Web-based Technical Appendix for
"Unequal Effects of Trade on Workers with Different Abilities"

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A.1 Problem of the Firm

As discussed in detail in Helpman, Itskhoki and Redding (2008; henceforth HIR), the problem of the firm is given by:

$$\pi(\theta) \equiv \max_{\substack{n\geq 0,\\ a_c\geq a_{\min},\\ I_x\in\{0,1\}}} \left\{ \frac{1}{1+\beta\gamma} r(n,a_c,I_x) - bn - \frac{c}{\delta}a_c^{\delta} - f_d - I_x f_x \right\}, \tag{A1}$$

subject to revenues from sales in the domestic and foreign market given by[1]

$$r(n,a_c,I_x) = \left[1 + I_x \tau^{-\frac{\beta}{1-\beta}} \left(\frac{A^*}{A}\right)^{\frac{1}{1-\beta}}\right]^{1-\beta} A y(n,a_c)^{\beta}$$

and output given by

$$y(n,a_c) = \left(\kappa_y \theta n^{\gamma} a_c^{1-\gamma k}\right),$$

as explained in the text of the paper. The firm's variables of optimization are the measure of workers to match with (n), the screening cutoff (a_c) and export status (I_x with $I_x = 1$ if the firm exports and $I_x = 0$ otherwise). The firm will stay in the industry if and only if its operating profit is non-negative, $\pi(\theta) \geq 0$. Note that the operating profit in (A1) equals the revenues

[1] Total revenues equal $r(q_d,q_x) = Aq_d^{\beta} + \tau^{-\beta}A^* q_x^{\beta}$, where τ is the iceberg trade cost, q_d is sales at home and q_x are shipments abroad so that total output $y = q_d + q_x$. Maximizing over q_d and q_x yields the expression in the text.

that accrue to the firm as a result of bargaining with the workers (a share $1/(1+\beta\gamma)$ of total revenues) minus all labor market costs (search and screening) and fixed costs of production and exporting.

The firm's first-order conditions for the choice of n and a_c are given by:

$$\frac{\beta\gamma}{1+\beta\gamma} r(\theta) = bn(\theta),$$
$$\frac{\beta(1-\gamma k)}{1+\beta\gamma} r(\theta) = ca_c(\theta)^{\delta},$$

where $r(\theta)$ denotes the optimal revenues of a firm with productivity θ. These first-order conditions imply that more productive firms have higher revenues, sample more workers and have higher screening thresholds. Using the expression for revenues as a function of (n, a_c, I_x) we can solve for closed-expressions for $n(\theta)$ and $a_c(\theta)$ given the export status I_x. Firm employment is obtained as $h(\theta) = n(\theta)\left[a_{\min}/a_c(\theta)\right]^k$ and it is also increasing in firm productivity as long as $\delta > k$.

Firm profits are given by

$$\pi(\theta) = \frac{\Gamma}{1+\beta\gamma} r(\theta) - f_d - I_x(\theta) f_x,$$

where $\Gamma \equiv 1 - \beta\gamma - \beta(1-\gamma k)/\delta > 0$. The zero-profit cutoff productivity is then defined by $\pi(\theta_d) = 0$ with $I_x(\theta_d) = 0$. The exporting cutoff productivity is defined by $\pi(\theta_x^-) = \pi(\theta_x^+)$ given that $I_x(\theta_x^-) = 0$ and $I_x(\theta_x^+) = 1$. The explicit conditions for θ_d and θ_x are provided in HIR. We focus on the range of parameter values for which $\theta_x > \theta_d$.

As a result of bargaining, the wage rate equals a fraction $\beta\gamma/(1+\beta\gamma)$ of average revenues per worker hired, as shown in HIR. Using the firm's first-order conditions this implies:

$$w(\theta) = \frac{\beta\gamma}{1+\beta\gamma}\frac{r(\theta)}{h(\theta)} = \frac{b}{h(\theta)/n(\theta)} = b\left(\frac{a_c(\theta)}{a_{\min}}\right)^k. \tag{A2}$$

Therefore, the wage rate is also increasing in the productivity of the firm.

These relationships together with the zero-profit cutoff condition $\pi(\theta_d) = 0$ allow us to obtain the closed-form expressions for the firm-specific variables provided in (1) in the paper.

Finally, we briefly discuss the determination of labor market tightness (x) and the search cost (b) given expected worker income (ω). Detailed discussion is provided in HIR. From (A2), expected worker income upon being matched with a θ-firm equals the search cost and is independent of θ:

$$w(\theta)\frac{h(\theta)}{n(\theta)} = b.$$

Labor market tightness x corresponds to the matching probability for a worker. Therefore, expected worker income from job search in the industry equals bx. In order for a worker to choose to search for a job in this industry, it has to equal expected income from his outside option, ω:

$$bx = \omega.$$

Finally, using a Cobb-Douglas matching function and a cost of posting vacancies for firms, we can show that the search cost is an increasing power function of labor market tightness (and parameters of the matching technology):

$$b = \alpha_0 x_1^{\alpha}.$$

Combining these two expressions allows us to solve for both x and b as functions of ω only. Therefore, holding expected worker income ω constant implies constant values of labor market tightness x and the search cost b.

A.2 Proof of Result 2

Consider the ability-specific hiring rate in the open economy relative to autarky, $\sigma^T(a)/\sigma^A(a)$, where the expressions for $\sigma^T(a)$ and $\sigma^A(a)$ are given in the paper. For the range of abilities $a \in \left[a_d, a_x^-\right]$, we have a constant relative hiring rate:

$$\frac{\sigma^T(a)}{\sigma^A(a)} = \frac{1}{1 + \rho^{z - \frac{\beta}{\Gamma}}\left[\Upsilon_x^{\frac{1-\beta}{\Gamma}} - 1\right]} \equiv \chi_0 < 1.$$

For ability $a \in \left(a_x^-, a_x^+\right)$, this ratio is decreasing and strictly below χ_0:

$$\frac{\sigma^T(a)}{\sigma^A(a)} = \chi_0 \frac{1 - \left(a_d/a_x^-\right)^{k/\mu}}{1 - \left(a_d/a\right)^{k/\mu}} < \chi_0 < 1,$$

where we used the facts that $a_d/a_x^- = \rho^{\beta/(\delta\Gamma)}$ and $\mu = \beta k/[\delta(z\Gamma - \beta)]$. Finally, for abilities $a \geq a_x^+$, the relative hiring rate increases monotonically from below χ_0 to 1 as $a \to \infty$:

$$\frac{\sigma^T(a)}{\sigma^A(a)} = \frac{1 - \chi_1\left(a_d/a\right)^{k/\mu}}{1 - \left(a_d/a\right)^{k/\mu}} \leq 1,$$

where $\chi_1 = \chi_0 \Upsilon_x^{z(1-\beta)/\beta} > 1$ since $\rho < 1$, $\Upsilon_x > 1$, and we assume $z\Gamma > \beta$. One can verify directly using the definitions of a_x^- and a_x^+ that $\sigma^T(a)/\sigma^A(a)$ is a continuous function of a on $[a_d, \infty)$. Therefore, there exists $\hat{a}_x > a_x^+$ such that:

$$\frac{\sigma^T(a)}{\sigma^A(a)} < \chi_0 \qquad \forall a \in \left(a_x^-, \hat{a}_x\right) \qquad \text{and} \qquad \frac{\sigma^T(a)}{\sigma^A(a)} > \chi_0 \qquad \forall a > \hat{a}_x.$$

Specifically, $\hat{a}_x$ solves

$$\frac{1-\chi_1\big(a_d/\hat{a}_x\big)^{k/\mu}}{1-\big(a_d/\hat{a}_x\big)^{k/\mu}}=\chi_0 \qquad \Leftrightarrow \qquad \left(\frac{a_d}{\hat{a}_x}\right)^{k/\mu}=\frac{1-\chi_0}{\chi_1-\chi_0}.$$

Using the expressions for χ_0, χ_1 and the definition of a_x^+, one can verify directly that $\hat{a}_x > a_x^+$.

Recall that the ability-specific unemployment rate is given by $u(a) = 1 - x\sigma(a)$. Under the assumption that expected worker income ω is constant, labor market tightness x is also constant, as discussed in Appendix A.1 above. Therefore, the normalized change in the unemployment rate equals one minus the relative hiring rate:

$$\frac{u^T(a)-u^A(a)}{1-u^A(a)}=1-\frac{\sigma^T(a)}{\sigma^A(a)}.$$

Result 2 then follows directly from the properties of $\sigma^T(a)/\sigma^A(a)$ discussed above and illustrated in Figure 1 in the paper. ■

A.3 Conditional Wage Distributions

Consider a wage distribution conditional on ability $a \geq a_d$. As explained in the paper, workers with ability a receive wages in the range $\big[w_d, w\big(\theta_c(a)\big)\big]$, where $w(\theta)$ is a firm-specific wage defined in (1) in the paper and $\theta_c(a)$ is the most productive firm that hires workers with ability a.

In the closed economy, for any $\tilde{w} \in \big[w_d, w\big(\theta_c(a)\big)\big]$ there exists $\tilde{\theta} \geq \theta_d$ such that $w(\tilde{\theta}) = \tilde{w}$. Denote the inverse of $w(\cdot)$ by $\theta_w(\cdot)$ so that $\theta_w(\tilde{w}) = \tilde{\theta}$. Then workers with ability a are hired by all firms with $\theta \in [\theta_d, \theta_c(a)]$, while those hired by $\theta \in [\theta_d, \theta_w(w)]$ receive wages in $[w_d, w]$. A firm with productivity θ in these ranges hires workers with ability a proportional to $n(\theta)$ since matching is random.[2] Therefore, we can express the conditional wage distribution as:

$$F_w^A(w|a)=\frac{\int_{\theta_d}^{\theta_w(w)} n(\theta)\mathrm{d}G_\theta(\theta)}{\int_{\theta_d}^{\theta_c(a)} n(\theta)\mathrm{d}G_\theta(\theta)} \qquad \text{for} \quad w \in \big[w_d, w\big(\theta_c(a)\big)\big]. \tag{A3}$$

Using the firm-specific solution for $n(\theta)$ and the Pareto productivity distribution, we can integrate to obtain expression (4) in the paper, where note from (1) in the paper that $w\big(\theta_c(a)\big) = w_d \cdot \big(a/a_d\big)^k$. We reproduce the closed economy wage distribution here:

$$\forall a \geq a_d \qquad F_w^A(w|a)=\frac{1-\big(w_d/w\big)^{1/\mu}}{1-\big(a_d/a\big)^{k/\mu}} \qquad \text{for} \quad w_d \leq w \leq w_d\big(a/a_d\big)^k.$$

[2]This means that the ratio of a-workers in firms θ' and $\theta'' \in [\theta_d, \theta_c(a)]$ equals $n(\theta')/n(\theta'')$.

Now consider the open economy. For workers with ability $a \in [a_d, a_x^-]$ employed only by non-exporting firms, the same logic as above applies. Therefore, we have

$$F_w^T(w|a) = F_w^A(w|a) \qquad \text{for} \quad a_d \leq a \leq a_x^- \quad \text{and} \quad w_d \leq w \leq w_d\big(a/a_d\big)^k.$$

Workers with ability $a \in (a_x^-, a_x^+)$ experience the same labor market outcomes (in terms of the wage distribution and unemployment rate) as workers with ability $a = a_x^-$. Therefore,

$$F_w^T(w|a) = F_w^T\left(w|a_x^-\right) \qquad \text{for} \quad a_x^- < a < a_x^+ \quad \text{and} \quad w_d \leq w \leq w_x^- = w_d\big(a_x^-/a_d\big)^k.$$

Finally, consider workers with abilities $a \geq a_x^+$ employed both by non-exporters and some exporters. These workers receive wages on $[w_d, w_x^-] \cup [w_x^+, w(\theta_c(a))]$, where $w_x^- = w_d \rho^{-\beta k/(\delta\Gamma)}$ and $w_x^+ = w_x^- \Upsilon_x^{(1-\beta)k/(\delta\Gamma)}$ are the wages paid by most productive non-exporter and least productive exporter respectively. We can still define $\theta_w(w)$ as the productivity of firm paying wage w with the additional requirement that $\theta_w(w) = \theta_x$ for $w \in [w_x^-, w_x^+]$. Then the characterization in (A3) also applies for the open economy conditional wage distribution. Computing the integrals in (A3) using the solution for firm-specific allocations and the productivity distribution, we obtain:

$$\forall a \geq a_x^+ \qquad F_w^T(w|a) = \begin{cases} \dfrac{1-\left(w_d/w\right)^{1/\mu}}{1+\rho^{z-\beta/\Gamma}\left[\Upsilon_x^{(1-\beta)/\Gamma}-1\right]-\Upsilon_x^{z(1-\beta)/\beta}\left(a_d/a\right)^{k/\mu}}, & w_d \leq w \leq w_x^-, \\[2ex] \dfrac{1-\rho^{z-\beta/\Gamma}}{1+\rho^{z-\beta/\Gamma}\left[\Upsilon_x^{(1-\beta)/\Gamma}-1\right]-\Upsilon_x^{z(1-\beta)/\beta}\left(a_d/a\right)^{k/\mu}}, & w_x^- < w < w_x^+, \\[2ex] 1-\dfrac{\Upsilon_x^{z(1-\beta)/\beta}\left[\left(w_d/w\right)^{1/\mu}-\left(a_d/a\right)^{k/\mu}\right]}{1+\rho^{z-\beta/\Gamma}\left[\Upsilon_x^{(1-\beta)/\Gamma}-1\right]-\Upsilon_x^{z(1-\beta)/\beta}\left(a_d/a\right)^{k/\mu}}, & w_x^+ \leq w \leq w_d\big(a/a_d\big)^k. \end{cases}$$

A.4 Proof of Result 3

Expression (4) in the paper for the autarky conditional wage distribution immediately implies that for any $a_2 > a_1 \geq a_d$, $F_w^A(w|a_2) < F_w^A(w|a_1)$ for all $w \in \left(w_d, w_d(a_2/a_d)^k\right)$. Therefore, the wage distribution for workers with higher ability first-order stochastically dominates the wage distribution for workers with lower abilities. A direct implication of this is that the conditional average wage $\bar{w}(a)$ increases with $a \geq a_d$. A closed-form expression for the conditional average wage can be computed as:

$$\bar{w}^A(a) = \frac{w_d}{1-\mu}\frac{1-\left(a_d/a\right)^{k(1-\mu)/\mu}}{1-\left(a_d/a\right)^{k/\mu}}, \tag{A4}$$

which can be verified to increase in $a \geq a_d$. In Helpman, Itskhoki and Redding (2008) we additionally compute the Coefficient of Variation and the Theil Index of wage inequality for $F_w^A(w|a)$ and show that they both increase in a. ■

Similar results obtain for the open economy conditional wage distribution, $F_w^T(w|a)$, derived in Appendix A.3. First of all, as long as $a_1 < a_2$ and both a_1 and a_2 do not belong to $\left[a_x^-, a_x^+\right]$, $F_w^T(w|a_2) < F_w^T(w|a_1)$, so that there is a similar first-order stochastic dominance property for workers of higher ability. This again implies that the conditional average wage increases (weakly) in a. The closed-form expression for the average wage in the open economy is:

$$\bar{w}^T(a) = \frac{w_d}{1-\mu} \begin{cases} \frac{1-\left(a_d/a\right)^{k(1-\mu)/\mu}}{1-\left(a_d/a\right)^{k/\mu}}, & a_d \leq a \leq a_x^-, \\ \frac{1-\rho^{z-\beta(1-k/\delta)/\Gamma}}{1-\rho^{z-\beta/\Gamma}}, & a_x^- < a < a_x^+, \\ \frac{1-\rho^{z-\beta(1-k/\delta)/\Gamma}+\Upsilon_x^{\frac{z(1-\beta)}{\beta}}(a_d/a_x)^{\frac{k(1-\mu)}{\mu}}\left(1-(a_x^+/a)^{k(1-\mu)/\mu}\right)}{1-\rho^{z-\beta/\Gamma}+\Upsilon_x^{\frac{z(1-\beta)}{\beta}}(a_d/a_x)^{\frac{k}{\mu}}\left(1-(a_x^+/a)^{k/\mu}\right)}, & a \geq a_x^+. \end{cases} \tag{A5}$$

We use (A4) and (A5) to construct Figure 2 in the paper.

One can also calculate in closed-form the coefficient of variation and the Theil index for the open economy conditional wage distribution.

A.5 Proof of Result 4

Consider the autarky and open economy conditional wage distributions derived in Appendix A.3, $F_w^A(w|a)$ and $F_w^T(w|a)$. Part (i) follows immediately. Part (ii) follows from Result 3 and the fact that $F_w^T(w|a) = F_w^A\left(w|a_x^-\right)$ for $a \in \left[a_x^-, a_x^+\right]$.

To prove part (iii) we need to show that there exists $\tilde{a}_x$ such that $F_w^T(w|a) < F_w^A(w|a)$ for all $w > w_d$ when $a > \tilde{a}_x$. Consider the autarky and open economy wage distributions for a worker of ability a over the interval $\left[w_d, w_x^-\right]$. From the expressions for $F_w^A(w|a)$ and $F_w^T(w|a)$ above, these distributions take the same value when:

$$1 + \rho^{z-\beta/\Gamma}\left[\Upsilon_x^{(1-\beta)/\Gamma} - 1\right] - \Upsilon_x^{z(1-\beta)/\beta}\left(a_d/a\right)^{k/\mu} = 1 - \left(a_d/a\right)^{k/\mu}.$$

Denote the a which solves this equation by $\tilde{a}_x$. Using the definitions of a_x^- and a_x^+, one can solve for:

$$\left(\frac{\tilde{a}_x}{a_x^+}\right)^{k/\mu} = \frac{\Upsilon_x^{\frac{z(1-\beta)}{\beta}} - 1}{\Upsilon_x^{\frac{z(1-\beta)}{\beta}} - \Upsilon_x^{\frac{1-\beta}{\beta}\frac{z\Gamma-\beta}{\Gamma}}} > 1,$$

since $z\Gamma > \beta$. On the range $[w_d, w_x^-]$, the autarky wage distribution lies below that in the open economy for $a < \tilde{a}_x$ and above that in the open economy for $a > \tilde{a}_x$. Therefore, for the open economy wage distribution to first-order stochastically dominate that in autarky, a necessary condition is that $a > \tilde{a}_x$. We now show that this is also sufficient. For $w \in [w_x^-, w_x^+]$, the open economy wage cdf is flat while the autarky wage cdf is increasing. Therefore, we only need to verify that for any $a > \tilde{a}_x$ and for all $w > w_x^+$ the open economy wage cdf still lies below that in autarky. This is straightforward to establish since both cdfs reach 1 at $w\big(\theta_c(a)\big) = w_d\big(a/a_d\big)^k$ and the slope of the open economy cdf is always steeper in w for $w_x^+ < w < w_d\big(a/a_d\big)^k$.[3] This establishes the first-order stochastic dominance of the open economy wage distribution for $a \geq \tilde{a}_x > a_x^+$. Another corollary of this argument is that for $a \in (a_x^+, \tilde{a}_x)$, there is no first-order stochastic ordering of the two distributions. ■

An immediate corollary of Result 4 is that $\bar{w}^T(a) = \bar{w}^A(a)$ for $a_d \leq a \leq a_x^-$; $\bar{w}^T(a) < \bar{w}^A(a)$ for $a_x^- < a < a_x^+$ and $\bar{w}^T(a) > \bar{w}^A(a)$ for $a > \tilde{a}_x$. Moreover, there exists $a_x^+ < \tilde{\tilde{a}}_x < \tilde{a}_x$ such that $\bar{w}^T(a) - \bar{w}^A(a)$ switches sign from negative to positive at $a = \tilde{\tilde{a}}_x$. These features are illustrated in Figure 2.

A.6 Construction of Figure 3

To construct Figure 3 we pick a quantile q of the aggregate wage distribution and find the corresponding wage rate w_q. Formally, we have $[0,1] \ni q = F_w^j(w_q^j)$, where $j \in \{A, T\}$ and $F_w(w)$ represents the unconditional industry wage distribution. Given w_q^j we can recover the abilities which are consistent with this wage rate. Specifically, only workers with ability above $a_c\big(\theta_w^j(w_q^j)\big)$ can receive a wage w_q^j, where $\theta_w^j(w)$ denotes the productivity of a firm which pays wage rate w (in autarky and open economy respectively), as defined in Appendix A.3. Therefore, the distribution of ability consistent with wage rate w_q^j is a Pareto with shape parameter k and cutoff $a_c\big(\theta_w^j(w_q^j)\big)$. Higher q corresponds to a higher w_q^j which also corresponds to a higher ability cutoff. With the obtained distribution of ability for each quantile, we can compute expected unemployment rates $\bar{u}^j(q) \equiv \mathbb{E}\left\{u^j(a)|a \geq a_c\big(\theta_w^j(w_q^j)\big)\right\}$, using the expressions for $u^j(a)$ and

[3] To see this, rewrite the autarky conditional wage distribution for comparability as

$$F_w^A(w|a) = 1 - \frac{\left[\left(w_d/w\right)^{1/\mu} - \left(a_d/a\right)^{k/\mu}\right]}{1 - \left(a_d/a\right)^{k/\mu}}.$$

Then for any a, $z\Gamma > \beta$ immediately implies a larger coefficient (in absolute value) in front of $\left[\left(w_d/w\right)^{1/\mu} - \left(a_d/a\right)^{k/\mu}\right]$ for the open economy wage distribution. In other words, the open economy wage distribution is always steeper for $w > w_x^+$ and has to lie below that in autarky.

$\sigma^j(a)$ provided in the text of the paper. We then plot $\bar{u}^j(q)$ against q in Figure 3.

References

[1] Helpman, Elhanan, Oleg Itskhoki and Stephen Redding (2008) "Wages, Unemployment and Inequality with Heterogeneous Firms and Workers," *NBER Working Paper No. 14122.*

[2] Helpman, Elhanan, Oleg Itskhoki and Stephen Redding (2009) "Inequality and Unemployment in a Global Economy," *CEPR Discussion Paper No. 7353.*