Online Appendix for "Dynamic Spatial General Equilibrium"*

Benny Kleinman†
Princeton University
Ernest Liu‡
Princeton University and NBER
Stephen J. Redding§
Princeton University, NBER and CEPR

December 2022

A Introduction

In this Online Appendix, we report the detailed derivations for our baseline model with a single traded sector from Section 2 of the paper.

B Baseline Dynamic Spatial Model

The model environment is summarized in Table 1 in the paper. We begin by providing additional derivations for capital accumulation decisions.

B.1 Capital Accumulation

Combining landlords' intertemporal utility (5) and budget constraint (6), the landlord's intertemporal optimization problem is:

$$\max_{\left\{c^k_{t+s}, k_{t+s+1}\right\}_{s=0}^{\infty}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^{t+s} \frac{\left(c^k_{it+s}\right)^{1-1/\psi}}{1-1/\psi}, \tag{B.1}$$

$$\text{subject to} \qquad p_{it} c^k_{it} + p_{it}\left(k_{it+1} - (1-\delta) k_{it}\right) = r_{it} k_{it}.$$

Lemma. *(Lemma 1 in the paper) We denote $R_{it} \equiv 1 - \delta + r_{it}/p_{it}$ as the gross return on capital. The optimal consumption of location i's landlords satisfies $c_{it} = \varsigma_{it} R_{it} k_{it}$, where ς_{it} is defined recursively as*

$$\varsigma_{it}^{-1} = 1 + \beta^{\psi}\left(\mathbb{E}_t\left[R_{it+1}^{\frac{\psi-1}{\psi}} \varsigma_{it+1}^{-\frac{1}{\psi}}\right]\right)^{\psi}.$$

Landlord's optimal saving and investment satisfies $k_{it+1} = (1 - \varsigma_{it}) R_{it} k_{it}$.

*The latest version of the paper can be downloaded from here. The latest version of this Online Appendix can be downloaded from here. A separate Online Supplement containing further theoretical extensions, additional empirical results and the data appendix can be downloaded from here. A toolkit illustrating our spectral analysis for a model economy can be downloaded from here.

†Dept. Economics, JRR Building, Princeton, NJ 08544. Email: binyamin@princeton.edu.
‡Dept. Economics, JRR Building, Princeton, NJ 08544. Email: ernestliu@princeton.edu.
§Dept. Economics and SPIA, JRR Building, Princeton, NJ 08544. Email: reddings@princeton.edu.

Proof. For notational simplicity we drop the locational subscript. Consider a landlord facing linear returns R_t on wealth k_t for all t. Let $v(k_t;t)$ denote the value function at time t; we can rewrite the landlord's consumption-saving problem recursively as:

$$v(k_t;t) = \max_{\{c_t,k_{t+1}\}} \frac{c_t^{1-1/\psi}}{1-1/\psi} + \beta \mathbb{E}_t v(k_{t+1};t+1) \qquad \text{s.t. } c_t + k_{t+1} = R_t k_t,$$

where, with a slight abuse of notation, we denote landlord consumption as c instead of c^k for the purpose of this proof. We guess-and-verify that there exists a_t, ς_t such that $v(k;t) = \frac{(a_t R_t k_t)^{1-1/\psi}}{1-1/\psi}$, and that optimal $c_t = \varsigma_t R_t k_t$.

Under the conjecture, $v_k(k_t;t) = \frac{a_t^{1-1/\psi} R_t^{1-1/\psi} k_t^{-1/\psi}}{1-1/\psi}$, we set up the Lagrangian as:

$$\mathcal{L}_t = \frac{c_t^{1-1/\psi}}{1-1/\psi} + \beta \mathbb{E}_t v(k_{t+1};t+1) + \xi_t \left[R_t k_t - c_t - k_{t+1}\right].$$

The first-order conditions imply:

$$\{c_t\} \qquad c_t^{-1/\psi} = \xi_t,$$

$$\{k_{t+1}\} \qquad \xi_t = \beta k_{t+1}^{-1/\psi} \mathbb{E}_t \left[a_{t+1}^{1-1/\psi} R_{t+1}^{1-1/\psi}\right].$$

Hence:

$$c_t = \beta^{-\psi} k_{t+1} \mathbb{E}_t \left[a_{t+1}^{1-1/\psi} R_{t+1}^{1-1/\psi}\right]^{-\psi}. \tag{B.2}$$

The Envelope condition $v_k(k_t;t) = \xi_t R_t$ implies

$$a_t^{1-1/\psi} R_t^{1-1/\psi} k_t^{-1/\psi} = c_t^{-1/\psi} R_t. \tag{B.3}$$

Substituting our guess that $c_t \equiv \varsigma_t R_t k_t$ into the Envelope condition (B.3), we obtain:

$$a_t^{1-\psi} = \varsigma_t.$$

The budget constraint implies $k_{t+1} = (1-\varsigma_t) R_t k_t$, and substituting this result into (B.2), we get:

$$\varsigma_t = \beta^{-\psi} \mathbb{E}_t \left[a_{t+1}^{1-1/\psi} R_{t+1}^{1-1/\psi}\right]^{-\psi} (1-\varsigma_t)$$

$$\iff \varsigma_t^{-1} = 1 + \beta^{\psi} \mathbb{E}_t \left[R_{t+1}^{\frac{\psi-1}{\psi}} \varsigma_{t+1}^{-1/\psi}\right]^{\psi}. \tag{B.4}$$

□

In the special case of logarithmic flow utility ($\psi = 1$), landlord's optimal consumption and saving rate is independent of future returns to capital, and $\varsigma_t = (1-\beta)$ for all t, as in Moll (2014).

B.2 Existence and Uniqueness (Proof of Proposition 1 in the Paper)

We now use the system of general equilibrium equations (10)-(16) in the paper to characterize the existence and uniqueness of a deterministic steady-state equilibrium with time-invariant fundamentals $\{z_i, b_i, \tau_{ni}, \kappa_{ni}\}$ and endogenous variables $\{v_i^*, w_i^*, R_i^*, \ell_i^*, k_i^*\}$. Given these time-invariant fundamentals, we can drop the expectation over future fundamentals, such that $\mathbb{E}_t v_{gt+1}^w = v_{gt+1}^w$.

B.2.1 Capital Labor Ratio

In steady-state, $k_{it+1} = k_{it} = k_i^*$, $c_{it+1}^k = c_{it}^k = c_i^{k*}$, and $\varsigma_{it+1} = \varsigma_{it} = \varsigma_i^*$, which implies: $1-\varsigma_i^* = \beta$. Using these results and the capital accumulation condition in equation (11) in the paper, we can solve for the steady-state capital-labor ratio:

$$\frac{k_i^*}{\ell_i^*} = \frac{\beta}{1-\beta\left(1-\delta\right)} \frac{1-\mu}{\mu} \frac{w_i^*}{p_i^*}. \tag{B.5}$$

B.2.2 Price Index

Using this result for the steady-state capital-labor ratio, we can re-write the price index in equation (10) in the paper as follows:

$$\left(p_n^*\right)^{-\theta} = \sum_{i=1}^{N} \psi \widetilde{\tau}_{ni} \left(w_i^*\right)^{-\theta\mu} \left(p_i^*\right)^{-\theta(1-\mu)}, \tag{B.6}$$

$$\psi \equiv \left(\frac{1-\beta\left(1-\delta\right)}{\beta}\right)^{-\theta(1-\mu)}, \qquad \widetilde{\tau}_{ni} \equiv \left(\tau_{ni}/z_i\right)^{-\theta}.$$

B.2.3 Goods Market Clearing Condition

Using this result for the steady-state capital-labor ratio, we can also re-write the goods market clearing condition in equation (12) in the paper as follows:

$$\ell_i^* \left(w_i^*\right)^{1+\theta\mu} \left(p_i^*\right)^{\theta(1-\mu)} = \sum_{n=1}^{N} \psi \widetilde{\tau}_{ni} \left(p_n^*\right)^{\theta} w_n^* \ell_n^*. \tag{B.7}$$

B.2.4 Value Function

The value function in equation (14) in the paper can be re-written as follows:

$$\exp\left(\frac{\beta}{\rho} v_n^{w*}\right) = \left(\frac{w_n^*}{p_n^*}\right)^{\beta/\rho} \phi_n^{\beta}, \qquad \phi_n \equiv \sum_{g=1}^{N} \widetilde{\kappa}_{gn} \exp\left(\frac{\beta}{\rho} v_g^{w*}\right). \tag{B.8}$$

Using this solution in the definition of ϕ_n immediately above, we have:

$$\phi_n = \sum_{g=1}^{N} \widetilde{\kappa}_{gn} \left(p_g^*\right)^{-\beta/\rho} \left(w_g^*\right)^{\beta/\rho} \phi_g^{\beta}. \tag{B.9}$$

B.2.5 Population Flow Condition

The population flow condition in equation (15) in the paper can be re-written as follows:

$$\ell_g^* = \sum_{i=1}^{N} \widetilde{\kappa}_{gi} \exp\left(\frac{\beta}{\rho} v_g^{w*}\right) \phi_i^{-1} \ell_i^*, \qquad \phi_i \equiv \sum_{m=1}^{N} \widetilde{\kappa}_{mi} \exp\left(\frac{\beta}{\rho} v_m^{w*}\right).$$

Now using the value function result (B.8) above, we have:

$$\left(p_g^*\right)^{\beta/\rho}\left(w_g^*\right)^{-\beta/\rho}\ell_g^*\phi_g^{-\beta}=\sum_{i=1}^{N}\widetilde{\kappa}_{gi}\ell_i^*\phi_i^{-1}. \tag{B.10}$$

B.2.6 System of Equations

Collecting together these results, the steady-state equilibrium of the model $\{p_i^*, w_i^*, \ell_i^*, \phi_i^*\}$ can be expressed as the solution to the following system of equations:

$$\left(p_i^*\right)^{-\theta}=\sum_{n=1}^{N}\psi\widetilde{\tau}_{in}\left(p_n^*\right)^{-\theta(1-\mu)}\left(w_n^*\right)^{-\theta\mu}, \tag{B.11}$$

$$\left(p_i^*\right)^{\theta(1-\mu)}\left(w_i^*\right)^{1+\theta\mu}\ell_i^*=\sum_{n=1}^{N}\psi\widetilde{\tau}_{ni}\left(p_n^*\right)^{\theta}w_n^*\ell_n^*, \tag{B.12}$$

$$\left(p_i^*\right)^{\beta/\rho}\left(w_i^*\right)^{-\beta/\rho}\ell_i^*\left(\phi_i^*\right)^{-\beta}=\sum_{n=1}^{N}\widetilde{\kappa}_{in}\ell_n^*\left(\phi_n^*\right)^{-1}, \tag{B.13}$$

$$\phi_i^*=\sum_{n=1}^{N}\widetilde{\kappa}_{ni}\left(p_n^*\right)^{-\beta/\rho}\left(w_n^*\right)^{\beta/\rho}\left(\phi_n^*\right)^{\beta}, \tag{B.14}$$

where we have the following definitions:

$$\psi\equiv\left(\frac{1-\beta\left(1-\delta\right)}{\beta}\right)^{-\theta(1-\mu)},\quad\widetilde{\tau}_{ni}\equiv\left(\tau_{ni}/z_i\right)^{-\theta},\quad\phi_i^*\equiv\sum_{n=1}^{N}\widetilde{\kappa}_{ni}\exp\left(\frac{\beta}{\rho}v_n^{w*}\right),\quad\widetilde{\kappa}_{in}\equiv\left(\kappa_{in}/b_n^{\beta}\right)^{-1/\rho}.$$

We now provide a sufficient condition for the existence of a unique steady-state equilibrium in terms of the properties of a coefficient matrix ($\boldsymbol{A}$) of model parameters $\{\psi, \theta, \beta, \rho, \mu, \delta\}$ following the approach of Allen, Arkolakis and Li (2020).

Proposition. ***Existence and Uniqueness (Proposition 1 in the paper).*** *A sufficient condition for the existence of a unique steady-state spatial distribution of economic activity $\{\ell_i^*, k_i^*, w_i^*, R_i^*, v_i^*\}$ (up to a choice of units) given time-invariant locational fundamentals $\{z_i^*, b_i^*, \tau_{ni}^*, \kappa_{ni}^*\}$ is that the spectral radius of a coefficient matrix ($\boldsymbol{A}$) of model parameters $\{\psi, \theta, \beta, \rho, \mu, \delta\}$ is less than or equal to one.*

Proof. We begin by deriving the sufficient condition for the existence a unique steady-state spatial distribution of economic activity $\{\ell_i^*, k_i^*, w_i^*, R_i^*, v_i^*\}$. The exponents on the variables on the left-hand side of the system of equations (B.11)-(B.14) can be represented as the following matrix:

$$\boldsymbol{\Lambda}=\begin{bmatrix}-\theta & 0 & 0 & 0\\ \theta\left(1-\mu\right) & \left(1+\theta\mu\right) & 1 & 0\\ \beta/\rho & -\beta/\rho & 1 & -\beta\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

The exponents on the variables on the right-hand side of the system of equations (B.11)-(B.14) can be represented as the following matrix:

$$\mathbf{\Gamma} = \begin{bmatrix} -\theta(1-\mu) & -\theta\mu & 0 & 0 \\ \theta & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -\beta/\rho & \beta/\rho & 0 & \beta \end{bmatrix}.$$

Let $\boldsymbol{A} \equiv |\mathbf{\Gamma}\mathbf{\Lambda}^{-1}|$ and denote the spectral radius (eigenvalue with the largest absolute value) of this matrix by $\rho(\boldsymbol{A})$. From Theorem 1 in Allen, Arkolakis and Li (2020), a sufficient condition for the existence of a unique equilibrium (up to a choice of units) is $\rho(\boldsymbol{A}) \leq 1$.

We next derive a sharper sufficient condition for the case of quasi-symmetric trade and migration costs: $\tau_{in} = \widetilde{\tau}_{in}\widetilde{\tau}_i^a\widetilde{\tau}_n^b$ and $\kappa_{in} = \widetilde{\kappa}_{in}\widetilde{\kappa}_i^c\widetilde{\kappa}_n^d$, where $\widetilde{\tau}_{in} = \widetilde{\tau}_{ni}$ and $\widetilde{\kappa}_{in} = \widetilde{\kappa}_{ni}$, as assumed in our empirical application. In this case of quasi-symmetric trade and migration costs, we can re-write the system of equations (B.11)-(B.14) as follows:

$$p_i^{-\theta}\left(\widetilde{\tau}_i^a\right)^{-1} = \sum_{n=1}^{N} \widetilde{\tau}_{in}\widetilde{\tau}_n^b p_n^{-\theta} q_n^{-\theta\mu}, \tag{B.15}$$

$$p_i^{\theta-1} q_i^{1+\theta\mu} \ell_i \left(\widetilde{\tau}_i^b\right)^{-1} = \sum_{n=1}^{N} \widetilde{\tau}_{in}\widetilde{\tau}_n^a p_n^{\theta-1} q_n \ell_n, \tag{B.16}$$

$$q_i^{-\beta/\rho} \ell_i \phi_i^{-\beta}\left(\widetilde{\kappa}_i^c\right)^{-1} = \sum_{n=1}^{N} \widetilde{\kappa}_{in}\widetilde{\kappa}_n^d \ell_n \phi_n^{-1}, \tag{B.17}$$

$$\phi_i\left(\widetilde{\kappa}_i^d\right)^{-1} = \sum_{n=1}^{N} \widetilde{\kappa}_{in}\widetilde{\kappa}_n^c q_n^{\beta/\rho}\phi_n^{\beta}. \tag{B.18}$$

From equation (B.18), we know:

$$1 = \sum_{n=1}^{N} \frac{\widetilde{\kappa}_{in}\widetilde{\kappa}_n^c q_n^{\beta/\rho}\phi_n^{\beta}}{\phi_i\left(\widetilde{\kappa}_i^d\right)^{-1}}.$$

Multiply the left-hand side of equation (B.17) by $\sum_{n=1}^{N} \frac{\widetilde{\kappa}_{in}\widetilde{\kappa}_n^c q_n^{\beta/\rho}\phi_n^{\beta}}{\phi_i\left(\widetilde{\kappa}_i^d\right)^{-1}}$, and move $\left(\widetilde{\kappa}_i^c\right)^{-1}\phi_i^{-\beta} q_i^{-\beta/\rho}$ to the right-hand side to obtain:

$$\sum_{n=1}^{N} \frac{\widetilde{\kappa}_{in}\widetilde{\kappa}_n^c q_n^{\beta/\rho}\phi_n^{\beta}}{\phi_i\left(\widetilde{\kappa}_i^d\right)^{-1}} \ell_i = \sum_{n=1}^{N} \frac{\widetilde{\kappa}_{in}\widetilde{\kappa}_i^c \phi_i^{\beta} q_i^{\beta/\rho}}{\phi_n\left(\widetilde{\kappa}_n^d\right)^{-1}} \ell_n,$$

$$\iff \frac{\ell_i\widetilde{\kappa}_i^d/\phi_i}{\sum_{n=1}^{N}\widetilde{\kappa}_{in}\ell_n\widetilde{\kappa}_n^d/\phi_n} = \frac{\widetilde{\kappa}_i^c\phi_i^{\beta} q_i^{\beta/\rho}}{\sum_{n=1}^{N}\widetilde{\kappa}_{in}\widetilde{\kappa}_n^c q_n^{\beta/\rho}\phi_n^{\beta}}.$$

Let $\gamma_i \equiv \frac{\ell_i\widetilde{\kappa}_i^d/\phi_i}{\sum_{n=1}^{N}\widetilde{\kappa}_{in}\ell_n\widetilde{\kappa}_n^d/\phi_n}$, then:

$$\ell_i\widetilde{\kappa}_i^d/\phi_i = \sum_{n=1}^{N} \gamma_i\widetilde{\kappa}_{in}\ell_n\widetilde{\kappa}_n^d/\phi_n,$$

$$\widetilde{\kappa}_i^c \phi_i^\beta q_i^{\beta/\rho} = \sum_{n=1}^N \gamma_i \widetilde{\kappa}_{in} \widetilde{\kappa}_n^c q_n^{\beta/\rho} \phi_n^\beta.$$

By the Perron-Frobenius theorem, $\ell_i \widetilde{\kappa}_i^d / \phi_i = x \widetilde{\kappa}_i^c \phi_i^\beta q_i^{\beta/\rho}$ for some constant x. Since the scale of ℓ_i is not pinned down by the system of equations—if $\{\ell_i\}$ is part of a solution to the system of equations, so is $\{2\ell_i\}$—we can without loss of generality set $x = 1$. Hence:

$$\ell_i = \widetilde{\kappa}_i^c \left(\widetilde{\kappa}_i^d\right)^{-1} \phi_i^{1+\beta} q_i^{\beta/\rho}. \tag{B.19}$$

Now we use the same strategy to reduce equations (B.15) and (B.16) down to one. Re-write equation (B.15) as:

$$1 = \sum_{n=1}^N \frac{\widetilde{\tau}_{in} \widetilde{\tau}_n^b p_n^{-\theta} q_n^{-\theta\mu}}{p_i^{-\theta} \left(\widetilde{\tau}_i^a\right)^{-1}}.$$

Substitute (B.15) into equation (B.16), then multiply the left-hand side by $\sum_{n=1}^N \frac{\widetilde{\tau}_{in} \widetilde{\tau}_n^b p_n^{-\theta} q_n^{-\theta\mu}}{p_i^{-\theta} \left(\widetilde{\tau}_i^a\right)^{-1}}$:

$$\sum_{n=1}^N \frac{\widetilde{\tau}_{in} \widetilde{\tau}_n^b p_n^{-\theta} q_n^{-\theta\mu}}{p_i^{-\theta} \left(\widetilde{\tau}_i^a\right)^{-1}} p_i^{\theta-1} q_i^{1+\theta\mu+\beta/\rho} \phi_i^{1+\beta} \left(\widetilde{\tau}_i^b\right)^{-1} \widetilde{\kappa}_i^c \left(\widetilde{\kappa}_i^d\right)^{-1} = \sum_{n=1}^N \widetilde{\tau}_{in} \widetilde{\tau}_n^a p_n^{\theta-1} \phi_n^{1+\beta} q_n^{1+\beta/\rho} \widetilde{\kappa}_n^c \left(\widetilde{\kappa}_n^d\right)^{-1},$$

$$\iff \frac{\widetilde{\tau}_i^a p_i^{\theta-1} q_i^{1+\beta/\rho} \phi_i^{1+\beta} \widetilde{\kappa}_i^c \left(\widetilde{\kappa}_i^d\right)^{-1}}{\sum_{n=1}^N \widetilde{\tau}_{in} \widetilde{\tau}_n^a p_n^{\theta-1} \phi_n^{1+\beta} q_n^{1+\beta/\rho} \widetilde{\kappa}_n^c \left(\widetilde{\kappa}_n^d\right)^{-1}} = \frac{\widetilde{\tau}_i^b p_i^{-\theta} q_i^{-\theta\mu}}{\sum_{n=1}^N \widetilde{\tau}_{in} \widetilde{\tau}_n^b p_n^{-\theta} q_n^{-\theta\mu}}.$$

Now let $\varphi_i \equiv \frac{\widetilde{\tau}_i^a p_i^{\theta-1} q_i^{1+\beta/\rho} \phi_i^{1+\beta} \widetilde{\kappa}_i^c \left(\widetilde{\kappa}_i^d\right)^{-1}}{\sum_{n=1}^N \widetilde{\tau}_{in} \widetilde{\tau}_n^a p_n^{\theta-1} \phi_n^{1+\beta} q_n^{1+\beta/\rho} \widetilde{\kappa}_n^c \left(\widetilde{\kappa}_n^d\right)^{-1}}$. We know:

$$\widetilde{\tau}_i^a p_i^{\theta-1} q_i^{1+\beta/\rho} \phi_i^{1+\beta} \widetilde{\kappa}_i^c \left(\widetilde{\kappa}_i^d\right)^{-1} = \sum_{n=1}^N \varphi_i \widetilde{\tau}_{in} \widetilde{\tau}_n^a p_n^{\theta-1} \phi_n^{1+\beta} q_n^{1+\beta/\rho} \widetilde{\kappa}_n^c \left(\widetilde{\kappa}_n^d\right)^{-1},$$

$$\widetilde{\tau}_i^b p_i^{-\theta} q_i^{-\theta\mu} = \sum_{n=1}^N \varphi_i \widetilde{\tau}_{in} \widetilde{\tau}_n^b p_n^{-\theta} q_n^{-\theta\mu}.$$

Again by the Perron-Frobenius theorem, $\widetilde{\tau}_i^a p_i^{\theta-1} q_i^{1+\beta/\rho} \phi_i^{1+\beta} \widetilde{\kappa}_i^c \left(\widetilde{\kappa}_i^d\right)^{-1} = y \widetilde{\tau}_i^b p_i^{-\theta} q_i^{-\theta\mu}$ for some constant y. Since p_i is a nominal variable, we can without loss of generality set $y = 1$. Hence:

$$p_i^{\theta-1} q_i^{1+\beta/\rho} \phi_i^{1+\beta} \widetilde{\tau}_i^a \widetilde{\kappa}_i^c \left(\widetilde{\kappa}_i^d\right)^{-1} \left(\widetilde{\tau}_i^b\right)^{-1} = p_i^{-\theta} q_i^{-\theta\mu},$$

$$\iff p_i^{-\theta} = q_i^{-\theta \frac{1+\beta/\rho+\theta\mu}{1-2\theta}} \phi_i^{-\theta \frac{1+\beta}{1-2\theta}} e_i, \tag{B.20}$$

where $e_i \equiv \left(\widetilde{\tau}_i^a \widetilde{\kappa}_i^c \left(\widetilde{d}_i^d\right)^{-1} \left(\widetilde{\tau}_i^b\right)^{-1}\right)^{\frac{-\theta}{1-2\theta}}$.

Now substitute (B.19) and (B.20) into (B.15) and (B.18):

$$q_i^{-\theta \frac{1+\beta/\rho+\theta\mu}{1-2\theta}} \phi_i^{-\theta \frac{1+\beta}{1-2\theta}} e_i \left(\widetilde{\tau}_i^a\right)^{-1} = \sum_{n=1}^N \widetilde{\tau}_{in} \widetilde{\tau}_n^b q_n^{-\theta \frac{1+\beta/\rho+\theta\mu}{1-2\theta} - \theta\mu} \phi_n^{-\theta \frac{1+\beta}{1-2\theta}} e_n,$$

$$\phi_i \left(\kappa_i^d\right)^{-1} = \sum_{n=1}^{N} \widetilde{\kappa}_{in} \widetilde{\kappa}_n^c q_n^{\beta/\rho} \phi_n^{\beta}.$$

We now have two sets of equations in two sets of endogenous variables q_i, ϕ_i. We now again apply Theorem 1 in Allen, Arkolakis and Li (2020) for this system of two equations:

$$\boldsymbol{\Lambda} = \begin{bmatrix} -\theta \frac{1+\theta\mu+\beta/\rho}{1-2\theta} & -\theta \frac{1+\beta}{1-2\theta} \\ 0 & 1 \end{bmatrix},$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} -\theta \frac{1+\theta\mu+\beta/\rho}{1-2\theta} - \theta\mu & -\theta \frac{1+\beta}{1-2\theta} \\ \beta/\rho & \beta \end{bmatrix},$$

$$\boldsymbol{A} \equiv \left|\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1}\right| = \left|\begin{bmatrix} \frac{\beta+\rho+\mu\rho-\mu\rho\theta}{\beta+\rho+\mu\rho\theta} & \frac{\mu\rho\theta(\beta+1)}{\beta+\rho+\mu\rho\theta} \\ \frac{\beta(2\theta-1)}{\theta(\beta+\rho+\mu\rho\theta)} & \beta - \frac{\beta(\beta+1)}{\beta+\rho+\mu\rho\theta} \end{bmatrix}\right|.$$

□

A sufficient condition for a unique equilibrium is again that the spectral radius of $\boldsymbol{A}$ is less than or equal to one ($\rho(\boldsymbol{A}) \leq 1$), which is satisfied for our baseline parameter values and symmetric trade and migration costs in our empirical application.

B.3 Dynamic Exact-hat Algebra (Proof of Proposition 2 in the Paper)

Given an initial allocation of the economy $\left(\{l_{i0}\}_{i=1}^N, \{k_{i0}\}_{i=1}^N, \{k_{i1}\}_{i=1}^N, \{S_{ni0}\}_{n,i=1}^N, \{D_{ni,-1}\}_{n,i=1}^N\right)$, and an anticipated sequence of changes in fundamentals, $\left\{\{\dot{z}_{it}\}_{i=1}^N, \left\{\dot{b}_{it}\right\}_{i=1}^N, \{\dot{\tau}_{ijt}\}_{i,j=1}^N, \{\dot{\kappa}_{ijt}\}_{i,j=1}^N\right\}_{t=1}^{\infty}$, the solution to the sequential equilibrium in time differences solves the following system of nonlinear equations:

$$\dot{D}_{igt+1} = \frac{\dot{u}_{gt+2} / \left(\dot{\kappa}_{git+1}\right)^{1/\rho}}{\sum_{m=1}^{N} D_{imt} \dot{u}_{mt+2} / \left(\dot{\kappa}_{mit+1}\right)^{1/\rho}},$$

$$\dot{u}_{it+1} = \left(\dot{b}_{it+1} \frac{\dot{w}_{it+1}}{\dot{p}_{it+1}}\right)^{\frac{\beta}{\rho}} \left(\sum_{g=1}^{N} D_{igt} \dot{u}_{gt+2} / \left(\dot{\kappa}_{git+1}\right)^{\frac{1}{\rho}}\right)^{\beta},$$

$$\dot{p}_{it+1} = \left(\sum_{m=1}^{N} S_{imt} \left(\dot{\tau}_{imt+1} \dot{w}_{mt+1} \left(\dot{l}_{mt+1} / \dot{k}_{mt+1}\right)^{1-\mu} / \dot{z}_{mt+1}\right)^{-\theta}\right)^{-1/\theta},$$

$$\ell_{gt+1} = \sum_{i=1}^{N} D_{igt} \ell_{it},$$

$$\dot{w}_{it+1} \dot{\ell}_{it+1} = \sum_{n=1}^{N} \frac{S_{nit+1} w_{nt} \ell_{nt}}{\sum_{k=1}^{N} S_{kit} w_{kt} \ell_{kt}} \dot{w}_{nt+1} \dot{\ell}_{nt+1},$$

$$\dot{S}_{nit+1} \equiv \frac{\left(\dot{\tau}_{nit+1}\dot{w}_{it+1}\left(\dot{l}_{it+1}/\dot{k}_{it+1}\right)^{1-\mu}/\dot{z}_{it+1}\right)^{-\theta}}{\sum_{k=1}^{N} S_{nkt}\left(\dot{\tau}_{nkt+1}\dot{w}_{kt+1}\left(\dot{l}_{kt+1}/\dot{k}_{kt+1}\right)^{1-\mu}/\dot{z}_{kt+1}\right)^{-\theta}},$$

$$\varsigma_{it+1} = \beta^{\psi} R_{it+1}^{\psi-1}\frac{\varsigma_{it}}{1-\varsigma_{it}},$$

$$k_{it+1} = (1-\varsigma_{it}) R_{it} k_{it},$$

$$(R_{it} - (1-\delta)) = \frac{\dot{p}_{it+1}\dot{k}_{it+1}}{\dot{w}_{it+1}\dot{l}_{it+1}}(R_{it+1} - (1-\delta)),$$

where we define $u_{it} \equiv \exp\left(\frac{\beta}{\rho}v_{it}^{w}\right)$,[1] and we use a dot above a variable to denote a time difference: $\dot{x}_{it+1} = x_{it+1}/x_{it}$. Note that the solution to this system of equations does not require information on the level of fundamentals, $\left\{\{z_{it}\}_{i=1}^{N}, \{b_{it}\}_{i=1}^{N}, \{\tau_{ijt}\}_{i,j=1}^{N}, \{\kappa_{ijt}\}_{i,j=1}^{N}\right\}_{t=0}^{\infty}$.

B.4 Linearization

We now derive our main linearization results for the comparative statics of the economy's steady-state and its transition path.

B.4.1 Comparative Statics

Expenditure Shares Totally differentiating expenditure shares (s_{nt}), we get:

$$\mathrm{d}\ln S_{nit} = \theta\left(\sum_{h=1}^{N} S_{nht}\,\mathrm{d}\ln p_{nht} - \mathrm{d}\ln p_{nit}\right). \tag{B.21}$$

Prices Totally differentiating the pricing rule from equation (2) in the paper, using equations (9) and (2) in the paper, we have:

$$\mathrm{d}\ln p_{nit} = \mathrm{d}\ln \tau_{nit} + \mathrm{d}\ln w_{it} - (1-\mu)\,\mathrm{d}\ln \chi_{it} - \mathrm{d}\ln z_{it}. \tag{B.22}$$

Price Indices Totally differentiating the price index in equation (4) in the paper, we have:

$$\mathrm{d}\ln p_{nt} = \sum_{m=1}^{N} S_{nmt}\,\mathrm{d}\ln p_{nmt}. \tag{B.23}$$

Real Income. Totally differentiating real income we have:

$$\mathrm{d}\ln\left(\frac{w_{it}}{p_{it}}\right) = \mathrm{d}\ln w_{it} - \sum_{m=1}^{N} S_{nmt}\left[\mathrm{d}\ln \tau_{nmt} + \mathrm{d}\ln w_{mt} - (1-\mu)\,\mathrm{d}\ln \chi_{mt} - \mathrm{d}\ln z_{mt}\right], \tag{B.24}$$

[1] Note that we express the set of equilibrium conditions in terms of transformed workers utility $u_{it} \equiv \exp\left(\frac{\beta}{\rho}v_{it}^{w}\right)$, whereas in Caliendo et al. (2018), the equilibrium conditions are expressed in terms of $\exp\left(v_{it}^{w}\right)$.

Migration Shares Totally differentiating the outmigration share in equation (16) in the paper, we get:

$$\mathrm{d}\ln D_{igt} = \frac{1}{\rho}\left[(\beta\mathbb{E}_t\,\mathrm{d}v_{gt+1} - \mathrm{d}\ln\kappa_{git}) - \sum_{h=1}^{N} D_{iht}\left(\beta\mathbb{E}_t\,\mathrm{d}v_{ht+1} - \mathrm{d}\ln\kappa_{hit}\right)\right]. \tag{B.25}$$

Goods Market Clearing Totally differentiating the goods market clearing condition from equation (12) in the paper, and using equations (B.21) and (B.22), we have:

$$\begin{bmatrix} \mathrm{d}\ln w_{it} \\ +\mathrm{d}\ln \ell_{it} \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{N} T_{int}\left(\mathrm{d}\ln w_{nt} + \mathrm{d}\ln \ell_{nt}\right) \\ +\theta\sum_{n=1}^{N}\sum_{m=1}^{N} T_{int}S_{nmt}\left(\mathrm{d}\ln\tau_{nmt} + \mathrm{d}\ln w_{mt} - (1-\mu)\,\mathrm{d}\ln\chi_{mt} - \mathrm{d}\ln z_{mt}\right) \\ -\theta\sum_{n=1}^{N} T_{int}\left(\mathrm{d}\ln\tau_{nit} + \mathrm{d}\ln w_{it} - (1-\mu)\,\mathrm{d}\ln\chi_{it} - \mathrm{d}\ln z_{it}\right) \end{bmatrix}. \tag{B.26}$$

$$T_{int} \equiv \frac{S_{nit}w_{nt}\ell_{nt}}{w_{it}\ell_{it}}.$$

Population Flow. Totally differentiating the population flow condition in equation (15) in the paper we have:

$$\mathrm{d}\ln\ell_{gt+1} = \sum_{i=1}^{N} E_{git}\left[\mathrm{d}\ln\ell_{it} + \frac{1}{\rho}\left(\beta\mathbb{E}_t\,\mathrm{d}v_{gt+1} - \mathrm{d}\ln\kappa_{gi} - \sum_{m=1}^{N} D_{imt}\left(\beta\mathbb{E}_t\,\mathrm{d}v_{mt+1} - \mathrm{d}\ln\kappa_{mit}\right)\right)\right]. \tag{B.27}$$

Value Function. Totally differentiating the value function, we have:

$$\mathrm{d}v_{it} = -\frac{1}{\theta}\,\mathrm{d}\ln S_{iit} + \mathrm{d}\ln w_{it} - \mathrm{d}\ln p_{iit} + \mathrm{d}\ln b_{it} + \beta\mathbb{E}_t\,\mathrm{d}v_{it+1} - \rho\,\mathrm{d}\ln D_{iit}.$$

Using the total derivatives of $\mathrm{d}\ln S_{iit}$ and $\mathrm{d}\ln D_{iit}$ in this expression for $\mathrm{d}v_{it}$ above, we have:

$$\mathrm{d}v_{it} = \begin{bmatrix} \mathrm{d}\ln w_{it} - \sum_{m=1}^{N} S_{imt}\,\mathrm{d}\ln p_{imt} \\ +\mathrm{d}\ln b_{it} + \sum_{m=1}^{N} D_{imt}\left(\beta\mathbb{E}_t\,\mathrm{d}v_{mt+1} - \mathrm{d}\ln\kappa_{mit}\right) \end{bmatrix},$$

where we have used $\mathrm{d}\ln\kappa_{iit} = 0$. Using the total derivative of the pricing rule (B.22), we can re-write this derivative of the value function as follows:

$$\mathrm{d}v_{it} = \begin{bmatrix} \mathrm{d}\ln w_{it} - \sum_{m=1}^{N} S_{imt}\left(\mathrm{d}\ln\tau_{nmt} + \mathrm{d}\ln w_{mt} - (1-\mu)\,\mathrm{d}\ln\chi_{mt} - \mathrm{d}\ln z_{mt}\right) \\ +\mathrm{d}\ln b_{it} + \sum_{m=1}^{N} D_{imt}\left(\beta\mathbb{E}_t\,\mathrm{d}v_{mt+1} - \mathrm{d}\ln\kappa_{mit}\right) \end{bmatrix}. \tag{B.28}$$

B.4.2 Steady-State Sufficient Statistics

Suppose that the economy starts from an initial steady-state with constant values of the endogenous variables: $k_{it+1} = k_{it} = k_i^*$, $\ell_{it+1} = \ell_{it} = \ell_i^*$, $w_{it+1}^* = w_{it}^* = w_i^*$ and $v_{it+1}^* = v_{it}^* = v_i^*$, where we use an asterisk to denote a steady-state value, and drop the time subscript for the remainder of this subsection, since we are concerned with steady-states. We consider small shocks to productivity ($\mathrm{d}\ln\boldsymbol{z}$) and amenities ($\mathrm{d}\ln\boldsymbol{b}$) in each location, holding constant the economy's aggregate labor endowment ($\mathrm{d}\ln\bar{\ell} = 0$), trade costs ($\mathrm{d}\ln\boldsymbol{\tau} = 0$) and commuting costs ($\mathrm{d}\ln\boldsymbol{\kappa} = 0$).

Capital Accumulation. From the capital accumulation equation (11) in the paper, the steady-state stock of capital solves:

$$(1-\beta(1-\delta))\,\chi_i^* = (1-\beta(1-\delta))\,\frac{k_i^*}{\ell_i^*} = \beta\frac{1-\mu}{\mu}\frac{w_i^*}{p_i^*}.$$

Totally differentiating, we have:

$$\mathrm{d}\ln\chi_i^* = \mathrm{d}\ln\left(\frac{w_i^*}{p_i^*}\right).$$

Using the total derivative of real income (B.24) above, this becomes:

$$\mathrm{d}\ln\chi_i^* = \mathrm{d}\ln w_i^* - \sum_{m=1}^{N} S_{im}^*\left[\mathrm{d}\ln w_m^* - (1-\mu)\,\mathrm{d}\ln\chi_m^* - \mathrm{d}\ln z_m\right],$$

where we have used and $\mathrm{d}\ln\tau_{nm} = 0$. This relationship has the matrix representation:

$$(\boldsymbol{I} - (1-\mu)\,\boldsymbol{S})\,\mathrm{d}\ln\boldsymbol{\chi}^* = (\boldsymbol{I}-\boldsymbol{S})\,\mathrm{d}\ln\boldsymbol{w}^* + \boldsymbol{S}\,\mathrm{d}\ln\boldsymbol{z}. \tag{B.29}$$

Goods Market Clearing. The total derivative of the goods market clearing condition (B.26) has the following matrix representation:

$$\mathrm{d}\ln\boldsymbol{w_t} + \mathrm{d}\ln\boldsymbol{\ell_t} = \boldsymbol{T}(\mathrm{d}\ln\boldsymbol{w_t} + \mathrm{d}\ln\boldsymbol{\ell_t}) + \theta(\boldsymbol{TS}-\boldsymbol{I})(\mathrm{d}\ln\boldsymbol{w_t} - (1-\mu)\,\mathrm{d}\ln\boldsymbol{\chi_t} - \mathrm{d}\ln\boldsymbol{z}),$$

where we have used $\mathrm{d}\ln\boldsymbol{\tau} = 0$. We can re-write this relationship as:

$$[\boldsymbol{I}-\boldsymbol{T}+\theta(\boldsymbol{I}-\boldsymbol{TS})]\,\mathrm{d}\ln\boldsymbol{w_t} = -(\boldsymbol{I}-\boldsymbol{T})\,\mathrm{d}\ln\boldsymbol{\ell_t} + \theta(\boldsymbol{I}-\boldsymbol{TS})(\mathrm{d}\ln\boldsymbol{z} + (1-\mu)\,\mathrm{d}\ln\boldsymbol{\chi_t}).$$

In steady-state we have:

$$[\boldsymbol{I}-\boldsymbol{T}+\theta(\boldsymbol{I}-\boldsymbol{TS})]\,\mathrm{d}\ln\boldsymbol{w}^* = [-(\boldsymbol{I}-\boldsymbol{T})\,\mathrm{d}\ln\boldsymbol{\ell}^* + \theta(\boldsymbol{I}-\boldsymbol{TS})(\mathrm{d}\ln\boldsymbol{z} + (1-\mu)\,\mathrm{d}\ln\boldsymbol{\chi}^*)]. \tag{B.30}$$

Population Flow. The total derivative of the population flow condition (B.27) has the following matrix representation:

$$\mathrm{d}\ln\boldsymbol{\ell_{t+1}} = \boldsymbol{E}\,\mathrm{d}\ln\boldsymbol{\ell_t} + \frac{\beta}{\rho}(\boldsymbol{I}-\boldsymbol{ED})\,\mathrm{d}\boldsymbol{v_{t+1}}.$$

In steady-state, we have:

$$\mathrm{d}\ln\boldsymbol{\ell}^* = \boldsymbol{E}\,\mathrm{d}\ln\boldsymbol{\ell}^* + \frac{\beta}{\rho}(\boldsymbol{I}-\boldsymbol{ED})\,\mathrm{d}\boldsymbol{v}^*. \tag{B.31}$$

Value function. The total derivative of the value function (B.28) has the following matrix representation:

$$\mathrm{d}\boldsymbol{v_t} = (\boldsymbol{I}-\boldsymbol{S})\,\mathrm{d}\ln\boldsymbol{w_t} + \boldsymbol{S}(\mathrm{d}\ln\boldsymbol{z} + (1-\mu)\,\mathrm{d}\ln\boldsymbol{\chi_t}) + \mathrm{d}\ln\boldsymbol{b} + \beta\boldsymbol{D}\,\mathrm{d}\boldsymbol{v_{t+1}},$$

where we have used $\mathrm{d}\ln\boldsymbol{\tau} = \mathrm{d}\ln\boldsymbol{\kappa} = 0$. In steady-state, we have:

$$\mathrm{d}\boldsymbol{v}^* = (\boldsymbol{I}-\boldsymbol{S})\,\mathrm{d}\ln\boldsymbol{w}^* + \boldsymbol{S}(\mathrm{d}\ln\boldsymbol{z} + (1-\mu)\,\mathrm{d}\ln\boldsymbol{\chi}^*) + \mathrm{d}\ln\boldsymbol{b} + \beta\boldsymbol{D}\,\mathrm{d}\boldsymbol{v}^*. \tag{B.32}$$

System of Steady-State Equations. Collecting together the system of steady-state equations, we have:

$$\mathrm{d}\ln\boldsymbol{\chi}^{*} = \left(\boldsymbol{I} - (1-\mu)\,\boldsymbol{S}\right)^{-1}\left((\boldsymbol{I}-\boldsymbol{S})\,\mathrm{d}\ln\boldsymbol{w}^{*} + \boldsymbol{S}\,\mathrm{d}\ln\boldsymbol{z}\right). \tag{B.33}$$

$$\mathrm{d}\ln\boldsymbol{w}^{*} = \left(\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\right)^{-1}\left(-\left(\boldsymbol{I}-\boldsymbol{T}\right)\mathrm{d}\ln\boldsymbol{\ell}^{*} + \left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\left(\mathrm{d}\ln\boldsymbol{z} + (1-\mu)\,\mathrm{d}\ln\boldsymbol{\chi}^{*}\right)\right). \tag{B.34}$$

$$\mathrm{d}\ln\boldsymbol{\ell}^{*} = \frac{\beta}{\rho}\left(\boldsymbol{I}-\boldsymbol{E}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{ED}\right)\mathrm{d}\boldsymbol{v}^{*}. \tag{B.35}$$

$$\mathrm{d}\boldsymbol{v}^{*} = \left(\boldsymbol{I}-\beta\boldsymbol{D}\right)^{-1}\left\{\mathrm{d}\ln\boldsymbol{w}^{*} - \boldsymbol{S}\left(\mathrm{d}\ln\boldsymbol{w}^{*} - \mathrm{d}\ln\boldsymbol{z} - (1-\mu)\,\mathrm{d}\ln\boldsymbol{\chi}^{*}\right) + \mathrm{d}\ln\boldsymbol{b}\right\}. \tag{B.36}$$

B.4.3 Steady-State Elasticities

We now use equation (B.33) to substitute for $\mathrm{d}\ln\boldsymbol{\chi}^{*}$ in the value function (B.36) to obtain:

$$\begin{aligned}
\mathrm{d}\boldsymbol{v}^{*} &= \left(\boldsymbol{I}-\beta\boldsymbol{D}\right)^{-1}\left\{\mathrm{d}\ln\boldsymbol{w}^{*} - \boldsymbol{S}\left(\mathrm{d}\ln\boldsymbol{w}^{*} - \mathrm{d}\ln\boldsymbol{z} - (1-\mu)\,\mathrm{d}\ln\boldsymbol{\chi}^{*}\right) + \mathrm{d}\ln\boldsymbol{b}\right\}, \\
&= \left(\boldsymbol{I}-\beta\boldsymbol{D}\right)^{-1}\left\{(\boldsymbol{I}-\boldsymbol{S})\,\mathrm{d}\ln\boldsymbol{w}^{*} + \boldsymbol{S}\,\mathrm{d}\ln\boldsymbol{z} + \boldsymbol{S}\,(1-\mu)\,\mathrm{d}\ln\boldsymbol{\chi}^{*} + \mathrm{d}\ln\boldsymbol{b}\right\}, \\
&= \left(\boldsymbol{I}-\beta\boldsymbol{D}\right)^{-1}\left(\boldsymbol{I} + \boldsymbol{S}\,(1-\mu)\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\right)\left[(\boldsymbol{I}-\boldsymbol{S})\,\mathrm{d}\ln\boldsymbol{w}^{*} + \boldsymbol{S}\,\mathrm{d}\ln\boldsymbol{z} + \mathrm{d}\ln\boldsymbol{b}\right], \\
&= \left(\boldsymbol{I}-\beta\boldsymbol{D}\right)^{-1}\left[\mathrm{d}\ln\boldsymbol{b} + \left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\left((\boldsymbol{I}-\boldsymbol{S})\,\mathrm{d}\ln\boldsymbol{w}^{*} + \boldsymbol{S}\,\mathrm{d}\ln\boldsymbol{z}\right)\right].
\end{aligned} \tag{B.37}$$

We now use equation (B.33) to substitute for $\mathrm{d}\ln\boldsymbol{\chi}^{*}$ in the wage equation (B.34) to obtain:

$$\left(\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\right)\mathrm{d}\ln\boldsymbol{w}^{*} = -\left(\boldsymbol{I}-\boldsymbol{T}\right)\mathrm{d}\ln\boldsymbol{\ell}^{*} + \left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\left(\mathrm{d}\ln\boldsymbol{z} + (1-\mu)\,\mathrm{d}\ln\boldsymbol{\chi}^{*}\right),$$

$$\left(\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\right)\mathrm{d}\ln\boldsymbol{w}^{*} = \begin{bmatrix} -\left(\boldsymbol{I}-\boldsymbol{T}\right)\mathrm{d}\ln\boldsymbol{\ell}^{*} + \left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\,\mathrm{d}\ln\boldsymbol{z} \\ +\left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\,(1-\mu)\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\left((\boldsymbol{I}-\boldsymbol{S})\,\mathrm{d}\ln\boldsymbol{w}^{*} + \boldsymbol{S}\,\mathrm{d}\ln\boldsymbol{z}\right) \end{bmatrix},$$

$$\left(\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\right)\mathrm{d}\ln\boldsymbol{w}^{*} = \begin{bmatrix} -\left(\boldsymbol{I}-\boldsymbol{T}\right)\mathrm{d}\ln\boldsymbol{\ell}^{*} + \left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\left(\boldsymbol{I} + \left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}(1-\mu)\,\boldsymbol{S}\right)\mathrm{d}\ln\boldsymbol{z} \\ +\left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\,(1-\mu)\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{S}\right)\mathrm{d}\ln\boldsymbol{w}^{*} \end{bmatrix},$$

$$\begin{aligned}
\left(\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\right)\mathrm{d}\ln\boldsymbol{w}^{*} &= -\left(\boldsymbol{I}-\boldsymbol{T}\right)\mathrm{d}\ln\boldsymbol{\ell}^{*} + \left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\left(\mathrm{d}\ln\boldsymbol{z} + (1-\mu)\,\mathrm{d}\boldsymbol{\chi}^{*}\right) \\
&= -\left(\boldsymbol{I}-\boldsymbol{T}\right)\mathrm{d}\ln\boldsymbol{\ell}^{*} + \left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\,\mathrm{d}\ln\boldsymbol{z} \\
&\quad +\left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\,(1-\mu)\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\left((\boldsymbol{I}-\boldsymbol{S})\,\mathrm{d}\ln\boldsymbol{w}^{*} + \boldsymbol{S}\,\mathrm{d}\ln\boldsymbol{z}\right) \\
&= -\left(\boldsymbol{I}-\boldsymbol{T}\right)\mathrm{d}\ln\boldsymbol{\ell}^{*} + \left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\left(\boldsymbol{I} + \left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}(1-\mu)\,\boldsymbol{S}\right)\mathrm{d}\ln\boldsymbol{z} \\
&\quad +\left(\boldsymbol{I}-\boldsymbol{TS}\right)\theta\,(1-\mu)\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{S}\right)\mathrm{d}\ln\boldsymbol{w}^{*}
\end{aligned}$$

$$\begin{aligned}
&\left(\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\left(\boldsymbol{I}-(1-\mu)\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{S}\right)\right)\right)\mathrm{d}\ln\boldsymbol{w}^{*} \\
&\qquad = -\left(\boldsymbol{I}-\boldsymbol{T}\right)\mathrm{d}\ln\boldsymbol{\ell}^{*} + \theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\mathrm{d}\ln\boldsymbol{z},
\end{aligned}$$

$$\begin{aligned}
&\left(\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\left(\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1} - (1-\mu)\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\right)\right)\mathrm{d}\ln\boldsymbol{w}^{*} \\
&\qquad = -\left(\boldsymbol{I}-\boldsymbol{T}\right)\mathrm{d}\ln\boldsymbol{\ell}^{*} + \theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\mathrm{d}\ln\boldsymbol{z},
\end{aligned}$$

$$\begin{aligned}
&\left(\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\mu\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\right)\mathrm{d}\ln\boldsymbol{w}^{*} \\
&\qquad = -\left(\boldsymbol{I}-\boldsymbol{T}\right)\mathrm{d}\ln\boldsymbol{\ell}^{*} + \theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\left(\boldsymbol{I}-(1-\mu)\,\boldsymbol{S}\right)^{-1}\mathrm{d}\ln\boldsymbol{z}.
\end{aligned} \tag{B.38}$$

Collecting together the capital accumulation equation (B.33), the population equation (B.35), the value function (B.37) and the wage equation (B.38), we have:

$$\mathrm{d}\boldsymbol{v}^{*} = (\boldsymbol{I} - \beta \boldsymbol{D})^{-1} \left[\mathrm{d}\ln \boldsymbol{b} + (\boldsymbol{I} - (1-\mu)\,\boldsymbol{S})^{-1} \left((\boldsymbol{I} - \boldsymbol{S})\ \mathrm{d}\ln \boldsymbol{w}^{*} + \boldsymbol{S}\,\mathrm{d}\ln \boldsymbol{z} \right) \right], \tag{B.39}$$

$$\mathrm{d}\ln \boldsymbol{w}^{*} = \left[\boldsymbol{I} - \boldsymbol{T} + \theta\,(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{S})\,\mu\,(\boldsymbol{I} - (1-\mu)\,\boldsymbol{S})^{-1} \right]^{-1} \left[\begin{array}{c} -\,(\boldsymbol{I} - \boldsymbol{T})\ \mathrm{d}\ln \boldsymbol{\ell}^{*} \\ +\theta\,(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{S})\,(\boldsymbol{I} - (1-\mu)\,\boldsymbol{S})^{-1}\ \mathrm{d}\ln \boldsymbol{z} \end{array} \right], \tag{B.40}$$

$$\mathrm{d}\ln \boldsymbol{\chi}^{*} = (\boldsymbol{I} - (1-\mu)\,\boldsymbol{S})^{-1} \left[(\boldsymbol{I} - \boldsymbol{S})\ \mathrm{d}\ln \boldsymbol{w}^{*} + \boldsymbol{S}\,\mathrm{d}\ln \boldsymbol{z} \right], \tag{B.41}$$

$$\mathrm{d}\ln \boldsymbol{\ell}^{*} \quad = \quad \frac{\beta}{\rho}\,(\boldsymbol{I} - \boldsymbol{E})^{-1}\,(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D})\ \mathrm{d}\boldsymbol{v}^{*}. \tag{B.42}$$

We now show that we can further simplify this system of equations. We begin by defining the following composite matrices:

$$\begin{aligned} \boldsymbol{G} &\equiv (\boldsymbol{I} - \boldsymbol{E})^{-1}\,(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D})\,(\boldsymbol{I} - \beta \boldsymbol{D})^{-1}, \\ \boldsymbol{O} &\equiv (\boldsymbol{I} - (1-\mu)\,\boldsymbol{S})^{-1}, \\ \boldsymbol{M} &\equiv (\boldsymbol{T}\boldsymbol{S} - \boldsymbol{I}). \end{aligned} \tag{B.43}$$

which implies the following relationships:

$$\boldsymbol{I} + (1-\mu)\,\boldsymbol{S}\boldsymbol{O} = \boldsymbol{O},$$

$$\boldsymbol{I} - (1-\mu)\,\boldsymbol{O}\,(\boldsymbol{I} - \boldsymbol{S}) = \boldsymbol{I} + (1-\mu)\,\boldsymbol{O}\boldsymbol{S} - (1-\mu)\,\boldsymbol{O} = \mu \boldsymbol{O}.$$

Using these definitions and relationships, we can re-write the wage equation (B.40) as:

$$(\boldsymbol{I} - \boldsymbol{T} - \theta \boldsymbol{M})\ \mathrm{d}\ln \boldsymbol{w}^{*} = -\,(\boldsymbol{I} - \boldsymbol{T})\ \mathrm{d}\ln \boldsymbol{\ell}^{*} - \theta \boldsymbol{M} \left[\mathrm{d}\ln \boldsymbol{z} + (1-\mu)\,\boldsymbol{O}\,(\boldsymbol{I} - \boldsymbol{S})\ \mathrm{d}\ln \boldsymbol{w}^{*} + (1-\mu)\,\boldsymbol{O}\boldsymbol{S}\,\mathrm{d}\ln \boldsymbol{z} \right],$$

$$\left[\boldsymbol{I} - \boldsymbol{T} - \theta \boldsymbol{M}\,(\boldsymbol{I} - (1-\mu)\,\boldsymbol{O}\,(\boldsymbol{I} - \boldsymbol{S})) \right] \mathrm{d}\ln \boldsymbol{w}^{*} = -\,(\boldsymbol{I} - \boldsymbol{T})\ \mathrm{d}\ln \boldsymbol{\ell}^{*} - \theta \boldsymbol{M}\boldsymbol{O}\,\mathrm{d}\ln \boldsymbol{z},$$

$$\mathrm{d}\ln \boldsymbol{w}^{*} = \left[\boldsymbol{I} - \boldsymbol{T} + \theta\,(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{S})\,\mu\,(\boldsymbol{I} - (1-\lambda)\,\boldsymbol{S})^{-1} \right]^{-1} \left[\begin{array}{c} -\,(\boldsymbol{I} - \boldsymbol{T})\ \mathrm{d}\ln \boldsymbol{\ell}^{*} \\ +\theta\,(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{S})\,(\boldsymbol{I} - (1-\mu)\,\boldsymbol{S})^{-1}\ \mathrm{d}\ln \boldsymbol{z} \end{array} \right].$$

$$\mathrm{d}\ln \boldsymbol{w}^{*} = \left[\boldsymbol{I} - \boldsymbol{T} - \theta \mu \boldsymbol{M}\boldsymbol{O} \right]^{-1} \left[-\,(\boldsymbol{I} - \boldsymbol{T})\ \mathrm{d}\ln \boldsymbol{\ell}^{*} - \theta \boldsymbol{M}\boldsymbol{O}\,\mathrm{d}\ln \boldsymbol{z} \right]. \tag{B.44}$$

Using the value function (B.39), we can re-write the employment equation (B.42) as:

$$\mathrm{d}\ln \boldsymbol{\ell}^{*} \quad = \quad \frac{\beta}{\rho}\,(\boldsymbol{I} - \boldsymbol{E})^{-1}\,(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D})\,(\boldsymbol{I} - \beta \boldsymbol{D})^{-1} \left[\mathrm{d}\ln \boldsymbol{b} + (\boldsymbol{I} - (1-\mu)\,\boldsymbol{S})^{-1} \left[(\boldsymbol{I} - \boldsymbol{S})\ \mathrm{d}\ln \boldsymbol{w}^{*} + \boldsymbol{S}\,\mathrm{d}\ln \boldsymbol{z} \right] \right].$$

Using the capital accumulation equation (B.41) and our definitions (B.43), we can further re-write this employment equation as:

$$\mathrm{d}\ln \boldsymbol{\ell}^{*} = \frac{\beta}{\rho} \boldsymbol{G} \left[\mathrm{d}\ln \boldsymbol{\chi}^{*} + \ \mathrm{d}\ln \boldsymbol{b} \right]. \tag{B.45}$$

Using the definitions (B.43), we can re-write the capital accumulation equation (B.41) as follows:

$$\mathrm{d}\ln \boldsymbol{\chi}^{*} = \boldsymbol{O} \left[(\boldsymbol{I} - \boldsymbol{S})\,(\boldsymbol{I} - \boldsymbol{T} - \theta \boldsymbol{M} \mu \boldsymbol{O})^{-1} \left(-\,(\boldsymbol{I} - \boldsymbol{T})\,\frac{\beta}{\rho} \boldsymbol{G} \left[\mathrm{d}\ln \boldsymbol{\chi}^{*} + \ \mathrm{d}\ln \boldsymbol{b} \right] - \theta \boldsymbol{M}\boldsymbol{O}\,\mathrm{d}\ln \boldsymbol{z} \right) + \boldsymbol{S}\,\mathrm{d}\ln \boldsymbol{z} \right],$$

$$\left[\boldsymbol{I}+\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\right)\frac{\beta}{\rho}\boldsymbol{G}\right]\mathrm{d}\ln\boldsymbol{\chi}^{*}$$
$$=\begin{bmatrix}\left[\boldsymbol{O}\boldsymbol{S}-\theta\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\boldsymbol{M}\boldsymbol{O}\right]\mathrm{d}\ln\boldsymbol{z}\\ -\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\right)\frac{\beta}{\rho}\boldsymbol{G}\,\mathrm{d}\ln\boldsymbol{b}\end{bmatrix}. \tag{B.46}$$

We thus obtain the following representation of the steady-state elasticity of the endogenous variables in each location with respect to a shock in any location (omitted from the paper for brevity).

Proposition A.1. *The general equilibrium response of the steady-state distribution of economic activity* $\{w_i^*, v_i^*, \ell_i^*, k_i^*\}$ *to small productivity* ($\mathrm{d}\ln\boldsymbol{z}$) *and amenity shocks* ($\mathrm{d}\ln\boldsymbol{b}$) *is uniquely determined by the matrices* $\{\boldsymbol{L}^{z*}, \boldsymbol{K}^{z*}, \boldsymbol{W}^{z*}, \boldsymbol{V}^{z*}, \boldsymbol{L}^{b*}, \boldsymbol{K}^{b*}, \boldsymbol{W}^{b*}, \boldsymbol{V}^{b*}\}$*, which depend solely on the structural parameters* $\{\theta, \beta, \rho, \mu, \delta\}$ *and the observed matrices of expenditure shares* ($\boldsymbol{S}$)*, income shares* ($\boldsymbol{T}$)*, outmigration shares* ($\boldsymbol{D}$) *and inmigration shares* ($\boldsymbol{E}$)*:*

$$\begin{bmatrix}\mathrm{d}\ln\boldsymbol{\ell}^{*}\\ \mathrm{d}\ln\boldsymbol{k}^{*}\\ \mathrm{d}\ln\boldsymbol{w}^{*}\\ \mathrm{d}\ln\boldsymbol{v}^{*}\end{bmatrix}=\begin{bmatrix}\boldsymbol{L}^{z*}\\ \boldsymbol{K}^{z*}\\ \boldsymbol{W}^{z*}\\ \boldsymbol{V}^{z*}\end{bmatrix}\mathrm{d}\ln\boldsymbol{z}+\begin{bmatrix}\boldsymbol{L}^{b*}\\ \boldsymbol{K}^{b*}\\ \boldsymbol{W}^{b*}\\ \boldsymbol{V}^{b*}\end{bmatrix}\mathrm{d}\ln\boldsymbol{b}, \tag{B.47}$$

Proof. The proposition follows from the value function (B.39), wage equation (B.44), population equation (B.45), and capital-labor equation (B.46). In particular, from the population equation (B.45) and the capital-labor equation (B.46), we have:

$$\boldsymbol{L}^{z*}\equiv\frac{\beta}{\rho}\boldsymbol{G}\left[\boldsymbol{I}+\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\right)\frac{\beta}{\rho}\boldsymbol{G}\right]^{-1}$$
$$\times\left(\boldsymbol{O}\boldsymbol{S}-\theta\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\boldsymbol{M}\boldsymbol{O}\right),$$

$$\boldsymbol{L}^{b*}\equiv\begin{array}{c}\frac{\beta}{\rho}\boldsymbol{G}-\frac{\beta}{\rho}\boldsymbol{G}\left[\boldsymbol{I}+\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\right)\frac{\beta}{\rho}\boldsymbol{G}\right]^{-1}\\ \times\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\right)\frac{\beta}{\rho}\boldsymbol{G}\end{array}.$$

From the capital-labor equation (B.46) and population equation (B.45), we have:

$$\boldsymbol{K}^{\boldsymbol{z*}}\equiv\left[\boldsymbol{I}+\frac{\beta}{\rho}\boldsymbol{G}\right]\left[\boldsymbol{I}+\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\right)\frac{\beta}{\rho}\boldsymbol{G}\right]^{-1}$$
$$\times\left(\boldsymbol{O}\boldsymbol{S}-\theta\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\boldsymbol{M}\boldsymbol{O}\right),$$

$$\boldsymbol{K}^{b*}\equiv\begin{array}{c}\boldsymbol{L}^{b*}-\left[\boldsymbol{I}+\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\right)\frac{\beta}{\rho}\boldsymbol{G}\right]^{-1}\\ \times\boldsymbol{O}\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\right)\frac{\beta}{\rho}\boldsymbol{G}\end{array}.$$

From the wage equation (B.44) and population equation (B.45), we have:

$$\boldsymbol{W}^{z*}\equiv\left[\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right]^{-1}\left[-\left(\boldsymbol{I}-\boldsymbol{T}\right)L^{z*}-\theta\boldsymbol{M}\boldsymbol{O}\right],$$

$$\boldsymbol{W}^{b*}\equiv\left[\boldsymbol{I}-\boldsymbol{T}-\theta\boldsymbol{M}\mu\boldsymbol{O}\right]^{-1}\left[-\left(\boldsymbol{I}-\boldsymbol{T}\right)\boldsymbol{L}^{b*}\right].$$

From the value function (B.39) and the wage equation (B.44), we have:

$$\boldsymbol{V}^{z*}\equiv\left(\boldsymbol{I}-\beta\boldsymbol{D}\right)^{-1}\left(\boldsymbol{I}-\left(1-\mu\right)\boldsymbol{S}\right)^{-1}\left[\left(\boldsymbol{I}-\boldsymbol{S}\right)\boldsymbol{W}^{z*}+\boldsymbol{S}\right],$$

$$\boldsymbol{V}^{b*} \equiv (\boldsymbol{I}-\beta\boldsymbol{D})^{-1} + (\boldsymbol{I}-\beta\boldsymbol{D})^{-1}(\boldsymbol{I}-(1-\mu)\boldsymbol{S})^{-1}(\boldsymbol{I}-\boldsymbol{S})\boldsymbol{W}^{b*}.$$

Note that the matrices of steady-state elasticities $\{\boldsymbol{L}^{z*}, \boldsymbol{K}^{z*}, \boldsymbol{W}^{z*}, \boldsymbol{V}^{z*}, \boldsymbol{L}^{b*}, \boldsymbol{K}^{b*}, \boldsymbol{W}^{b*}, \boldsymbol{V}^{b*}\}$ are linear combinations of the structural parameters $\{\theta, \beta, \rho, \mu, \delta\}$ and the observed matrices of expenditure shares ($\boldsymbol{S}$), income shares ($\boldsymbol{T}$), outmigration shares ($\boldsymbol{D}$) and inmigration shares ($\boldsymbol{E}$). Therefore, the steady-state changes in the endogenous variables $\{w_i^*, v_i^*, \ell_i^*, k_i^*\}$ in response to productivity and amenity shocks are unique (up to a choice of numeraire for wages). □

As the expenditure shares ($\boldsymbol{S}$) and income shares ($\boldsymbol{T}$) are homogeneous of degree zero in factor prices, we require a numeraire in order for solve for changes in wages. We choose the total income of all locations as our numeraire ($\sum_{i=1}^N w_i^*\ell_i^* = \sum_{i=1}^N q_i^* = \bar{q} = 1$), which implies $\boldsymbol{q^*}\,\mathrm{d}\ln\boldsymbol{q^*} = \sum_{i=1}^N q_i^*\,\mathrm{d}\ln q_i^* = \sum_{i=1}^N q_i^*\frac{\mathrm{d}q_i^*}{q_i^*} = \sum_{i=1}^N \mathrm{d}q_i^* = 0$, where $\boldsymbol{q^*}$ is a row vector of the steady-state income of each location. Similarly, the outmigration shares ($\boldsymbol{D}$) and inmigration shares ($\boldsymbol{E}$) are homogeneous of degree zero in the total population of all locations, which requires a choice of units to solve for population levels. We solve for population shares, imposing the requirement that the population shares sum to one: $\sum_{i=1}^N \ell_i = \bar{\ell} = 1$, which implies $\boldsymbol{\ell^*}\,\mathrm{d}\ln\boldsymbol{\ell^*} = \sum_{i=1}^N \ell_i^*\,\mathrm{d}\ln\ell_i^* = 0$, where $\boldsymbol{\ell^*}$ is a row vector of the steady-state population of each location.

B.4.4 Derivations of the Linearized Equilibrium Conditions

We suppose that we observe the initial values of the state variables ($\boldsymbol{\ell}_0$, $\boldsymbol{k}_0$) and the trade and migration share matrices ($\boldsymbol{S}$, $\boldsymbol{T}$, $\boldsymbol{D}$, $\boldsymbol{E}$) at time $t = 0$, which need not correspond to a steady-state of the model. Throughout the following, we use a tilde above a variable to denote a log deviation from the steady-state implied by the initial fundamentals (the "initial steady-state"), such that $\widetilde{\chi}_{it+1} = \ln\chi_{it+1} - \ln\chi_i^*$, for all variables except for the worker value function v_{it}; with a slight abuse of notation we use $\widetilde{v}_{it} \equiv v_{it} - v_i^*$ to denote the deviation in levels for the worker value function. We consider stochastic shocks to productivity ($\mathrm{d}\ln\boldsymbol{z_t}$) and amenities ($\mathrm{d}\ln\boldsymbol{b_t}$) in each location, holding constant the economy's aggregate labor endowment ($\mathrm{d}\ln\bar{\ell} = 0$), trade costs ($\mathrm{d}\ln\boldsymbol{\tau_t} = 0$) and commuting costs ($\mathrm{d}\ln\boldsymbol{\kappa_t} = 0$).

Population Flow (equation (20) in the Paper). The total derivative of the population flow condition (B.27) relative to the initial steady-state has the following matrix representation:

$$\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+1}} = \boldsymbol{E}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}} + \frac{\beta}{\rho}(\boldsymbol{I}-\boldsymbol{ED})\,\mathbb{E}_t\widetilde{\boldsymbol{v}}_{\boldsymbol{t+1}}. \tag{B.48}$$

Capital Accumulation (equation (18) in the Paper). Note that in a deterministic steady-state, $\beta R^* = 1$, and $\varsigma^{*-1} = 1 + \beta^{\psi}(R^*)^{\psi-1}\varsigma^{*-1}$, thereby implying $\varsigma^* = 1-\beta$. We now linearize (B.4) relative to the deterministic steady-state (let $\widetilde{x}_t \equiv \ln x_t - \ln x^*$),

$$\begin{aligned}
\widetilde{\varsigma}_t &\approx -\mathbb{E}_t \ln\frac{1+\beta^{\psi}(R^*)^{\psi-1}(R_{t+1}/R^*)^{\psi-1}\varsigma_{t+1}^{-1}}{1+\beta\varsigma_{t+1}^{-1}} \\
&= -\mathbb{E}_t \ln\frac{1+\frac{\beta}{1-\beta}(R_{t+1}/R^*)^{\psi-1}(\varsigma_{t+1}/\varsigma^*)^{-1}}{1+\beta/(1-\beta)} \\
&\approx -\mathbb{E}_t \ln\left(1+\beta\left((R_{t+1}/R^*)^{\psi-1}-1\right)+\beta\left((\varsigma_{t+1}/\varsigma^*)^{-1}-1\right)\right) \\
&= \beta\mathbb{E}_t\widetilde{\varsigma}_{t+1} - (\psi-1)\beta\mathbb{E}_t\widetilde{R}_{t+1}
\end{aligned}$$

$$\widetilde{c}_t = \widetilde{k}_t + \widetilde{R}_t + \widetilde{\varsigma}_t = \widetilde{k}_t + \widetilde{R}_t - (\psi - 1)\, \mathbb{E}_t \sum_{s=1}^{\infty} \beta^s \widetilde{R}_{t+s}.$$

$$\begin{aligned} \widetilde{\boldsymbol{k}}_{t+1} &= \widetilde{\boldsymbol{k}}_t + \widetilde{\boldsymbol{R}}_t + \widetilde{(1-\varsigma_t)} = \widetilde{\boldsymbol{k}}_t + \widetilde{\boldsymbol{R}}_t - \tfrac{1-\beta}{\beta}\widetilde{\varsigma}_t, \\ &= \widetilde{\boldsymbol{k}}_t + \widetilde{\boldsymbol{R}}_t + \tfrac{1-\beta}{\beta}(\psi-1)\, \mathbb{E}_t \textstyle\sum_{s=1}^{\infty} \beta^s \widetilde{\boldsymbol{R}}_{t+s}. \end{aligned} \tag{B.49}$$

We now derive $\widetilde{\boldsymbol{R}}_{t+s}$. Note $R_{it} = 1-\delta+r_{it}/p_{it}$, and we know in steady-state $\beta\,(1-\delta+r^*/p^*) = 1$ and $r^*/p^* = \beta^{-1}+\delta-1$. Thus

$$\begin{aligned} \widetilde{R}_{it} &= \ln\left(\tfrac{1-\delta+r_{it}/p_{it}}{1-\delta+r^*/p^*}\right), \\ &= \ln\left(\beta\left(1-\delta+r^*\left(r_{it}/r^*-1+1\right)\left(p^{*-1}\left(p^*/p_{it}-1+1\right)\right)\right)\right), \\ &\approx \ln\left(1+\beta r^*/p^*\left(\left(r_{it}/r^*-1\right)+\left(p^{*-1}\left(p^*/p_{it}-1\right)\right)\right)\right) = \beta r^*/p^*\left(\widetilde{r}_{it}-\widetilde{p}_{it}\right), \\ &= \left(1-\beta\left(1-\delta\right)\right)\left(\widetilde{r}_{it}-\widetilde{p}_{it}\right) = \left(1-\beta\left(1-\delta\right)\right)\left(\widetilde{w}_{it}-\widetilde{p}_{it}-\widetilde{\chi}_{it}\right). \end{aligned} \tag{B.50}$$

where we have used $\chi_{it} \equiv k_{it}/\ell_{it}$ and $r_{it} = \frac{1-\mu}{\mu} w_{it}\ell_{it}/k_{it}$. Note (B.49) and (B.50) imply:

$$\widetilde{\boldsymbol{k}}_{t+1} = \widetilde{\boldsymbol{k}}_t + (1-\beta(1-\delta))\left[(\widetilde{\boldsymbol{w}}_t - \widetilde{\boldsymbol{p}}_t - \widetilde{\boldsymbol{\chi}}_t) + \frac{1-\beta}{\beta}(\psi-1)\,\mathbb{E}_t\sum_{s=1}^{\infty}\beta^s\left(\widetilde{\boldsymbol{w}}_{t+s} - \widetilde{\boldsymbol{p}}_{t+s} - \widetilde{\boldsymbol{\chi}}_{t+s}\right)\right]. \tag{B.51}$$

Value Function (equation (21) in the Paper). The total derivative of the value function (B.28) relative to the initial steady-state has the following matrix representation:

$$\widetilde{\boldsymbol{v}}_t = \widetilde{\boldsymbol{w}}_t - \widetilde{\boldsymbol{p}}_t + \widetilde{\boldsymbol{b}}_t + \beta \boldsymbol{D}\mathbb{E}_t\widetilde{\boldsymbol{v}}_{t+1}. \tag{B.52}$$

Goods Market Clearing (equation (19) in the Paper). The total derivative of the goods market clearing condition (B.26) relative to the initial steady-state has the following matrix representation:

$$\widetilde{\boldsymbol{w}}_t + \widetilde{\boldsymbol{\ell}}_t = \boldsymbol{T}\left(\widetilde{\boldsymbol{w}}_t + \widetilde{\boldsymbol{\ell}}_t\right) + \theta\left(\boldsymbol{TS}-\boldsymbol{I}\right)\left(\widetilde{\boldsymbol{w}}_t - (1-\mu)\,\widetilde{\boldsymbol{\chi}}_t - \widetilde{\boldsymbol{z}}_t\right),$$

where we have used $\mathrm{d}\ln\boldsymbol{\tau} = 0$. We can re-write this relationship as:

$$\left[\boldsymbol{I} - \boldsymbol{T} + \theta\left(\boldsymbol{I} - \boldsymbol{TS}\right)\right]\widetilde{\boldsymbol{w}}_t = \left[-\left(\boldsymbol{I}-\boldsymbol{T}\right)\widetilde{\boldsymbol{\ell}}_t + \theta\left(\boldsymbol{I}-\boldsymbol{TS}\right)\left(\widetilde{\boldsymbol{z}}_t + (1-\mu)\,\widetilde{\boldsymbol{\chi}}_t\right)\right]. \tag{B.53}$$

Price Index (equation (17) in the Paper). We obtain the equation (17) by substituting (B.22) into (B.23) and stack into a matrix to obtain:

$$\widetilde{\boldsymbol{p}}_t = \boldsymbol{S}\left(\widetilde{\boldsymbol{w}}_t - \widetilde{\boldsymbol{z}}_t - (1-\mu)\left(\widetilde{\boldsymbol{k}}_t - \widetilde{\boldsymbol{\ell}}_t\right)\right) \tag{B.54}$$

System of Equations for Transition Dynamics Relative to the Initial Steady-State. Collecting together capital dynamics (B.51), goods market clearing (B.53), the population flow condition (B.48), the value function (B.52), and the price index equation (B.54), the system of equations for the transition dynamics relative to the initial steady-state is:

$$\begin{aligned} \widetilde{\boldsymbol{k}}_{t+1} &= \widetilde{\boldsymbol{k}}_t + (1-\beta(1-\delta))\left(\widetilde{\boldsymbol{w}}_t - \widetilde{\boldsymbol{p}}_t - \widetilde{\boldsymbol{k}}_t + \widetilde{\boldsymbol{\ell}}_t\right) \\ &\quad + (1-\beta(1-\delta))\,\frac{1-\beta}{\beta}(\psi-1)\,\mathbb{E}_t\sum_{s=1}^{\infty}\beta^s\left(\widetilde{\boldsymbol{w}}_{t+s} - \widetilde{\boldsymbol{p}}_{t+s} - \widetilde{\boldsymbol{k}}_{t+s} + \widetilde{\boldsymbol{\ell}}_{t+s}\right) \end{aligned} \tag{B.55}$$

$$\widetilde{\boldsymbol{w}}_{\boldsymbol{t}} = [\boldsymbol{I}-\boldsymbol{T}+\theta(\boldsymbol{I}-\boldsymbol{TS})]^{-1}\left[-(\boldsymbol{I}-\boldsymbol{T})\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}}+\theta(\boldsymbol{I}-\boldsymbol{TS})(\widetilde{\boldsymbol{z}}_{\boldsymbol{t}}+(1-\mu)\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t}})\right]. \tag{B.56}$$

$$\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+1}} = \boldsymbol{E}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}}+\frac{\beta}{\rho}(\boldsymbol{I}-\boldsymbol{ED})\mathbb{E}_t\widetilde{\boldsymbol{v}}_{\boldsymbol{t+1}}. \tag{B.57}$$

$$\widetilde{\boldsymbol{v}}_{\boldsymbol{t}} = (\boldsymbol{I}-\boldsymbol{S})\widetilde{\boldsymbol{w}}_{\boldsymbol{t}}+\boldsymbol{S}\widetilde{\boldsymbol{z}}_{\boldsymbol{t}}+(1-\mu)\boldsymbol{S}\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t}}+\widetilde{\boldsymbol{b}}_{\boldsymbol{t}}+\beta\boldsymbol{D}\mathbb{E}_t\widetilde{\boldsymbol{v}}_{\boldsymbol{t+1}}. \tag{B.58}$$

$$\widetilde{\boldsymbol{p}}_t = \boldsymbol{S}\left(\widetilde{\boldsymbol{w}}_{\boldsymbol{t}}-\widetilde{\boldsymbol{z}}_{\boldsymbol{t}}-(1-\mu)\left(\widetilde{\boldsymbol{k}}_{\boldsymbol{t}}-\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}}\right)\right). \tag{B.59}$$

B.4.5 Equilibrium Conditions in terms of the State Variables

We now re-express the equilibrium conditions (B.55) through (B.59) and solve for the law of motion of the endogenous state variables ($\boldsymbol{\ell}_t$ and $\boldsymbol{k}_t$). For notational convenience, we re-express the state variables as labor and the capital-labor ratio ($\boldsymbol{\ell}_t$ and $\boldsymbol{\chi}_t$), but note that a law of motion for capital can always recovered since $k_{it} = \ell_{it}\chi_{it}$. We begin by using the wage equation (B.56) to substitute for $\mathrm{d}\ln\widetilde{\boldsymbol{w}}_{\boldsymbol{t}}$ in the value function (B.58):

$$\widetilde{\boldsymbol{v}}_{\boldsymbol{t}} = \left[\begin{array}{c}(\boldsymbol{I}-\boldsymbol{S})[\boldsymbol{I}-\boldsymbol{T}+\theta(\boldsymbol{I}-\boldsymbol{TS})]^{-1}\left[\begin{array}{c}-(\boldsymbol{I}-\boldsymbol{T})\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}}\\ +\theta(\boldsymbol{I}-\boldsymbol{TS})(\widetilde{\boldsymbol{z}}_{\boldsymbol{t}}+(1-\mu)\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t}})\end{array}\right]\\ +\boldsymbol{S}\widetilde{\boldsymbol{z}}_{\boldsymbol{t}}+(1-\mu)\boldsymbol{S}\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t}}+\widetilde{\boldsymbol{b}}_{\boldsymbol{t}}+\beta\boldsymbol{D}\mathbb{E}_t\widetilde{\boldsymbol{v}}_{\boldsymbol{t+1}}\end{array}\right], \tag{B.60}$$

$$\widetilde{\boldsymbol{v}}_{\boldsymbol{t}} = \left[\begin{array}{c}-(\boldsymbol{I}-\boldsymbol{S})[\boldsymbol{I}-\boldsymbol{T}+\theta(\boldsymbol{I}-\boldsymbol{TS})]^{-1}(\boldsymbol{I}-\boldsymbol{T})\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}}\\ +(1-\mu)\left[\boldsymbol{S}+\theta(\boldsymbol{I}-\boldsymbol{S})[\boldsymbol{I}-\boldsymbol{T}+\theta(\boldsymbol{I}-\boldsymbol{TS})]^{-1}(\boldsymbol{I}-\boldsymbol{TS})\right]\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t}}\\ +\left[\boldsymbol{S}+\theta(\boldsymbol{I}-\boldsymbol{S})[\boldsymbol{I}-\boldsymbol{T}+\theta(\boldsymbol{I}-\boldsymbol{TS})]^{-1}(\boldsymbol{I}-\boldsymbol{TS})\right]\widetilde{\boldsymbol{z}}_{\boldsymbol{t}}\\ +\widetilde{\boldsymbol{b}}_{\boldsymbol{t}}+\beta\boldsymbol{D}\mathbb{E}_t\widetilde{\boldsymbol{v}}_{\boldsymbol{t+1}}\end{array}\right]$$

which can be re-written more compactly as:

$$\widetilde{\boldsymbol{v}}_{\boldsymbol{t}} = \boldsymbol{A}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}}+\boldsymbol{B}\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t}}+\boldsymbol{C}\widetilde{\boldsymbol{z}}_{\boldsymbol{t}}+\widetilde{\boldsymbol{b}}_{\boldsymbol{t}}+\beta\boldsymbol{D}\mathbb{E}_t\widetilde{\boldsymbol{v}}_{\boldsymbol{t+1}}, \tag{B.61}$$

$$\boldsymbol{A} \equiv -(\boldsymbol{I}-\boldsymbol{S})[\boldsymbol{I}-\boldsymbol{T}+\theta(\boldsymbol{I}-\boldsymbol{TS})]^{-1}(\boldsymbol{I}-\boldsymbol{T}),$$

$$\boldsymbol{B} \equiv (1-\mu)\left\{\boldsymbol{S}+\theta(\boldsymbol{I}-\boldsymbol{S})[\boldsymbol{I}-\boldsymbol{T}+\theta(\boldsymbol{I}-\boldsymbol{TS})]^{-1}(\boldsymbol{I}-\boldsymbol{TS})\right\},$$

$$\boldsymbol{C} \equiv \boldsymbol{S}+\theta(\boldsymbol{I}-\boldsymbol{S})[\boldsymbol{I}-\boldsymbol{T}+\theta(\boldsymbol{I}-\boldsymbol{TS})]^{-1}(\boldsymbol{I}-\boldsymbol{TS}).$$

Iterating equation (B.61) forward in time, we have:

$$\widetilde{\boldsymbol{v}}_{\boldsymbol{t}} = \mathbb{E}_t\sum_{s=0}^{\infty}(\beta\boldsymbol{D})^s\left(\boldsymbol{A}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+s}}+\boldsymbol{B}\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t+s}}+\boldsymbol{C}\widetilde{\boldsymbol{z}}_{\boldsymbol{t+s}}+\widetilde{\boldsymbol{b}}_{\boldsymbol{t+s}}\right). \tag{B.62}$$

Using equation (B.62) to substitute for $\widetilde{\boldsymbol{v}}_{\boldsymbol{t+1}}$ in equation (B.57), we obtain the following autoregressive representation of the log deviations of population from steady-state value ($\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}}$):

$$\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+1}}-\boldsymbol{E}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}} = \left[\frac{\beta}{\rho}(\boldsymbol{I}-\boldsymbol{ED})\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\boldsymbol{D})^s\left(\boldsymbol{A}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+s+1}}+\boldsymbol{B}\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t+s+1}}+\boldsymbol{C}\widetilde{\boldsymbol{z}}_{\boldsymbol{t+s+1}}+\widetilde{\boldsymbol{b}}_{\boldsymbol{t+s+1}}\right)\right]. \tag{B.63}$$

Likewise, capital dynamics (B.55) can be re-written as (noting $\widetilde{\boldsymbol{w}}_t-\widetilde{\boldsymbol{p}}_t = \boldsymbol{A}\widetilde{\boldsymbol{\ell}}_t+\boldsymbol{B}\widetilde{\boldsymbol{\chi}}_t+\boldsymbol{C}\widetilde{\boldsymbol{z}}_t$):

$$\begin{aligned}\widetilde{\boldsymbol{\chi}}_{t+1}+\widetilde{\boldsymbol{\ell}}_{t+1} \quad = \quad & \widetilde{\boldsymbol{\chi}}_t+\widetilde{\boldsymbol{\ell}}_t+(1-\beta(1-\delta))\left(\boldsymbol{A}\widetilde{\boldsymbol{\ell}}_t+(\boldsymbol{B}-\boldsymbol{I})\widetilde{\boldsymbol{\chi}}_t+\boldsymbol{C}\widetilde{\boldsymbol{z}}_t\right)\\ & +(1-\beta(1-\delta))\frac{1-\beta}{\beta}(\psi-1)\mathbb{E}_t\sum_{s=1}^{\infty}\beta^s\left(\boldsymbol{A}\widetilde{\boldsymbol{\ell}}_{t+s}+(\boldsymbol{B}-\boldsymbol{I})\widetilde{\boldsymbol{\chi}}_{t+s}+\boldsymbol{C}\widetilde{\boldsymbol{z}}_{t+s}\right).\end{aligned} \tag{B.64}$$

B.4.6 Proof of Proposition 3 in the Paper

We suppose that agents learn at time $t=0$ about a one-time, unexpected, and permanent change in productivity and amenities from time $t=1$ onwards. Under this assumption, we can write the sequence of future fundamentals (productivities and amenities) relative to the initial level as $\left(\widetilde{\boldsymbol{z}}_t, \widetilde{\boldsymbol{b}}_t\right) = \left(\widetilde{\boldsymbol{z}}, \widetilde{\boldsymbol{b}}\right)$ for $t \geq 1$.

Proposition. ***Transition Path (Proposition 3 in the paper).*** *There exists a $2N \times 2N$ transition matrix ($\boldsymbol{P}$) and a $2N \times 2N$ impact matrix ($\boldsymbol{R}$) such that the second-order difference equation system in (22) has a closed-form solution of the form:*

$$\widetilde{\boldsymbol{x}}_{t+1} = \boldsymbol{P}\widetilde{\boldsymbol{x}}_t + \boldsymbol{R}\widetilde{\boldsymbol{f}}_{\boldsymbol{t}} \quad \textit{for } t \geq 0, \tag{B.65}$$

where $\widetilde{\boldsymbol{x}}_{\boldsymbol{t}} = \begin{bmatrix} \widetilde{\boldsymbol{\ell}}_t \\ \widetilde{\boldsymbol{k}}_t \end{bmatrix}$ is a $2N \times 2N$ vector of the state variables; $\widetilde{\boldsymbol{f}}_{\boldsymbol{t}} = \begin{bmatrix} \widetilde{\boldsymbol{z}}_{\boldsymbol{t}} \\ \widetilde{\boldsymbol{b}}_{\boldsymbol{t}} \end{bmatrix}$ is a $2N \times 2N$ vector of the shocks to fundamentals; and {$\boldsymbol{P}$, $\boldsymbol{R}$} are $2N \times 2N$ matrices that depend only on the structural parameters {ψ, θ, β, ρ, μ, δ} and the observed trade and migration matrices {$\boldsymbol{S}$, $\boldsymbol{T}$, $\boldsymbol{D}$, $\boldsymbol{E}$}.

Proof. We prove the proposition using the equivalent representation of $\widetilde{\boldsymbol{\ell}}_t$ and $\widetilde{\boldsymbol{\chi}}_t \equiv \widetilde{\boldsymbol{k}}_t - \widetilde{\boldsymbol{\ell}}_t$ as the state variables, where $\widetilde{\boldsymbol{\chi}}_t$ is the vector of capital-labor ratios in each location. Since agents expect fundamentals to be constant for all $t \geq 1$, we can drop the expectation signs in equations (B.63) and (B.64) and write $\left(\widetilde{\boldsymbol{z}}_t, \widetilde{\boldsymbol{b}}_t\right) = \left(\widetilde{\boldsymbol{z}}, \widetilde{\boldsymbol{b}}\right)$:

$$(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D})^{-1}\left(\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+1}} - \boldsymbol{E}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}}\right) = \frac{\beta}{\rho}\sum_{s=0}^{\infty}(\beta\boldsymbol{D})^s\left(\boldsymbol{A}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+s+1}} + \boldsymbol{B}\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t+s+1}} + \boldsymbol{C}\widetilde{\boldsymbol{z}} + \widetilde{\boldsymbol{b}}\right). \tag{B.66}$$

$$\begin{aligned} \widetilde{\boldsymbol{\chi}}_{t+1} + \widetilde{\boldsymbol{\ell}}_{t+1} \quad = \quad & \widetilde{\boldsymbol{\chi}}_t + \widetilde{\boldsymbol{\ell}}_t + (1-\beta(1-\delta))\left(\boldsymbol{A}\widetilde{\boldsymbol{\ell}}_t + (\boldsymbol{B}-\boldsymbol{I})\widetilde{\boldsymbol{\chi}}_t + \boldsymbol{C}\widetilde{\boldsymbol{z}}\right) \\ & + (1-\beta(1-\delta))\frac{1-\beta}{\beta}(\psi-1)\sum_{s=1}^{\infty}\beta^s\left(\boldsymbol{A}\widetilde{\boldsymbol{\ell}}_{t+s} + (\boldsymbol{B}-\boldsymbol{I})\widetilde{\boldsymbol{\chi}}_{t+s} + \boldsymbol{C}\widetilde{\boldsymbol{z}}\right). \end{aligned} \tag{B.67}$$

Analogously,

$$(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D})^{-1}\left(\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+2}} - \boldsymbol{E}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+1}}\right) = \frac{\beta}{\rho}\sum_{s=0}^{\infty}(\beta\boldsymbol{D})^s\left(\boldsymbol{A}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+s+2}} + \boldsymbol{B}\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t+s+2}} + \boldsymbol{C}\widetilde{\boldsymbol{z}} + \widetilde{\boldsymbol{b}}\right). \tag{B.68}$$

$$\begin{aligned} \widetilde{\boldsymbol{\chi}}_{t+2} + \widetilde{\boldsymbol{\ell}}_{t+2} \quad = \quad & \widetilde{\boldsymbol{\chi}}_{t+1} + \widetilde{\boldsymbol{\ell}}_{t+1} + (1-\beta(1-\delta))\left(\boldsymbol{A}\widetilde{\boldsymbol{\ell}}_{t+1} + (\boldsymbol{B}-\boldsymbol{I})\widetilde{\boldsymbol{\chi}}_{t+1} + \boldsymbol{C}\widetilde{\boldsymbol{z}}\right) \\ & + (1-\beta(1-\delta))\frac{1-\beta}{\beta}(\psi-1)\sum_{s=1}^{\infty}\beta^s\left(\boldsymbol{A}\widetilde{\boldsymbol{\ell}}_{t+s+1} + (\boldsymbol{B}-\boldsymbol{I})\widetilde{\boldsymbol{\chi}}_{t+s+1} + \boldsymbol{C}\widetilde{\boldsymbol{z}}\right). \end{aligned} \tag{B.69}$$

Multiply (B.68) by $\beta\boldsymbol{D}$, subtract from (B.66), and re-arrange to obtain:

$$\beta\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D})^{-1}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+2}} = \begin{bmatrix} \left[\beta\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{E}\boldsymbol{D})^{-1}\boldsymbol{E} + (\boldsymbol{I}-\boldsymbol{E}\boldsymbol{D})^{-1} - \frac{\beta}{\rho}\boldsymbol{A}\right]\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t+1}} \\ -(\boldsymbol{I}-\boldsymbol{E}\boldsymbol{D})^{-1}\boldsymbol{E}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}} \\ -\frac{\beta}{\rho}\boldsymbol{B}\widetilde{\boldsymbol{\chi}}_{\boldsymbol{t+1}} - \frac{\beta}{\rho}\boldsymbol{C}\widetilde{\boldsymbol{z}} - \frac{\beta}{\rho}\widetilde{\boldsymbol{b}} \end{bmatrix}.$$

Likewise, multiply (B.69) by β, subtract from (B.67) to obtain:

$$\begin{aligned}\beta\left(\widetilde{\boldsymbol{\chi}}_{t+2}+\widetilde{\boldsymbol{\ell}}_{t+2}\right) &= \left(-\boldsymbol{I}-\left(1-\beta\left(1-\delta\right)\right)\boldsymbol{A}\right)\widetilde{\boldsymbol{\ell}}_t+\left(-\boldsymbol{I}-\left(1-\beta\left(1-\delta\right)\right)\left(\boldsymbol{B}-\boldsymbol{I}\right)\right)\widetilde{\boldsymbol{\chi}}_t \\ &\quad+\left(\left(1+\beta\right)\boldsymbol{I}-\left(1-\beta\left(1-\delta\right)\right)\left(\psi-1-\beta\psi\right)\left(\boldsymbol{B}-\boldsymbol{I}\right)\right)\widetilde{\boldsymbol{\chi}}_{t+1} \\ &\quad+\left(\left(1+\beta\right)\boldsymbol{I}-\left(1-\beta\left(1-\delta\right)\right)\left(\psi-1-\beta\psi\right)\boldsymbol{A}\right)\widetilde{\boldsymbol{\ell}}_{t+1} \\ &\quad-\left(1-\beta\left(1-\delta\right)\right)\psi\left(1-\beta\right)\boldsymbol{C}\widetilde{\boldsymbol{z}}.\end{aligned}$$

Stacking these two, second-order difference equations, we obtain:

$$\begin{bmatrix}\beta\boldsymbol{D}\left(\boldsymbol{I}-\boldsymbol{E}\boldsymbol{D}\right)^{-1} & \boldsymbol{0}\\ \beta\boldsymbol{I} & \beta\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\widetilde{\boldsymbol{\ell}}_{t+2}\\ \widetilde{\boldsymbol{\chi}}_{t+2}\end{bmatrix}=\begin{bmatrix}\boldsymbol{\Upsilon}_{11} & \boldsymbol{\Upsilon}_{12}\\ \boldsymbol{\Upsilon}_{21} & \boldsymbol{\Upsilon}_{22}\end{bmatrix}\begin{bmatrix}\widetilde{\boldsymbol{\ell}}_{t+1}\\ \widetilde{\boldsymbol{\chi}}_{t+1}\end{bmatrix}+\begin{bmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{0}\\ \boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22}\end{bmatrix}\begin{bmatrix}\widetilde{\boldsymbol{\ell}}_{t}\\ \widetilde{\boldsymbol{\chi}}_{t}\end{bmatrix}+\begin{bmatrix}-\frac{\beta}{\rho}\boldsymbol{C} & -\frac{\beta}{\rho}\boldsymbol{I}\\ -\boldsymbol{H} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\widetilde{\boldsymbol{z}}\\ \widetilde{\boldsymbol{b}}\end{bmatrix}. \tag{B.70}$$

$$\boldsymbol{\Upsilon}_{11}\equiv\beta\boldsymbol{D}\left(\boldsymbol{I}-\boldsymbol{E}\boldsymbol{D}\right)^{-1}\boldsymbol{E}+\left(\boldsymbol{I}-\boldsymbol{E}\boldsymbol{D}\right)^{-1}-\frac{\beta}{\rho}\boldsymbol{A},\qquad \boldsymbol{\Upsilon}_{12}\equiv-\frac{\beta}{\rho}\boldsymbol{B},$$

$$\boldsymbol{\Upsilon}_{21}\equiv\left[\left(1+\beta\right)\boldsymbol{I}+\left(1-\beta\left(1-\delta\right)\right)\left(\psi-1-\beta\psi\right)\left(\boldsymbol{I}-\boldsymbol{S}\right)\left[\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{T}\boldsymbol{S}\right)\right]^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\right)\right],$$

$$\boldsymbol{\Upsilon}_{22}\equiv\left[\begin{array}{c}\left(1+\beta\right)\boldsymbol{I}-\Big\{\left(1-\beta\left(1-\delta\right)\right)\left(\psi-1-\beta\psi\right)\times\\ \left[\left(1-\mu\right)\left\{\boldsymbol{S}+\theta\left(\boldsymbol{I}-\boldsymbol{S}\right)\left[\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{T}\boldsymbol{S}\right)\right]^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\boldsymbol{S}\right)\right\}-\boldsymbol{I}\right]\Big\}\end{array}\right],$$

$$\boldsymbol{\Theta}_{11}\equiv-\left(\boldsymbol{I}-\boldsymbol{E}\boldsymbol{D}\right)^{-1}\boldsymbol{E},\qquad \boldsymbol{\Theta}_{21}\equiv-\boldsymbol{I}+\left(1-\beta\left(1-\delta\right)\right)\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{T}\boldsymbol{S}\right)\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\right).$$

$$\boldsymbol{\Theta}_{22}\equiv-\boldsymbol{I}-\left(1-\beta\left(1-\delta\right)\right)\left(\left(1-\mu\right)\left\{\boldsymbol{S}+\theta\left(\boldsymbol{I}-\boldsymbol{S}\right)\left(\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{T}\boldsymbol{S}\right)\right)^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\boldsymbol{S}\right)\right\}-\boldsymbol{I}\right).$$

$$\boldsymbol{H}\equiv\psi\left(1-\beta\right)\left(1-\beta\left(1-\delta\right)\right)\left[\theta\left(\boldsymbol{I}-\boldsymbol{S}\right)\left[\boldsymbol{I}-\boldsymbol{T}+\theta\left(\boldsymbol{I}-\boldsymbol{T}\boldsymbol{S}\right)\right]^{-1}\left(\boldsymbol{I}-\boldsymbol{T}\boldsymbol{S}\right)+\boldsymbol{S}\right].$$

We first conjecture the linear closed-form solution (B.65) and substitute it into the second-order difference equation (B.70) to obtain a matrix system of quadratic equations. We next solve this matrix system of quadratic equations and confirm that our conjecture of a linear closed-form solution is indeed satisfied. Using our conjecture (B.65) in the system of second-order difference equations (B.70), we obtain:

$$\left(\boldsymbol{\Psi}\boldsymbol{P}^2-\boldsymbol{\Gamma}\boldsymbol{P}-\boldsymbol{\Theta}\right)\begin{bmatrix}\widetilde{\boldsymbol{\ell}}_t\\ \widetilde{\boldsymbol{\chi}}_t\end{bmatrix}+\left[\left(\boldsymbol{\Psi}\boldsymbol{P}+\boldsymbol{\Psi}-\boldsymbol{\Gamma}\right)\boldsymbol{R}-\boldsymbol{\Pi}\right]\begin{bmatrix}\widetilde{\boldsymbol{z}}\\ \widetilde{\boldsymbol{b}}\end{bmatrix}=0, \tag{B.71}$$

$$\boldsymbol{\Psi}\equiv\begin{bmatrix}\left(\beta D\right)\left(\boldsymbol{I}-\boldsymbol{E}\boldsymbol{D}\right)^{-1} & \boldsymbol{0}\\ \beta\boldsymbol{I} & \beta\boldsymbol{I}\end{bmatrix},\qquad \boldsymbol{\Gamma}\equiv\begin{bmatrix}\boldsymbol{\Upsilon}_{11} & \boldsymbol{\Upsilon}_{12}\\ \boldsymbol{\Upsilon}_{21} & \boldsymbol{\Upsilon}_{22}\end{bmatrix},$$

$$\boldsymbol{\Theta}\equiv\begin{bmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{0}\\ \boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22}\end{bmatrix},\qquad \boldsymbol{\Pi}\equiv\begin{bmatrix}-\frac{\beta}{\rho}\boldsymbol{C} & -\frac{\beta}{\rho}\boldsymbol{I}\\ -\boldsymbol{H} & \boldsymbol{0}\end{bmatrix}.$$

For the system (B.71) to have a solution for $\begin{bmatrix}\widetilde{\boldsymbol{\ell}}_t\\ \widetilde{\boldsymbol{\chi}}_t\end{bmatrix}\neq 0$ and $\begin{bmatrix}\widetilde{\boldsymbol{z}}\\ \widetilde{\boldsymbol{b}}\end{bmatrix}\neq 0$, we require:

$$\boldsymbol{\Psi}\boldsymbol{P}^2-\boldsymbol{\Gamma}\boldsymbol{P}-\boldsymbol{\Theta}=0, \tag{B.72}$$

$$\boldsymbol{R}=\left(\boldsymbol{\Psi}\boldsymbol{P}+\boldsymbol{\Psi}-\boldsymbol{\Gamma}\right)^{-1}\boldsymbol{\Pi}. \tag{B.73}$$

Following Uhlig (1999), we can write this first condition (B.72) as the following generalized eigenvector-eigenvalue problem, where e is a generalized eigenvector and ξ is a generalized eigenvalue of $\boldsymbol{\Xi}$ with respect to $\boldsymbol{\Delta}$:

$$\xi\boldsymbol{\Delta}e=\boldsymbol{\Xi}e,$$

where:

$$\boldsymbol{\Xi} \equiv \begin{bmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Theta} \\ \boldsymbol{I} & \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{\Delta} \equiv \begin{bmatrix} \boldsymbol{\Psi} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix}.$$

If e_h is a generalized eigenvector and ξ_h is a generalized eigenvalue of $\boldsymbol{\Xi}$ with respect to $\boldsymbol{\Delta}$, then e_h can be written for some $h \in \Re^N$ as:

$$e_h = \begin{bmatrix} \xi_h \bar{e}_h \\ \bar{e}_h \end{bmatrix}.$$

Assuming that the transition matrix has distinct eigenvalues, which we verify empirically, there are $2N$ linearly independent generalized eigenvectors $(e_1, \ldots, e_{2N})$ and corresponding stable eigenvalues $(\xi_1, \ldots, \xi_{2N})$, and the transition matrix ($\boldsymbol{P}$) is given by:

$$\boldsymbol{P} = \boldsymbol{\Omega}\boldsymbol{\Lambda}\boldsymbol{\Omega}^{-1},$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix of the $2N$ eigenvalues and $\boldsymbol{\Omega}$ is the matrix stacking the corresponding $2N$ eigenvectors $\{\bar{e}_h\}$. The impact matrix ($\boldsymbol{R}$) in the second condition (B.73) can be recovered using:

$$\boldsymbol{R} = (\boldsymbol{\Psi}\boldsymbol{P} + \boldsymbol{\Psi} - \boldsymbol{\Gamma})^{-1}\,\boldsymbol{\Pi},$$

and our conjecture (B.65) is satisfied. □

B.4.7 Properties of the Transition Path.

We now use the eigenvalue-eigenvector representation in Proposition 3 in the paper to establish some properties of the transition path towards the new steady-state.

B.4.8 Convergence Dynamics Versus Fundamental Shocks

In particular, we now consider the case in which agents at time $t = 0$ learn of a permanent change in fundamentals $(\widetilde{\boldsymbol{z}}, \widetilde{\boldsymbol{b}})$ at time $t = 1$. From Proposition 3 in the paper and equation (B.65) above, the initial impact of the productivity $(\widetilde{\boldsymbol{z}})$ and amenity $(\widetilde{\boldsymbol{b}})$ shocks in the first period is:

$$\widetilde{\boldsymbol{x}}_1 = \boldsymbol{R}\widetilde{\boldsymbol{f}}.$$

More generally, the impact of these productivity and amenity shocks in period $t \geq 1$ is:

$$\widetilde{\boldsymbol{x}}_{t+1} \;=\; \boldsymbol{P}\widetilde{\boldsymbol{x}}_t + \boldsymbol{R}\widetilde{\boldsymbol{f}} = \left(\sum_{s=0}^{t} \boldsymbol{P}^s\right)\boldsymbol{R}\widetilde{\boldsymbol{f}}. \tag{B.74}$$

If the spectral radius of $\boldsymbol{P}$ is less than one, a condition that we verify empirically, the summation $\lim_{t\to\infty}\sum_{s=0}^{t}\boldsymbol{P}^s$ converges, and we can re-write the impact of the productivity and amenity shocks in period $t \geq 1$ as:

$$\widetilde{\boldsymbol{x}}_{t+1} \;=\; \left(\sum_{s=0}^{\infty}\boldsymbol{P}^s - \sum_{s=t+1}^{\infty}\boldsymbol{P}^s\right)\boldsymbol{R}\widetilde{\boldsymbol{f}} = \left(\boldsymbol{I} - \boldsymbol{P}^{t+1}\right)(\boldsymbol{I} - \boldsymbol{P})^{-1}\,\boldsymbol{R}\widetilde{\boldsymbol{f}}.$$

From this relationship, the new steady-state must satisfy:

$$\lim_{t\to\infty}\widetilde{\boldsymbol{x}}_t = \boldsymbol{x}^*_{\text{new}} - \widetilde{x}^*_{\text{initial}} = (\boldsymbol{I} - \boldsymbol{P})^{-1}\,\boldsymbol{R}\widetilde{\boldsymbol{f}},$$

where $\left(\boldsymbol{I}-\boldsymbol{P}\right)^{-1}\boldsymbol{R}$ coincides with the explicit solution for the changes-in-steady-states in Proposition A.1 in Online Appendix B.4.3:

$$\left(\boldsymbol{I}-\boldsymbol{P}\right)^{-1}\boldsymbol{R}=\left[\begin{array}{cc}\boldsymbol{L}^{z} & \boldsymbol{L}^{b}\\ \boldsymbol{K}^{z} & \boldsymbol{K}^{b}\end{array}\right].$$

Using Proposition 3 in the paper, we can also decompose the evolution of the spatial distribution of economic activity across locations into the contributions of convergence towards steady-state and shocks to fundamentals. In particular, from Proposition 3 in the paper, we have:

$$\begin{aligned}\widetilde{x}_{t} &= \boldsymbol{P}\widetilde{x}_{t-1}+\boldsymbol{R}\tilde{\boldsymbol{f}},\\ \widetilde{x}_{t-1} &= \boldsymbol{P}\widetilde{x}_{t-2}+\boldsymbol{R}\tilde{\boldsymbol{f}},\\ \widetilde{x}_{1} &= \boldsymbol{P}\widetilde{x}_{0}+\boldsymbol{R}\widetilde{f},\\ \widetilde{x}_{0} &= \boldsymbol{P}\widetilde{x}_{-1},\end{aligned}$$

where the last equation at $t=0$ differs from others, because agents become aware at time $t=0$ of the shock to fundamentals a time $t=1$, after they have migrated between time $t=-1$ and time $t=0$. Taking the difference between the equations for time t and $t-1$, we have:

$$\ln x_{t}-\ln x_{t-1}=\boldsymbol{P}\left(\ln x_{t-1}-\ln x_{t-2}\right)=\boldsymbol{P}^{t-1}\left(\ln x_{1}-\ln x_{0}\right)=\boldsymbol{P}^{t}\left(\ln x_{0}-\ln x_{-1}\right)+\boldsymbol{P}^{t-1}\boldsymbol{R}\widetilde{f}.$$

Therefore, we have:

$$\begin{aligned}\ln x_{t}-\ln x_{-1} &= \left[\ln x_{t}-\ln x_{t-1}\right]+\left[\ln x_{t-1}-\ln x_{t-2}\right]+\cdots\\ &\quad+\left[\ln x_{1}-\ln x_{0}\right]+\left[\ln x_{0}-\ln x_{-1}\right]\\ &= \left[\boldsymbol{P}^{t}\left(\ln x_{0}-\ln x_{-1}\right)+\boldsymbol{P}^{t-1}\boldsymbol{R}\widetilde{f}\right]+\left[\boldsymbol{P}^{t-1}\left(\ln x_{0}-\ln x_{-1}\right)+\boldsymbol{P}^{t-2}\boldsymbol{R}\widetilde{f}\right]\\ &\quad+\cdots+\left[\boldsymbol{P}\left(\ln x_{0}-\ln x_{-1}\right)+\boldsymbol{R}\widetilde{f}\right]+\left[\ln x_{0}-\ln x_{-1}\right]\\ &= \sum_{s=0}^{t}\boldsymbol{P}^{s}\left(\ln x_{0}-\ln x_{-1}\right)+\sum_{s=0}^{t-1}\boldsymbol{P}^{s}\boldsymbol{R}\widetilde{f},\end{aligned}\tag{B.75}$$

which corresponds to equation (24) in the paper.

B.4.9 Spectral Analysis of the Transition Matrix $\boldsymbol{P}$

We now show that we can further characterize the economy's transition path in terms of the lower-dimensional components of the eigenvectors and eigenvalues of the transition matrix ($\boldsymbol{P}$). We have already shown in that we can decompose the dynamic path of the economy into one component capturing shocks to fundamentals and another component capturing convergence to the initial steady-state. Therefore, for the remainder of this subsection, we focus for expositional simplicity on an economy that is initially in steady-state.

Eigendecomposition of the Transition Matrix We use the eigendecomposition of the transition matrix, $\boldsymbol{P}\equiv\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix of eigenvalues arranged in decreasing order by absolute values, and $\boldsymbol{V}=\boldsymbol{U}^{-1}$. For each eigenvalue λ_h, the h-th column of $\boldsymbol{U}$ ($\boldsymbol{u_h}$) and the h-th row of $\boldsymbol{V}$ ($\boldsymbol{v}'_{\boldsymbol{h}}$) are the corresponding right- and left-eigenvectors of $\boldsymbol{P}$, respectively, such that

$$\lambda_{h}\boldsymbol{u}_{h}=\boldsymbol{P}\boldsymbol{u}_{h},\qquad\lambda_{h}\boldsymbol{v}_{h}'=\boldsymbol{v}_{h}'\boldsymbol{P}.$$

That is, $\boldsymbol{u}_h$ ($\boldsymbol{v}_h'$) is the vector that, when left-multiplied (right-multiplied) by $\boldsymbol{P}$, is proportional to itself but scaled by the corresponding eigenvalue λ_h.[2] We refer to $\boldsymbol{u}_h$ simply as eigenvectors. Both $\{\boldsymbol{u}_h\}$ and $\{\boldsymbol{v}_h'\}$ are bases that span the $2N$-dimensional vector space.

We next introduce a particular type of shock to productivity and amenities that proves useful for characterizing the model's transition dynamics. We define an *eigen-shock* as a shock to productivity and amenities ($\widetilde{\boldsymbol{f}}_{(h)}$) for which the initial impact of these shocks on the state variables ($\boldsymbol{R}\widetilde{\boldsymbol{f}}_{(h)}$) coincides with a real eigenvector of the transition matrix ($\boldsymbol{u}_h$) or the zero vector. The eigen-shock that corresponds to each eigenvector $\boldsymbol{u}_h$ can be recovered as $\widetilde{\boldsymbol{f}}_{(h)} = \boldsymbol{\Pi}^{-1}\left(\boldsymbol{\Psi}\boldsymbol{P} + \boldsymbol{\Psi} - \boldsymbol{\Gamma}\right)\boldsymbol{u}_h$. Recall that all matrices involved in this operation and the eigenvectors of the transition matrix ($\boldsymbol{u}_h$) can be computed using only our observed trade and migration share matrices ($\boldsymbol{S}$, $\boldsymbol{T}$, $\boldsymbol{D}$, $\boldsymbol{E}$) and the structural parameters of the model $\{\psi, \theta, \beta, \rho, \mu, \delta\}$. Therefore, we can solve for the eigen-shocks from these observed data and the structural parameters of the model.

Using our eigendecomposition and definition of an eigen-shock, we can undertake a spectral analysis of the economy's dynamic response to shocks.

Proposition. ***Spectral Analysis (Proposition 4 in the paper).*** *Consider an economy that is initially in steady-state at time $t = 0$ when agents learn about one-time, permanent shocks to productivity and amenities ($\widetilde{\boldsymbol{f}} = \begin{bmatrix} \widetilde{\boldsymbol{z}} \\ \widetilde{\boldsymbol{b}} \end{bmatrix}$) from time $t = 1$ onwards. The transition path of the state variables can be written as a linear combination the eigenvalues (λ_h) and eigenvectors ($\boldsymbol{u}_h$) of the transition matrix:*

$$\widetilde{\boldsymbol{x}}_t = \sum_{s=0}^{t-1} \boldsymbol{P}^s \boldsymbol{R}\widetilde{\boldsymbol{f}} = \sum_{h=1}^{2N} \frac{1-\lambda_h^t}{1-\lambda_h} \boldsymbol{u}_h \boldsymbol{v}_h' \boldsymbol{R}\widetilde{\boldsymbol{f}} = \sum_{h=2}^{2N} \frac{1-\lambda_h^t}{1-\lambda_h} \boldsymbol{u}_h a_h, \tag{B.76}$$

where the weights in this linear combination (a_h) can be recovered as the coefficients from a linear projection (regression) of the observed shocks ($\widetilde{\boldsymbol{f}}$) on the eigen-shocks ($\widetilde{\boldsymbol{f}}_{(h)}$).

Proof. The proposition follows from the eigendecomposition of the transition matrix: $\boldsymbol{P} \equiv \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$, which implies $P^s = \sum_{h=1}^{2N} \lambda_h^s u_h v_h'$ and hence:

$$\widetilde{x}_t = \sum_{s=0}^{t-1} \boldsymbol{P}^s \boldsymbol{R}\widetilde{\boldsymbol{f}} = \sum_{s=0}^{t-1} \left(\sum_{h=1}^{2N} \lambda_h^s u_h v_h' \right) \boldsymbol{R}\widetilde{\boldsymbol{f}} = \sum_{h=1}^{2N} \left(\sum_{s=0}^{t-1} \lambda_h^s \right) u_h v_h' \boldsymbol{R}\widetilde{\boldsymbol{f}} = \sum_{h=1}^{2N} \frac{1-\lambda_h^t}{1-\lambda_h} u_h v_h' \boldsymbol{R}\widetilde{\boldsymbol{f}}.$$

To decompose any observed shock $\widetilde{\boldsymbol{f}}$ as a linear combination $\boldsymbol{a}$ of the eigen-shocks $\left\{\widetilde{\boldsymbol{f}}_{(h)}\right\}$, let $\boldsymbol{F}$ denote the matrix whose h-th column is the h-th eigen-shock. Then $\boldsymbol{F}\boldsymbol{a} = \widetilde{\boldsymbol{f}} \iff \boldsymbol{a} = \left(\boldsymbol{F}'\boldsymbol{F}\right)^{-1} \boldsymbol{F}\widetilde{\boldsymbol{f}}$, which implies that $\boldsymbol{a}$ can be recovered as the coefficients from a regression of $\widetilde{\boldsymbol{f}}$ on the eigen-shocks. □

We now show how this proposition can be used to characterize both the speed of convergence to steady-state and the heterogeneous impact of shocks across locations.

[2] Note that $\boldsymbol{P}$ need not be symmetric. This eigendecomposition exists if the transition matrix has distinct eigenvalues, a condition that we verify is satisfied empirically. We construct the right-eigenvectors such that the 2-norm of $\boldsymbol{u}_h$ is equal to 1 for all h, where note that $\boldsymbol{v}_i'\boldsymbol{u}_h = 1$ if $i = h$ and is equal to zero otherwise.

Speed of Convergence We measure the speed of convergence to steady-state using the conventional measure of the half-life. In particular, we define the half-life of a shock $\widetilde{\boldsymbol{f}}$ for the i-th state variable as the time it takes for that state variable to converge half of the way to steady-state:

$$\arg\max_{t} \frac{|\widetilde{x}_{it} - \widetilde{x}_{i\infty}|}{\max_s |\widetilde{x}_{is} - \widetilde{x}_{i\infty}|} \geq \frac{1}{2}, \tag{B.77}$$

where $\widetilde{x}_{i\infty} = x^*_{i,\text{new}} - x^*_{i,\text{initial}}$.

We begin by considering the speed of convergence for nontrivial eigen-shocks, for which the initial impact on the state variables corresponds to a real eigenvector of the transition matrix. For these eigen-shocks, the state variables converge exponentially towards steady-state, and the speed of convergence depends solely on the corresponding eigenvalue (λ_h).

Proposition. ***Speed of Convergence (Proposition 5 in the paper).*** *Consider an economy that is initially in steady-state at time $t = 0$ when agents learn about one-time, permanent shocks to productivity and amenities ($\widetilde{\boldsymbol{f}} = \begin{bmatrix} \widetilde{\boldsymbol{z}} \\ \widetilde{\boldsymbol{b}} \end{bmatrix}$) from time $t = 1$ onwards. Suppose that these shocks are a nontrivial eigen-shock ($\widetilde{\boldsymbol{f}}_{(h)}$), for which the initial impact on the state variables at time $t = 1$ coincides with a real eigenvector ($\boldsymbol{u}_h$) of the transition matrix ($\boldsymbol{P}$): $\boldsymbol{R}\widetilde{\boldsymbol{f}}_{(h)} = \boldsymbol{u}_h$. The transition path of the state variables ($\boldsymbol{x_t}$) in response to such an eigen-shock ($\widetilde{\boldsymbol{f}}_{(h)}$) is :*

$$\widetilde{\boldsymbol{x}}_t = \sum_{j=2}^{2N} \frac{1-\lambda_j^t}{1-\lambda_j} \boldsymbol{u}_j \boldsymbol{v}_j' \boldsymbol{u}_h = \frac{1-\lambda_h^t}{1-\lambda_h} \boldsymbol{u}_h \qquad \Longrightarrow \qquad \ln \boldsymbol{x}_{t+1} - \ln \boldsymbol{x}_t = \lambda_h^t \boldsymbol{u}_h,$$

and the half-life is given by: $t_i^{(1/2)} \left(\widetilde{\boldsymbol{f}}\right) = -\left\lceil \frac{\ln 2}{\ln \lambda_h} \right\rceil$, for all state variables $i = 2, \cdots, 2N$, where $\lceil \cdot \rceil$ is the ceiling function. The eigen-shock with associated eigenvalue of zero has zero half-life.

Proof. If the initial impact impact of the shock to productivity and amenities on the state variables ($\boldsymbol{R}\widetilde{\boldsymbol{f}}$) coincides with a real eigenvector ($\boldsymbol{R}\widetilde{\boldsymbol{f}}_{(h)} = \boldsymbol{u}_h$), we can re-write equation (28) in Proposition 4 in the paper as follows:

$$\widetilde{\boldsymbol{x}}_t = \sum_{h=2}^{2N} \left(\frac{\lambda_h^t}{1-\lambda_h}\right) \boldsymbol{u}_h \boldsymbol{v}_h' \boldsymbol{R}\widetilde{\boldsymbol{f}} = \sum_{j=2}^{2N} \frac{1-\lambda_j^t}{1-\lambda_j} \boldsymbol{u}_j \boldsymbol{v}_j' \boldsymbol{u}_h = \frac{1-\lambda_h^t}{1-\lambda_h} \boldsymbol{u}_h,$$

where we have used $\boldsymbol{v}_i' \boldsymbol{u}_h = 0$ for $i \neq h$ and $\boldsymbol{v}_i' \boldsymbol{u}_h = 1$ for $i = h$. Taking differences between periods $t+1$ and t, we have:

$$\widetilde{\boldsymbol{x}}_{t+1} - \widetilde{\boldsymbol{x}}_t = \frac{1-\lambda_h^{t+1}}{1-\lambda_h} \boldsymbol{u}_h - \frac{1-\lambda_h^t}{1-\lambda_h} \boldsymbol{u}_h,$$

which simplifies to: $(1-\lambda_h)\left(\widetilde{\boldsymbol{x}}_{t+1} - \widetilde{\boldsymbol{x}}_t\right) = (1-\lambda_h)\lambda_h^t \boldsymbol{u}_h$. Therefore: $\left(\widetilde{\boldsymbol{x}}_{t+1} - \widetilde{\boldsymbol{x}}_t\right) = \lambda_h^t \boldsymbol{u}_h$. Noting that $\widetilde{\boldsymbol{x}}_t = \ln \boldsymbol{x}_t - \ln \boldsymbol{x}^*_{\text{initial}}$, we have: $\ln \boldsymbol{x}_{t+1} - \ln \boldsymbol{x}_t = \lambda_h^t \boldsymbol{u}_h$. This implies exponential convergence to steady-state, such that for each location i: $\frac{x_{it+1}}{x_{it}} = \exp\left(\lambda_h^t u_{ih}\right)$. Using the half-life definition (B.77), we can solve for the half-life as:

$$\frac{\frac{1-\lambda_h^t}{1-\lambda_h} u_h}{\frac{1}{1-\lambda_h} u_h} = \frac{1}{2}, \quad \Rightarrow \quad \lambda_h^t = \frac{1}{2}, \quad \Rightarrow \quad \ln\frac{1}{2} = t \ln \lambda_h, \quad \Rightarrow \quad t = -\frac{\ln 2}{\ln \lambda_h}.$$

Imposing the requirement that t is an integer, we obtain: $t = -\left\lceil \frac{\ln 2}{\ln \lambda_h} \right\rceil$, for all state variables $i = 2, \cdots, 2N$, where $\lceil \cdot \rceil$ is the ceiling function. □

B.4.10 Two-Region Example

In Section 3.3 of the paper, we illustrate our spectral analysis using a simple example of two symmetric locations that begin in steady-state. By location symmetry and trade and migration frictions, the expenditure and migration share matrices ($\boldsymbol{S}$ and $\boldsymbol{D}$) are both symmetric and diagonal-dominant, with $\boldsymbol{T} = \boldsymbol{S}$ and $\boldsymbol{E} = \boldsymbol{D}$. In this section of the Online Appendix, we provide a further characterization of the four eigenvectors of the transition matrix ($\boldsymbol{P}$) in this simple example. Following the Proof of Proposition 3 in Section B.4.6 of this appendix, we provide this characterization using the equivalent representation of $\widetilde{\boldsymbol{\ell}}_t$ and $\widetilde{\boldsymbol{\chi}}_t \equiv \widetilde{\boldsymbol{k}}_t - \widetilde{\boldsymbol{\ell}}_t$ as the state variables, where $\widetilde{\boldsymbol{\chi}}_t$ is the vector of capital-labor ratios in each location.

As discussed in the paper, the four eigenvectors of the transition matrix ($\boldsymbol{P}$) in this example take the following simple form:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ \zeta \\ -\zeta \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -\xi \\ \xi \end{bmatrix}, \tag{B.78}$$

for some constants ζ, ξ that depend on the model parameters and the trade and migration share matrices ($\boldsymbol{S} = \boldsymbol{T}$, $\boldsymbol{D} = \boldsymbol{E}$).

We now provide a further analytical characterization of the properties of these four eigenvectors. We know $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{P}$ iff

$$\lambda^2 \Psi u = \lambda \Gamma u + \Theta u \tag{B.79}$$

for some constant λ, which is the corresponding eigenvalue. Ψ, Γ, and Θ are all 4×4 matrices from equation (22) in the paper. It is thus easy to verify by brute force (for instance, using Matlab symbolic toolbox to express Ψ, Γ, Θ as a function of model parameters and the entries in the $\boldsymbol{S}$ and $\boldsymbol{D}$ matrices) that $[1, 1, 0, 0]'$ is an eigenvector with eigenvalue 0 and $[0, 0, 1, 1]'$ is also an eigenvector. The eigenvalue corresponding to the latter is $1 - \mu (1 - \beta (1 - \delta))$ if landlord's intertemporal elasticity of substitution (ψ) is equal to one (logarithmic preferences). More generally, for values of the intertemporal elasticity of substitution (ψ) different from one, the eigenvalue (λ) corresponding to the eigenvector $[0, 0, 1, 1]'$ is the solution to the following quadratic equation:

$$\lambda = \frac{(\beta + \psi (1 - \beta) (1 - X) + X) - \sqrt{(\beta + \psi (1 - \beta) (1 - X) + X)^2 - 4\beta X}}{2\beta},$$

where $X \equiv 1 - \mu (1 - \beta (1 - \delta))$.

We can similarly verify that $[1, -1, 0, 0]'$ and $[0, 0, 1, -1]'$ are not eigenvectors. By symmetry, and because the eigenvectors form a basis, the remaining eigenvectors must take the form $[1, -1, \zeta, -\zeta]'$ and $[1, -1, -\xi, \xi]'$ for some constants ζ, ξ. To find the corresponding eigen-shocks, use:

$$\boldsymbol{\Psi P}^2 - \boldsymbol{\Gamma P} - \boldsymbol{\Theta} = 0, \qquad (\boldsymbol{\Psi P} + \boldsymbol{\Psi} - \boldsymbol{\Gamma}) \boldsymbol{R} = \boldsymbol{\Pi}.$$

Hence, for any eigenvector $\boldsymbol{u}$ with the corresponding eigen-shock $\widetilde{\boldsymbol{f}}$ such that $\boldsymbol{R}\widetilde{\boldsymbol{f}} = \boldsymbol{u}$, it must be the case that

$$\boldsymbol{\Pi}\widetilde{\boldsymbol{f}} \;=\; (\boldsymbol{\Psi P} + \boldsymbol{\Psi} - \boldsymbol{\Gamma}) \boldsymbol{u} = \frac{1}{\lambda} \left(\lambda \boldsymbol{\Psi} + \boldsymbol{\Psi P}^2 - \boldsymbol{\Gamma P}\right) \boldsymbol{u} = \frac{1}{\lambda} (\lambda \boldsymbol{\Psi} + \boldsymbol{\Theta}) \boldsymbol{u}.$$

Because eigenvectors and eigen-shocks are scale-invariant, we can ignore the constant $\frac{1}{\lambda}$ and write eigen-shocks as

$$\widetilde{\boldsymbol{f}} = \boldsymbol{\Pi}^{-1} (\lambda \boldsymbol{\Psi} + \boldsymbol{\Theta}) \boldsymbol{u}.$$

One can then verify that the eigen-shock corresponding to $\boldsymbol{u} = [1,1,0,0]'$ is $\widetilde{\boldsymbol{f}} = [0,0,1,1]'$, while the eigen-shock corresponding to $\boldsymbol{u} = [0,0,1,1]'$ is $\widetilde{\boldsymbol{f}} = [1,1,0,0]'$. One can also verify that generically $[0,0,1,-1]$' is not an eigen-shock (since the first two entries of $\mathbf{\Pi}^{-1}(\lambda\mathbf{\Psi}+\mathbf{\Theta})[1,-1,\zeta,-\zeta]'$ are generically non-zero). Since the eigen-shocks must span the vector space, by symmetry the two remaining eigen-shocks must be of the form $[1,-1,c,-c]'$ and $[1,-1,d,-d]'$ for some constants c, d.

References

Allen, Treb, Costas Arkolakis and Xiangliang Li (2020) "On the Equilibrium Properties of Network Models with Heterogeneous Agents," *NBER Working Paper*, 27837.

Uhlig, Harald (1999) "A Toolkit for Analyzing Non-linear Dynamic Stochastic Models Easily," in *Computational Methods for the Study of Dynamic Economies*, ed. by R. Marimon and A. Scott, New York: Oxford University Press, 30-61.