$), and hence a too large OLS elasticity of substitution (too positive $\widehat{\sigma}^{OLS}$). As the standard deviation of appeal shocks becomes large, we find that the FB estimate remains close to the OLS estimate, whereas the RW estimate departs from it (consistent with our results in Section 4.4.3).

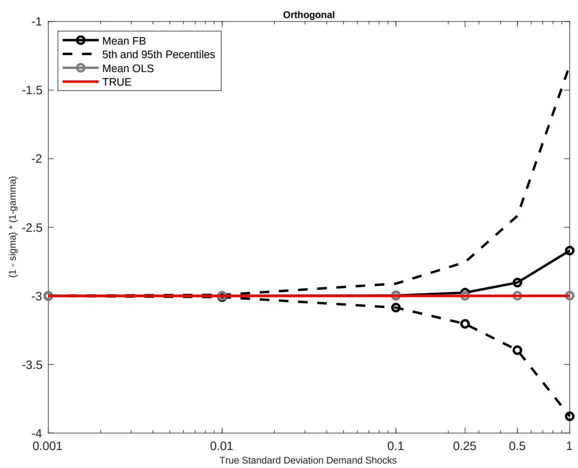
In the bottom panel where price and appeal shocks are positively correlated, the mean OLS estimate ($\widehat{\sigma}^{OLS}$) lies below the true elasticity of substitution, which is again consistent with conventional omitted variables bias. A positive correlation between prices and appeal implies that the negative impact of high prices on sales is offset by high appeal, resulting in an estimated OLS slope coefficient on prices that is biased away from negative values ($\widehat{\beta}^{OLS}$ less negative than the true parameter), which implies an OLS elasticity of substitution that is biased downwards ($\widehat{\sigma}^{OLS}$ less positive than the true parameter). As the standard deviation of appeal shocks becomes large, we again find that the FB estimate remains close to the OLS estimate, whereas the RW estimate departs from it (in line with our results in Section 4.4.3).

Figure 2: Specification Test for CES Preferences and Joint Log Normally Distributed Price and Appeal Shocks

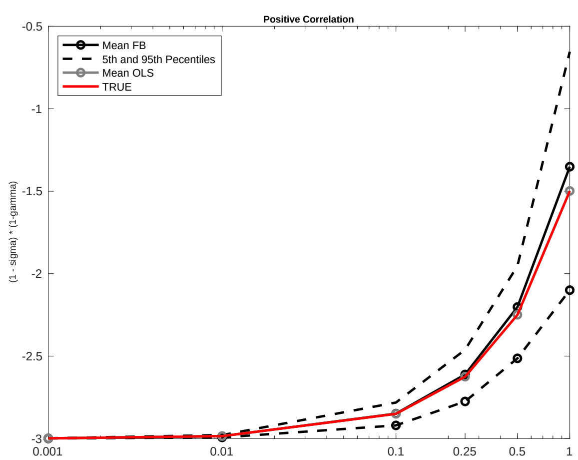
(a) Negative Correlation ($\rho = -0.5$)



(b) Orthogonal ($\rho = 0$)



(c) Positive Correlation ($\rho = 0.5$)



Note: 1,000 Monte Carlo simulations; 10,000 varieties; joint log normal distribution of price and appeal shocks; we set the standard deviation for initial prices to one ($\chi_p = 1$); we consider three different values for the correlation between initial prices and appeal ($\rho \in \{-0.5, 0, 0.5\}$); we consider five different values for the standard deviation of appeal shocks ($\chi_\varphi \in \{0.001, 0.01, 0.1, 0.5, 1\}$).

Figure 2 shows how these results can be used to develop a joint specification check for the assumptions of CES preferences and joint log normally distributed price and appeal shocks. In each panel, the vertical axis corresponds to the slope coefficient from a regression of log changes in expenditure shares on log changes in prices. We show the true value of of this parameter ($(1-\sigma)(1-\gamma)$), the mean estimated OLS slope coefficient across the 250 replications ($\widehat{\beta}^{OLS}$), one minus the mean FB estimate of the elasticity of substitution across the 250 replications ($\left(1-\widehat{\sigma}^{FB}\right)$), one minus the 5th percentile of the FB estimate of the elasticity of substitution across the 250 replications ($\left(1-\widehat{\sigma}^{FB}\right)$), and one minus the 95th percentile of the elasticity of substitution across the 250 replications ($\left(1-\widehat{\sigma}^{FB}\right)$). We display each of these variables against the standard deviation of appeal shocks (χ_{φ}) on the horizontal axis using a log scale. The top panel reports results for a negative correlation of price and appeal shocks ($\rho = -0.5$); the middle panel gives results for orthogonal price and appeal shocks ($\rho = 0$); and the bottom panel presents results for a positive correlation between price and appeal shocks ($\rho = 0.5$).

Across each of the panels, we find that the mean estimated OLS slope coefficient ($\widehat{\beta}^{OLS}$) lies close to the true parameter value ($(1-\sigma)(1-\gamma)$), consistent with the properties of OLS summarized in Proposition 4. This pattern holds regardless of whether price and appeal shocks are negatively correlated (top panel), orthogonal (middle panel), or positively correlated (bottom panel), and regardless of the standard deviation of appeal shocks (horizontal axis in each panel). Additionally, we find that one minus the mean FB estimate of the elasticity of substitution ($\left(1-\widehat{\sigma}^{FB}\right)$) lies close to the mean estimated OLS slope coefficient ($\widehat{\beta}^{OLS}$), with this mean OLS estimate well within the bootstrapped 95 percent confidence intervals around the mean FB estimate. This similarity of the OLS and FB estimates requires the full non-linear structure of CES preferences and log normally distributed price and appeal shocks and therefore provides a specification check on these functional form assumptions.

5 Logit

A well-known result in the discrete choice literature is that CES preferences can be derived as the aggregation of the choices of individual consumers with extreme-value-distributed idiosyncratic preferences, as shown in Anderson, de Palma and Thisse (1992) and Train (2009). In this section, we briefly use this result to show that our family of estimators also can be applied for logit preferences, as widely used in applied microeconometric research. Following McFadden (1974), we suppose that the utility of an individual consumer i who consumes c_{ik} units of product k at time t is given by:

$$U_{it} = \log \varphi_{kt} + \log c_{ikt} + z_{ikt}, \tag{39}$$

where φ_{kt} captures product appeal that is common across consumers; z_{ikt} captures idiosyncratic consumer tastes for each product that are drawn from an independent Type-I Extreme Value distribution, $G(z) = e^{-e^{(-z/\nu-\kappa)}}$, where ν is the shape parameter of the extreme value distribution and $\kappa \approx 0.577$ is the Euler-Mascheroni constant.

Each consumer is assumed to have the same expenditure (E_t) and to choose their preferred product given the observed realizations for idiosyncratic tastes. Under these assumptions, the probability that individual i

chooses product k is equal to the share of aggregate expenditure on product k and given by:

$$s_{ikt} = s_{kt} = \frac{\left(p_{kt}/\varphi_{kt}\right)^{-1/\nu}}{\sum_{\ell\in\Omega_t}\left(p_{\ell t}/\varphi_{\ell t}\right)^{-1/\nu}}, \tag{40}$$

as shown in Section A.10 of the online appendix. Integrating over the distribution for idiosyncratic tastes, the expected utility of each consumer is:

$$\mathbb{E}\left[U_{it}\right] = \log\left[\frac{E_t}{P_t}\right], \qquad P_t = \left[\sum_{k\in\Omega_t}\left(p_{kt}/\varphi_{kt}\right)^{-1/\nu}\right]^{-\nu}, \tag{41}$$

where P_t is the unit expenditure function, as also shown in Section A.10 of the online appendix.

Comparing equations (40) and (41) for logit preferences to equations (2) and (1) for CES preferences, the predictions of logit demand for aggregate expenditure shares and the unit expenditure function are the same as those of CES demand, where $1/\nu = \sigma - 1$. Therefore, our results for the FB and RW estimators for CES preferences also hold for logit preferences.

6 Conclusions

We provide new aggregation and estimation results for constant elasticity of substitution (CES) preferences. This specification for utility is not only one of the most commonly-used preferences structures in international trade and macroeconomics, but also provides a tractable approach to introducing variable elasticities of substitution through its extensions to mixed CES, mixed logit, nested CES, or nested logit specifications.

In our main aggregation result, we show that CES preferences have a fractal-like property: in the same way that the change in the appeal-adjusted price of an individual variety can be written as the ratio of changes in prices to changes in appeal, the change in the aggregate unit expenditure function can be written as the ratio of a price index to an appeal index. We show that this result holds regardless of the cardinality of utility and with the entry and exit of varieties. Analogous results hold for aggregate consumer behavior for logit preferences, for each nest of utility for nested CES or nested logit, and for each type of consumers for mixed CES or mixed logit.

We provide an analytical characterization of the properties of the appeal indexes in this aggregation result. We use these properties of the appeal indexes to develop four estimators of the elasticity of substitution: the forward, backward, forward-backward, and reverse-weighting estimators. We show that the forward-backward estimator is a consistent estimator of the true elasticity of substitution as the number of varieties becomes large, if price changes and appeal changes are independently distributed and uncorrelated across varieties. More generally, if preferences and CES, and price changes and appeal changes are correlated and joint log normally distributed, we show that the forward-backward estimator converges to the ordinary least squares (OLS) slope coefficient from regressing log changes in expenditure shares on log changes in prices.

In this case of correlated price changes and appeal changes, neither the forward-backward estimator nor the OLS estimator are consistent. But the comparison of the forward-backward estimator and the OLS slope coefficient provides a specification check on the combined assumptions of CES preferences and joint log normally distributed price changes and appeal changes. Although this last result requires a functional form assumption for both preferences and the distribution of price and appeal shocks, this joint log normal distributional assumption can be rationalized through a central limit theorem argument as the limiting distribution of correlated price and appeal shocks as the number of varieties becomes large, if these shocks are independently distributed across varieties.

More broadly, if price changes and appeal changes are correlated according to a general statistical distribution, the forward, backward, forward-backward, reverse-weighting and OLS estimators are all consistent estimators of the true elasticity of substitution as the number of varieties becomes large, as appeal shocks become small for each good. Therefore, the comparison of these four estimators provides a metric for assessing the empirical relevance of appeal shocks in the estimation of demand systems for CES preferences.

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