Online Appendix for "The Linear Algebra of Economic Geography Models"[*]

Benny Kleinman[†]
University of Chicago
Ernest Liu[‡]
Princeton University and NBER
Stephen J. Redding[§]
Princeton University, NBER and CEPR

July 2023

A Theoretical Framework

In this Online Appendix, we provide the derivations for the results reported in the paper. The world economy consists of a set of locations indexed by $i, n \in \{1, \ldots, N\}$. The economy as a whole has an exogenous supply of workers that we normalize to one ($\bar{\ell} = 1$). Each worker is endowed with one unit of labor that is supplied inelastically. Workers are perfectly mobile across locations, but have idiosyncratic preferences for each location.

A.1 Consumer Preferences

The preferences of worker ν who chooses to live in location n are characterized by the following indirect utility function:

$$u_n(\nu) = \frac{b_n \epsilon_n(\nu) w_n}{p_n}, \tag{A.1}$$

where w_n is the wage, p_n is the consumption goods price index; b_n captures amenities that are common for all workers (such as climate and scenic views); and $\epsilon_n(\nu)$ is an idiosyncratic amenity draw that is specific to each worker ν and location n. The consumption goods price index is assumed to take the following constant elasticity of substitution (CES) form:

$$p_n = \left[\sum_{i=1}^{N} p_{ni}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \qquad \sigma > 1. \tag{A.2}$$

[*]We are grateful to Princeton University for research support. We would to thank Andre Alcantra and Chenming Zhang for excellent research assistance. The usual disclaimer applies.
[†]Dept. Economics, Becker Friedman Institute, Chicago, IL 60637. Email: binyamin.kleinman@gmail.com.
[‡]Dept. Economics, JRR Building, Princeton, NJ 08544. Email: ernestliu@princeton.edu.
[§]Dept. Economics and SPIA, JRR Building, Princeton, NJ 08544. Email: reddings@princeton.edu.

Idiosyncratic amenities are drawn independently for each worker and location from the following independent Fréchet distribution:

$$F(\epsilon) = \exp\left(-\epsilon^{-\kappa}\right), \qquad \kappa > 1, \tag{A.3}$$

where we normalize the scale parameter to one, because it enters the model isomorphically to b_n; the shape parameter $\kappa > 1$ regulates the dispersion of idiosyncratic amenities, and determines the migration elasticity that captures the responsiveness of population shares to real wages.

A.2 Production Technology

Goods are produced with labor according to a constant returns to scale production technology and under conditions of perfect competition. These goods can be traded between locations subject to iceberg variable costs of trade, such that $\tau_{ni} \geq 1$ units must be shipped from location i to location n in order for one unit to arrive. Therefore, the cost to the consumer in location n of purchasing the good produced by location i is:

$$p_{ni} = \frac{\tau_{ni} w_i}{z_i}, \tag{A.4}$$

where z_i captures productivity in location i and iceberg variable trade costs satisfy $\tau_{ni} > 1$ for $n \neq i$ and $\tau_{nn} = 1$.

For comparability with the international trade literature, we focus on the case in which productivity (z_i) is exogenous. Nevertheless, it is straightforward to introduce agglomeration economies, whereby productivity in each location is increasing in its own population, or the population of surrounding locations.

A.3 General Equilibrium

General equilibrium can be referenced by the vectors of wages and population shares in each location $\{w_n, \ell_n\}$. The $2 \times N$ values of wages and populations shares are determined by the $2 \times N$ equilibrium conditions from goods market clearing and population mobility. Goods market clearing requires that income in each location equals expenditure on the goods produced by that location:

$$w_i \ell_i = \sum_{n=1}^{N} s_{ni} w_n \ell_n, \tag{A.5}$$

where s_{ni} is the share of expenditure of importer n on exporter i. From CES demand (A.2) and the production technology (A.4), this expenditure share is given by:

$$s_{ni} = \frac{\left(\tau_{ni} w_i / z_i\right)^{-\theta}}{\sum_{m=1}^{N} \left(\tau_{nm} w_m / z_m\right)^{-\theta}}, \tag{A.6}$$

where $\theta \equiv \sigma - 1$ is the trade elasticity.

We choose the total income of all locations as the numeraire:

$$\sum_{i=1}^{N} q_i = 1, \tag{A.7}$$

where $q_i \equiv w_i \ell_i$ is the nominal income of location i.

Using the properties of the Fréchet distribution (A.3), the probability that a worker chooses to live in location n is:

$$\ell_n = \frac{(b_n w_n / p_n)^{\kappa}}{\sum_{h=1}^{N} (b_h w_h / p_h)^{\kappa}}, \tag{A.8}$$

and expected utility conditional on choosing to live in a location is equalized across all locations and given by:

$$\bar{u} = \Gamma\left(\frac{\kappa - 1}{\kappa}\right)\left[\sum_{h=1}^{N} (b_h w_h / p_h)^{\kappa}\right]^{\frac{1}{\kappa}}, \tag{A.9}$$

where $\Gamma(\cdot)$ is the Gamma function.

Given our assumption of exogenous productivity, there are no agglomeration forces in the model. Therefore, the dispersion force from worker idiosyncratic preferences ensures the existence of a unique equilibrium distribution of wages and population shares $\{w_n, \ell_n\}$ across locations.

A.4 Comparative Statics

We begin by linearizing the general equilibrium conditions of the model.

A.4.1 Expenditure Shares

Totally differentiating the expenditure share (A.6), we get:

$$\frac{\mathrm{d}s_{ni}}{s_{ni}} = \theta\left(\sum_{h=1}^{N} s_{nh} \frac{\mathrm{d}p_{nh}}{p_{nh}} - \frac{\mathrm{d}p_{ni}}{p_{ni}}\right), \tag{A.10}$$

$$\mathrm{d}\ln s_{ni} = \theta\left(\sum_{h=1}^{N} s_{nh}\, \mathrm{d}\ln p_{nh} - \mathrm{d}\ln p_{ni}\right),$$

where from the expression for equilibrium prices (A.4) above, we have:

$$\frac{\mathrm{d}p_{ni}}{p_{ni}} = \frac{\mathrm{d}\tau_{ni}}{\tau_{ni}} + \frac{\mathrm{d}w_i}{w_i} - \frac{\mathrm{d}z_i}{z_i},$$

$$\mathrm{d}\ln p_{ni} = \mathrm{d}\ln \tau_{ni} + \mathrm{d}\ln w_i - \mathrm{d}\ln z_i. \tag{A.11}$$

A.4.2 Price Indices

Totally differentiating the consumption goods price index (A.2), we have:

$$\frac{\mathrm{d}p_n}{p_n} = \sum_{m=1}^{N} s_{nm} \frac{\mathrm{d}p_{nm}}{p_{nm}},$$

$$\mathrm{d}\ln p_n = \sum_{m=1}^{N} s_{nm}\, \mathrm{d}\ln p_{nm}. \tag{A.12}$$

A.4.3 Location Choice Probabilities

Totally differentiating the location choice probabilities (A.8), we have:

$$\frac{\mathrm{d}\ell_n}{\ell_n} = \kappa \left(\frac{\mathrm{d}b_n}{b_n} + \frac{\mathrm{d}w_n}{w_n} - \frac{\mathrm{d}p_n}{p_n} \right) - \kappa \sum_{\ell=1}^{N} \ell_h \left(\frac{\mathrm{d}b_h}{b_h} + \frac{\mathrm{d}w_h}{w_h} - \frac{\mathrm{d}p_h}{p_h} \right). \tag{A.13}$$

Using the total derivative of the consumption goods price index (A.12), we can rewrite this total derivative of the location choice probabilities as:

$$\frac{d\ell_n}{\ell_n} = \kappa \left(\frac{\mathrm{d}b_n}{b_n} + \frac{\mathrm{d}w_n}{w_n} - \sum_{m=1}^{N} s_{nm} \frac{\mathrm{d}p_{nm}}{p_{nm}} \right) - \kappa \sum_{h=1}^{N} \ell_h \left(\frac{\mathrm{d}b_h}{b_h} + \frac{\mathrm{d}w_h}{w_h} - \sum_{m=1}^{N} s_{hm} \frac{\mathrm{d}p_{hm}}{p_{hm}} \right),$$

which can be further rewritten as:

$$\mathrm{d}\ln \ell_n = \begin{bmatrix} \kappa \left(\mathrm{d}\ln b_n + \mathrm{d}\ln w_n - \sum_{m=1}^{N} s_{nm}\, \mathrm{d}\ln p_{nm} \right) \\ -\kappa \sum_{h=1}^{N} \ell_h \left(\mathrm{d}\ln b_h + \mathrm{d}\ln w_h - \sum_{m=1}^{N} s_{hm}\, \mathrm{d}\ln p_{hm} \right) \end{bmatrix}. \tag{A.14}$$

Totally differentiating expected utility, we have:

$$\frac{\mathrm{d}\bar{u}}{\bar{u}} = \sum_{h=1}^{N} \ell_h \left(\frac{\mathrm{d}b_h}{b_h} + \frac{\mathrm{d}w_h}{w_h} - \frac{\mathrm{d}p_h}{p_h} \right).$$

Using the total derivative of the consumption goods price index (A.12), we can rewrite this total derivative of expected utility as:

$$\frac{\mathrm{d}\bar{u}}{\bar{u}} = \sum_{h=1}^{N} \ell_h \left(\frac{\mathrm{d}b_h}{b_h} + \frac{\mathrm{d}w_h}{w_h} - \sum_{m=1}^{N} s_{hm} \frac{\mathrm{d}p_{hm}}{p_{hm}} \right),$$

which equivalently can be written as:

$$\mathrm{d}\ln \bar{u} = \sum_{h=1}^{N} \ell_h \left(\mathrm{d}\ln b_h + \mathrm{d}\ln w_h - \sum_{m=1}^{N} s_{hm}\, \mathrm{d}\ln p_{hm} \right). \tag{A.15}$$

A.4.4 Market Clearing

Recall the market clearing condition (A.5):

$$w_i \ell_i = \sum_{n=1}^{N} s_{ni} w_n \ell_n .$$

Totally differentiating this market clearing condition, we have:

$$\frac{\mathrm{d}w_i}{w_i} w_i \ell_i + \frac{\mathrm{d}\ell_i}{\ell_i} w_i \ell_i = \sum_{n=1}^{N} \frac{\mathrm{d}s_{ni}}{s_{ni}} s_{ni} w_n \ell_n + \sum_{n=1}^{N} \frac{\mathrm{d}w_n}{w_n} s_{ni} w_n \ell_n + \sum_{n=1}^{N} \frac{\mathrm{d}\ell_n}{\ell_n} s_{ni} w_n \ell_n ,$$

$$\frac{\mathrm{d}w_i}{w_i} w_i \ell_i + \frac{\mathrm{d}\ell_i}{\ell_i} w_i \ell_i = \sum_{n=1}^{N} s_{ni} w_n \ell_n \left(\frac{\mathrm{d}w_n}{w_n} + \frac{\mathrm{d}s_{ni}}{s_{ni}} + \frac{\mathrm{d}\ell_n}{\ell_n} \right) .$$

Using our total derivative of expenditure shares (A.10), this becomes:

$$\frac{\mathrm{d}w_i}{w_i} w_i \ell_i + \frac{\mathrm{d}\ell_i}{\ell_i} w_i \ell_i = \sum_{n=1}^{N} s_{ni} w_n \ell_n \left(\frac{\mathrm{d}w_n}{w_n} + \theta \left(\sum_{h=1}^{N} s_{nh} \frac{\mathrm{d}p_{nh}}{p_{nh}} - \frac{\mathrm{d}p_{ni}}{p_{ni}} \right) + \frac{\mathrm{d}\ell_n}{\ell_n} \right) ,$$

$$\frac{\mathrm{d}w_i}{w_i} + \frac{\mathrm{d}\ell_i}{\ell_i} = \sum_{n=1}^{N} \frac{s_{ni} w_n \ell_n}{w_i \ell_i} \left(\frac{\mathrm{d}w_n}{w_n} + \theta \left(\sum_{h=1}^{N} s_{nh} \frac{\mathrm{d}p_{nh}}{p_{nh}} - \frac{\mathrm{d}p_{ni}}{p_{ni}} \right) + \frac{\mathrm{d}\ell_n}{\ell_n} \right) ,$$

$$\frac{\mathrm{d}w_i}{w_i} + \frac{\mathrm{d}\ell_i}{\ell_i} = \sum_{n=1}^{N} t_{in} \left(\frac{\mathrm{d}w_n}{w_n} + \frac{\mathrm{d}\ell_n}{\ell_n} + \theta \left(\sum_{h=1}^{N} s_{nh} \frac{\mathrm{d}p_{nh}}{p_{nh}} - \frac{\mathrm{d}p_{ni}}{p_{ni}} \right) \right) ,$$

where we have defined t_{in} as the share of location i's income from market n:

$$t_{in} \equiv \frac{s_{ni} w_n \ell_n}{w_i \ell_i} ,$$

and equivalently we can write this expression as:

$$\mathrm{d}\ln w_i + \mathrm{d}\ln \ell_i = \sum_{n=1}^{N} t_{in} \left(\mathrm{d}\ln w_n + \mathrm{d}\ln \ell_n + \theta \left(\sum_{h=1}^{N} s_{nh} \,\mathrm{d}\ln p_{nh} - \mathrm{d}\ln p_{ni} \right) \right) . \tag{A.16}$$

A.5 Friend-Enemy Representation

We consider small productivity shocks, holding constant amenities, bilateral trade costs and the total population of all locations:

$$\begin{aligned} \mathrm{d}\ln b_i &= 0, && \forall\, i \in N, \\ \mathrm{d}\ln \tau_{ni} &= 0, && \forall\, n, i \in N, \\ \mathrm{d}\ln \bar{\ell} &= 0. \end{aligned} \tag{A.17}$$

A.5.1 Goods Market Clearing

Using the total derivative of prices (A.11) and our assumption (A.17), we can re-write the total derivative of the market clearing condition (A.16) as:

$$\mathrm{d}\ln w_i + \mathrm{d}\ln \ell_i = \sum_{n=1}^{N} t_{in} \left(\mathrm{d}\ln w_n + \mathrm{d}\ln \ell_n + \theta \left(\sum_{h=1}^{N} s_{nh} \left[\mathrm{d}\ln w_h - \mathrm{d}\ln z_h \right] - \left[\mathrm{d}\ln w_i - \mathrm{d}\ln z_i \right] \right) \right),$$

which can be further rewritten in matrix form as:

$$\mathrm{d}\ln \mathbf{w} + \mathrm{d}\ln \boldsymbol{\ell} = \mathbf{T}\left(\mathrm{d}\ln \mathbf{w} + \mathrm{d}\ln \boldsymbol{\ell} \right) + \theta \left(\mathbf{TS} - \mathbf{I} \right) \left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z} \right). \tag{A.18}$$

where $\boldsymbol{S}$ is a matrix with elements S_{ni} for the share of importer n's expenditure on exporter i; $\boldsymbol{T}$ is a matrix with elements $T_{in} = S_{ni} w_n \ell_n / \left(w_i \ell_i \right)$ equal to the share of location i's income from location n.

The first term on the right-hand side captures the *market-size effect* of the productivity shocks ($\mathbf{T}\left(\mathrm{d}\ln \mathbf{w} + \mathrm{d}\ln \boldsymbol{\ell} \right)$). An increase in the income of market n on the right-hand side (either through higher wages (w_n) or a higher population share (ℓ_n)) raises the income of location i on the left-hand side by an amount that is determined by the share of location i's income from market n ($\boldsymbol{T}_{in}$).

The second term on the right-hand side captures the *cross-substitution effect* of the productivity shocks ($\theta \left(\mathbf{TS} - \mathbf{I} \right) \left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z} \right)$). This consumer substitution depends on the product of the income share and expenditure share matrices ($\boldsymbol{M} \equiv \boldsymbol{TS} - \boldsymbol{I}$), where the in-th element of the cross-substitution matrix ($\boldsymbol{M} \equiv \boldsymbol{TS} - \boldsymbol{I}$) is given by $m_{in} \equiv \sum_{h=1}^{N} t_{ih} s_{hn} - 1_{n=i}$. For $i \neq n$, the sum $\sum_{h=1}^{N} t_{ih} s_{hn}$ captures the overall competitive exposure of country i to country n, through each of their common markets h, weighted by the importance of market h for country i's income (t_{ih}). As the competitiveness of country n increases, as measured by a decline in its wage relative to its productivity ($\mathrm{d}\ln w_n - \mathrm{d}\ln z_n$), consumers in all markets h substitute towards country n and away from other countries $i \neq n$. This substitution reduces income in country i and raises it in country n. With a constant elasticity import demand system, the magnitude of this cross-substitution effect in market h depends on the trade elasticity (θ) and the share of expenditure in market h on the goods produced by country n (s_{hn}): consumers in market h increase the expenditure share on country n by $(1 - s_{hn})$ and lower the expenditure share on country i by s_{hn}. Summing across all markets h, we obtain the overall impact on country i's income.

The goods market clearing condition (A.18) takes a similar form as in constant elasticity international trade models, as considered in Kleinman et al. (2020). The key difference is that population shares (ℓ_n) are endogenous and affect the income of each location on both the left and right-hand sides of the equation.

A.5.2 Population Shares

Using the total derivative of prices (A.11) and our assumption (A.17), we can re-write the endogenous changes in population shares (A.14) as:

$$\mathrm{d}\ln \ell_n = \kappa \left(\mathrm{d}\ln w_n - \sum_{m=1}^{N} s_{nm} \left(\mathrm{d}\ln w_m - \mathrm{d}\ln z_m \right) \right) - \kappa \sum_{h=1}^{N} \ell_h \left(\mathrm{d}\ln w_h - \sum_{m=1}^{N} s_{hm} \left(\mathrm{d}\ln w_m - \mathrm{d}\ln z_m \right) \right),$$

which can be further rewritten in matrix form as:

$$\mathrm{d}\ln \boldsymbol{\ell} = \kappa \left(\mathbf{I} - \mathbf{1}\boldsymbol{\ell}' \right) \left[\mathrm{d}\ln \mathbf{w} - \mathbf{S} \left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z} \right) \right]. \tag{A.19}$$

The first term inside the square parentheses on the right-hand side of equation (A.19) captures the impact of productivity shocks on population shares through nominal wages, while the second term captures their impact through consumption price indexes.

A.5.3 Expected Utility

Using the total derivatives of price indexes (A.12) and prices (A.11), and our assumption (A.17), we can re-write the change in the common level of expected utility across all locations (A.15) as:

$$\mathrm{d}\ln \bar{u} = \sum_{h=1}^{N} \ell_h \left[\mathrm{d}\ln w_h - \sum_{m=1}^{N} s_{hm} \left(d\ln w_m - d\ln z_m \right) \right],$$

which can be further re-written in matrix form as:

$$\mathrm{d}\ln \bar{u} = \boldsymbol{\ell}' \left[\mathrm{d}\ln \mathbf{w} - \mathbf{S} \left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z} \right) \right], \tag{A.20}$$

where the term inside the square parentheses is the change in the real wage in each location.

A.5.4 Nominal Income Exposure

We now derive our nominal income exposure measure. Our choice of numeraire (A.7) implies:

$$\sum_{i=1}^{N} q_i \mathrm{d}\ln w_i + \sum_{i=1}^{N} q_i \mathrm{d}\ln \ell_i = 0,$$

where $q_i \equiv w_i \ell_i$, and $\mathrm{d}\ln \bar{\ell} = \frac{\mathrm{d}\bar{\ell}}{\bar{\ell}} = \frac{\sum_{i=1}^{N} \mathrm{d}\ell_i}{\bar{\ell}} = 0$. We can equivalently write this implication of our choice of numeraire as:

$$\mathbf{Q} \left(\mathrm{d}\ln \boldsymbol{w} + \mathrm{d}\ln \boldsymbol{\ell} \right) = 0,$$

where $\mathbf{Q}$ is a $N \times N$ matrix with the nominal income row vector $\boldsymbol{q}'$ stacked N times.

Using this choice of numeraire, we can re-write the goods market clearing condition (A.18) as:

$$(\mathbf{I}+\mathbf{Q})\left(\mathrm{d}\ln\mathbf{w}+\mathrm{d}\ln\boldsymbol{\ell}\right)=\mathbf{T}\left(\mathrm{d}\ln\mathbf{w}+\mathrm{d}\ln\boldsymbol{\ell}\right)+\theta\left(\mathbf{TS}-\mathbf{I}\right)\left(\mathrm{d}\ln\mathbf{w}-\mathrm{d}\ln\mathbf{z}\right),$$
$$(\mathbf{I}-\mathbf{T}+\mathbf{Q})\left(\mathrm{d}\ln\mathbf{w}+\mathrm{d}\ln\boldsymbol{\ell}\right)=\theta\left(\mathbf{TS}-\mathbf{I}\right)\left(\mathrm{d}\ln\mathbf{w}-\mathrm{d}\ln\mathbf{z}\right),$$
$$\mathrm{d}\ln\mathbf{w}+\mathrm{d}\ln\boldsymbol{\ell}=(\mathbf{I}-\mathbf{T}+\mathbf{Q})^{-1}\theta\left(\mathbf{TS}-\mathbf{I}\right)\left(\mathrm{d}\ln\mathbf{w}-\mathrm{d}\ln\mathbf{z}\right). \tag{A.21}$$

From equation (A.19), we can re-write the log change in population share ($\mathrm{d}\ln\boldsymbol{\ell}$) as:

$$\mathrm{d}\ln\boldsymbol{\ell}=\kappa\left(\mathbf{I}-\mathbf{1}\boldsymbol{\ell}'\right)\left[\left(\mathbf{I}-\mathbf{S}\right)\mathrm{d}\ln\mathbf{w}+\mathbf{S}\,\mathrm{d}\ln\mathbf{z}\right],$$
$$\mathrm{d}\ln\boldsymbol{\ell}=\left[\begin{array}{c}\kappa\left(\mathbf{I}-\mathbf{1}\boldsymbol{\ell}'\right)\left(\mathbf{I}-\mathbf{S}\right)\mathrm{d}\ln\mathbf{w}\\+\kappa\left(\mathbf{I}-\mathbf{1}\boldsymbol{\ell}'\right)\mathbf{S}\,\mathrm{d}\ln\mathbf{z}\end{array}\right] \tag{A.22}$$

Using equation (A.22) to substitute for the log change in population share ($\mathrm{d}\ln\boldsymbol{\ell}$) in equation (A.21), we obtain:

$$\left[\mathbf{I}+\kappa\left(\mathbf{I}-\mathbf{1}\boldsymbol{\ell}'\right)\left(\mathbf{I}-\mathbf{S}\right)-\left(\mathbf{I}-\mathbf{T}+\mathbf{Q}\right)^{-1}\theta\left(\mathbf{TS}-\mathbf{I}\right)\right]\mathrm{d}\ln\mathbf{w},$$
$$-\left[\kappa\left(\mathbf{I}-\mathbf{1}\boldsymbol{\ell}'\right)\mathbf{S}+\left(\mathbf{I}-\mathbf{T}+\mathbf{Q}\right)^{-1}\theta\left(\mathbf{TS}-\mathbf{I}\right)\right]\mathrm{d}\ln\mathbf{z}.$$

We can re-write this goods market clearing condition in terms of the elasticity of wages with respect to productivity shocks as:

$$\mathrm{d}\ln\boldsymbol{w}=\mathbf{W}\mathrm{d}\ln\boldsymbol{z}, \tag{A.23}$$

where $\mathbf{W}$ is our friend-enemy matrix of bilateral income exposure to productivity shocks:

$$\mathbf{W}\equiv-\left(\left(1+\kappa\right)\mathbf{I}-\kappa\mathbf{1}\boldsymbol{\ell}'-\mathbf{V}\right)^{-1}\mathbf{V}, \tag{A.24}$$
$$\mathbf{V}\equiv\left[\kappa\left(\mathbf{I}-\mathbf{1}\boldsymbol{\ell}'\right)+\left(\mathbf{I}-\mathbf{T}+\mathbf{Q}\right)^{-1}\theta\left(\mathbf{TS}-\mathbf{I}\right)\right].$$

The presence of the term $\mathbf{Q}$ ensures that the matrices $(\mathbf{I}-\mathbf{T}+\mathbf{Q})$ and $((1+\kappa)\,\mathbf{I}-\kappa\mathbf{1}\boldsymbol{\ell}'-\mathbf{V})$ are invertible.[1]

We can also compute an analogous measure of real wage exposure to productivity shocks in all locations ($\mathbf{U}$), such that the common change in expected utility (A.20) across all locations can be written as:

$$\mathrm{d}\ln\bar{u}=\boldsymbol{\ell}'\mathbf{U}\mathrm{d}\ln\boldsymbol{z}, \tag{A.25}$$

where real income exposure ($\mathbf{U}$) is:

$$\mathbf{U}\equiv\left[\left(\mathbf{I}-\mathbf{S}\right)\mathbf{W}+\mathbf{S}\right], \tag{A.26}$$

[1]The expenditure and income shares both sum to one, which implies that the rows and columns of $\mathbf{S}$ and $\mathbf{TS}$ are not linearly independent. Therefore, without the inclusion of the term in $\mathbf{Q}$, the matrices are not invertible. Economically, this reflects the fact that expenditure and income shares are homogeneous of degree zero in wages, such that that level of wage exposure cannot be recovered from these expenditure and income shares without a choice of numeraire.

and is invariant to the choice of numeraire.

We thus obtain sufficient statistics for the exposure of nominal and real wages in each location to productivity shocks in all locations. These sufficient statistics depend solely on the observed expenditure share ($\mathbf{S}$) and income share ($\mathbf{T}$) matrices and the two parameters of the trade elasticity (θ) and the migration elasticity (κ).

References

Kleinman, Benny, Ernest Liu and Stephen J. Redding (2020) "International Friends and Enemies," *NBER Working Paper*, 27587.