Dynamic Spatial General Equilibrium

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Motivation

- How does the spatial distribution of economic activity respond to local shocks? (e.g. productivity, transport infrastructure, trade)
  - This response can be gradual because of migration frictions for mobile factors and the accumulation of immobile factors (capital structures)

We make four main contributions:
1. Develop a dynamic spatial model with forward-looking investment and migration and characterize existence/uniqueness of steady-state
2. Generalize existing dynamic exact-hat algebra results for counterfactuals in migration models to include capital investments
3. Linearize the model to characterize analytically determinants of speed of convergence (spectral analysis of transition matrix)
4. Apply our framework to examine income convergence across U.S. states over time (both capital dynamics and labor mobility)
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  – Investment and migration decisions in each location depend on one another and these decisions in all locations in all future periods

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  - Can solve for counterfactual values of the endogenous variables without information on the level of unobserved fundamentals

Economy's transition dynamics shaped by an interaction between migration and investment dynamics
- Adjustment: (a) slow when capital and labor are both above/below steady-state; (b) fast when one is above and the other is below

Linearize the model to obtain a closed-form solution for the transition path to analyze the determinants of the speed of convergence
- Speed of convergence depends on spectral properties of transition matrix
- Path of state variables determined by these spectral properties

- Decline in rate of income convergence over time ($\beta$-convergence)
- Slow convergence and heterogeneous impact of shocks
- Heterogeneity explained by the interaction of capital and labor dynamics
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• Theoretical work on economic geography

• Static quantitative spatial trade models

• Dynamic models of capital accumulation in international trade

• Dynamic models of trade and geography with labor mobility

• Research on sufficient statistics for welfare in static trade models

• Regional income convergence and persistence of local shocks
Related Literature

Economic geography models with dynamic migration decisions
- Caliendo et al. (2019)

Trade models with dynamic investment decisions
- Eaton et al. (2016), Ravikumar et al. (2019), Alessandria et al. (2021)

Linearization methods in DSGE

A multi-region model with migration and investment decisions
- Existence and uniqueness of steady-state
- Non-linear solution for transition path ("dynamic exact-hat")
- Non-trivial K-L interactions

Linearization and spectral analysis
- Eigenvalue-eigenvector representation
- Speed of convergence
- The full spectrum matters
- High-dimensional state space
Outline

- Dynamic Spatial Model
- Extensions
- Data
- Empirical Results
- Conclusions
Model Setup

- Multi-location, single-sector Armington model (extensions later)
- Economy consists of a set of locations $i \in \{1, \ldots, N\}$
- Locations differ in productivity, amenities, bilateral goods trade costs, and bilateral migration costs
- Two types of agents: workers and landlords
- Continuum of workers
  - Endowed with one unit of labor
  - Geographically mobile subject to migration costs
  - No savings-investment technology ("hand to mouth")
  - Make dynamic forward-looking migration decisions to maximize intertemporal utility
- Continuum of landlords in each location
  - Own the stock of local capital
  - Geographically immobile
  - Make dynamic forward-looking consumption-investment choices to maximize intertemporal utility
Worker Migration (CDP)

- At the beginning of period $t$, mass of workers $\ell_{it}$ in location $i$:
  - Produce and consume
  - Observe extreme value idiosyncratic mobility shocks $\{\epsilon_{gt}\}$
  - Choose optimal location for period $t+1$ given mobility costs $\kappa_{git}$

- Expected value of living in location $i$ in period $t$ depends on wage $(w_{it})$, cost of living $(p_{it})$, amenities $(b_{it})$ and the expected value of optimal location choice

$$v_{it} = \ln \left( \frac{w_{it}}{p_{it}} \right) + \ln b_{it} + \rho \ln \sum_{g=1}^{N} \left( \exp \left( \beta \mathbb{E}_t \nu_{gt+1} \right) / \kappa_{git} \right)^{1/\rho}$$

- Location choice probabilities

$$D_{igt} = \frac{\left( \exp \left( \beta \mathbb{E}_t \nu_{gt+1} \right) / \kappa_{git} \right)^{1/\rho}}{\sum_{k=1}^{N} \left( \exp \left( \beta \mathbb{E}_t \nu_{kt+1} \right) / \kappa_{kit} \right)^{1/\rho}}$$

- Population flow condition

$$\ell_{gt+1} = \sum_{i=1}^{N} D_{igt} \ell_{it}$$
Trade and Production

• Armington differentiation of goods by location of origin

\[ p_{nt} = \left[ \sum_{i=1}^{N} p_{nit}^{-\theta} \right]^{-1/\theta}, \quad \theta = \sigma - 1, \quad \sigma > 1 \]

• Competitive production and iceberg trade costs \( \tau_{nit} \geq 1 \)

• Cost in location \( n \) of sourcing a variety from location \( i \) is

\[ p_{nit} = \frac{\tau_{nit} w_{it}^\lambda r_{it}^{1-\lambda}}{z_{it}}, \quad 0 < \lambda < 1 \]

• Using profit maximization to substitute for equilibrium labor input, landlord income is linear in capital

\[ \Pi_{it} = \lambda \left( p_{it} z_{it} \right)^{\frac{1}{\lambda}} \left( \frac{1 - \lambda}{w_{it}} \right)^{\frac{1-\lambda}{\lambda}} k_{it} \]
Landlord Investment

- Landlords optimal intertemporal consumption-investment decision

\[ \nu_{it}^k = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^{t+s} \left( \frac{c_{it+s}^k}{1 - 1/\psi} \right) \]

- Landlords in a location can produce one unit of capital in that location using one unit of the local consumption index

- Local capital is geographically immobile once installed (buildings and structures) and depreciates at constant rate \( \delta \)

- Intertemporal budget constraint

\[ r_{it} k_{it} = p_{it} c_{it}^k + p_{it} (k_{it+1} - (1 - \delta) k_{it}) \]

- CRRA preferences and linear income in capital imply linear saving rate (as Angeletos 2007 and Moll 2014)

\[ k_{it+1} = (1 - \zeta_{it}) R_{it} k_{it}, \quad R_{it} \equiv 1 - \delta + \frac{r_{it}}{p_{it}} \]

\[ \zeta_{it}^{-1} = 1 + \beta^\psi \left( \mathbb{E}_t \left[ \frac{\psi - 1}{\psi} R_{it+1} \zeta_{t+1} - \frac{1}{\psi} \right] \right)^\psi \]
General Equilibrium

- Value function

\[ v_{it} = \ln \left( \frac{w_{it}}{p_{it}} \right) + \ln b_{it} + \rho \ln \sum_{g=1}^{N} \left( \exp \left( \beta \mathbb{E}_t v_{gt+1} \right) / \kappa_{git} \right)^{1/\rho} \]

\[ p_{nt} = \left[ \sum_{i=1}^{N} \left( w_{it} \left( \frac{1 - \lambda}{\lambda} \right)^{1-\lambda} \left( \ell_{it} / k_{it} \right)^{1-\lambda} \tau_{nit} / z_{it} \right) \right]^{-1/\theta} \]

- Goods market clearing

\[ w_{it} \ell_{it} = \sum_{n=1}^{N} S_{nit} w_{nt} \ell_{nt}, \quad S_{nit} \equiv \frac{\left( w_{it} \left( \ell_{it} / k_{it} \right)^{1-\lambda} \tau_{nit} / z_{it} \right)^{-\theta}}{\sum_{m=1}^{N} \left( w_{mt} \left( \ell_{mt} / k_{mt} \right)^{1-\lambda} \tau_{nmt} / z_{mt} \right)^{-\theta}}, \quad T_{int} \equiv \frac{S_{nit} w_{nt} \ell_{nt}}{w_{it} \ell_{it}} \]

- Labor market clearing

\[ \ell_{gt+1} = \sum_{i=1}^{N} D_{igt} \ell_{it}, \quad D_{igt} = \frac{\left( \exp \left( \beta \mathbb{E}_t v_{gt+1} \right) / \kappa_{git} \right)^{1/\rho}}{\sum_{m=1}^{N} \left( \exp \left( \beta \mathbb{E}_t v_{mt+1} \right) / \kappa_{mit} \right)^{1/\rho}}!, \quad E_{git} \equiv \frac{\ell_{it} D_{igt}}{\ell_{gt+1}} \]

- Capital market clearing and accumulation

\[ \frac{r_{it}}{p_{it}} = \frac{1 - \lambda}{\lambda} \frac{w_{it} \ell_{it}}{p_{it} k_{it}}, \quad k_{it+1} = (1 - \zeta_{it}) R_{it} k_{it}, \quad \zeta^{-1}_{it} = 1 + \beta^{\psi} \left( \mathbb{E}_t \left[ \frac{\psi-1}{\psi} \frac{1}{\psi} \right] \right)^{\psi} \]
Existence and Uniqueness

- Dynamic spatial model with many locations, rich geography of trade and migration costs, and two sources of dynamics

Proposition

A sufficient condition for the existence of a unique steady-state spatial distribution of economic activity \( \{\ell_i^*, k_i^*, w_i^*, R_i^*, v_i^*\} \) (up to a choice of units) given time-invariant locational fundamentals \( \{z_i^*, b_i^*, \tau_{ni}^*, \kappa_{ni}^*\} \) is that the spectral radius of a coefficient matrix \( (A) \) of model parameters \( \{\psi, \theta, \beta, \rho, \mu, \delta\} \) is less than or equal to one. \( \text{Proof} \)

- When we introduce agglomeration forces
  - Analogous condition for the existence of unique equilibrium
Dynamic Exact Hat Algebra

Proposition

Given an initial observed allocation of the economy,
\[ \left( \{ l_{i0} \}_{i=1}^N, \{ k_{i0} \}_{i=1}^N, \{ k_{i1} \}_{i=1}^N, \{ S_{n0} \}_{n,i=1}^N, \{ D_{n,-1} \}_{n,i=1}^N \right), \]
and an expected sequence of changes in fundamentals,
\[ \left\{ \{ \hat{z}_{it} \}_{i=1}^N \right\}, \left\{ \hat{b}_{it} \}_{i=1}^N \right\}, \left\{ \hat{\tau}_{ijt} \}_{i,j=1}^N, \left\{ \hat{\kappa}_{ijt} \}_{i,j=1}^N \right\}_{t=1}^\infty \]
the solution for the sequence of changes in the model’s endogenous variables does not require information on the level of fundamentals,
\[ \left\{ \{ z_{it} \}_{i=1}^N \right\}, \left\{ b_{it} \}_{i=1}^N \right\}, \left\{ \tau_{ijt} \}_{i,j=1}^N, \left\{ \kappa_{ijt} \}_{i,j=1}^N \right\}_{t=0}^\infty. \]

• Generalizes existing results for dynamic migration decisions to incorporate dynamic investment decisions

• Can undertake counterfactuals in the model without having to solve for the initial level of fundamentals

• Can invert the non-linear model to recover the unobserved shocks
\[ \left\{ \{ \hat{z}_{it} \}_{i=1}^N \right\}, \left\{ \hat{b}_{it} \}_{i=1}^N \right\}, \left\{ \hat{\tau}_{ijt} \}_{i,j=1}^N, \left\{ \hat{\kappa}_{ijt} \}_{i,j=1}^N \right\}_{t=1}^\infty \]
Linearization

- Linearize the model to characterize transition dynamics analytically
- Suppose that the economy at time $t = 0$ is on a convergence path towards an initial steady-state with constant fundamentals $(z, b, \kappa, \tau)$

1. At time $t = 0$, agents learn about one-time, permanent shocks to fundamentals $(f = \begin{bmatrix} \tilde{z} \\ \tilde{b} \end{bmatrix})$ from time $t = 1$ onwards that are revealed under perfect foresight.

2. At time $t = 0$, agents learn about a convergent sequence of shocks to fundamentals $\{f_s\} \geq 1 = \left\{ \begin{bmatrix} \tilde{z}_s \\ \tilde{b}_s \end{bmatrix} \right\} \geq 1$ from time $t = 1$ onwards that are revealed under perfect foresight.

3. Given the initial value state variables at time $t = 0 (x_0)$, suppose that productivity and amenities evolve stochastically according to an AR(1) process, and agents have rational expectations.

- Transition path: 2nd-order difference equation in state variables $(\tilde{\ell}_t, \tilde{\kappa}_t)$ that solve with method of undetermined coefficients (Uhlig 1999)
Closed-form Transition Path

Proposition

Suppose that the economy at time $t = 0$ is on a convergence path towards an initial steady-state with constant fundamentals ($z, b, \kappa, \tau$). At time $t = 0$, agents learn about one-time, permanent shocks to productivity and amenities ($\tilde{f} = \begin{bmatrix} \tilde{z} \\ \tilde{b} \end{bmatrix}$) from time $t = 1$ onwards. There exists a $2N \times 2N$ transition matrix ($P$) and a $2N \times 2N$ impact matrix ($R$) such that the second-order difference equation system has a closed-form solution of the form:

$$\tilde{x}_{t+1} = P\tilde{x}_t + R\tilde{f} \quad \text{for } t \geq 1.$$ 

where $\tilde{x}_t \equiv \begin{bmatrix} \tilde{\ell}_t \\ \tilde{k}_t \end{bmatrix}$ and a tilde denotes a log deviation from the initial steady-state: $\tilde{\ell}_t \equiv \ln \ell_t - \ln \ell_{\text{initial}}$ and $\{P, R\}$ can be recovered from the observed data $\{S, T, D, E\}$ and the structural parameters of the model $\{\theta, \beta, \rho, \lambda, \psi, \delta\}$.
Convergence Versus Fundamental Shocks

• Exact additive decomposition of the dynamics of the spatial distribution of economic activity:

\[
\ln x_t - \ln x_{-1} = \sum_{s=0}^{t} P^s (\ln x_0 - \ln x_{-1}) + \sum_{s=0}^{t-1} P^s R \tilde{f}
\]

for all \( t \geq 1 \),

- convergence given initial fundamentals
- dynamics from fundamental shocks

• With no shocks to productivity and amenities (\( \tilde{f} = 0 \)), we have:

\[
\ln x_{\text{initial}}^* = \lim_{t \to \infty} \ln x_t = \ln x_{-1} + (I - P)^{-1} (\ln x_0 - \ln x_{-1})
\]

• Using only initial state variables (for \( t = 0 \) and \( t = -1 \)) and trade and migration matrices (and hence \( P \) and \( R \)), we can compute implied steady-states with unchanged fundamentals

• Given counterfactual shocks to fundamentals (\( \tilde{f} \)), we can compute changes in steady-states, even without observing initial state variables
Spectral Analysis

• Use our linearization to characterize the economy’s transition path in terms of lower-dimensional components
• Undertake an eigendecomposition of the transition matrix

\[ P \equiv UV, \]

• where \( \Lambda \) is a diagonal matrix of eigenvalues arranged in decreasing order by absolute values, and \( V = U^{-1} \)
• For each eigenvalue \( \lambda_k \), the left-eigenvectors (\( u_k \)) and right-eigenvectors (\( \nu'_k \)) satisfy

\[ \lambda_k u_k = Pu_k, \quad \lambda_k \nu'_k = \nu'kP \]

• Define an eigen-shock as a shock to productivity and amenities (\( \tilde{f}_k \)) for which the initial impact of these shocks on the state variables (\( R\tilde{f}_k \)) coincides with a real eigenvector of the transition matrix (\( u_k \))

\[ \tilde{f}_k = R^{-1}u_k \]

• Can recover these eigen-shocks from \{S, T, D, E\} and \{\theta, \beta, \rho, \lambda, \psi, \delta\}
Proposition

Consider an economy that is initially in steady-state at \( t = 0 \) when agents learn about one-time, permanent shocks to productivity and amenities \( \tilde{f} = \begin{bmatrix} \tilde{z} \\ \tilde{b} \end{bmatrix} \) from \( t = 1 \) onwards. Suppose the initial impact of the shock to fundamentals at \( t = 1 \) coincides with an eigenvector \( (R\tilde{f} = u_k) \) of the transition matrix \( (P) \) (eigen-shock). The transition path of the state variables \( (\tilde{x}_t \equiv \begin{bmatrix} \tilde{\ell}_t \\ \tilde{k}_t \end{bmatrix}) \) reduces to:

\[
\tilde{x}_t = \frac{1 - \lambda_k^t}{1 - \lambda_k} u_k,
\]

and the half-life is given by:

\[
t_i^{(1/2)}(\tilde{f}) = -\left\lfloor \frac{\ln 2}{\ln \lambda_k} \right\rfloor
\]

for all state variables \( i = 1, \cdots, 2N \), where \( \lfloor \cdot \rfloor \) is the ceiling function.
Outline

• Dynamic Spatial Model

• Extensions
  – Trade deficits
  – Shocks to trade and migration costs
  – Agglomeration and dispersion forces
  – Housing capital
  – Multi-sector
  – Multi-sector and input-output linkages

• Data

• Empirical Results

• Conclusions
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Data

• Two empirical implementations
  – State-time data from 1965-2015 (decline Rust Belt and rise Sun Belt)
  – State-industry-time data from 1999-2015

• U.S. State GDP, population and capital stock
  – Bureau of Economic Analysis (BEA) 1965-2015

• Bilateral value of shipments between U.S. states
  – Commodity Flow Survey (CFS)
  – Commodity Transportation Survey (CTS)

• Bilateral migration flows between U.S. states
  – Population census and American Community Survey (ACS) 1960-2010
  – Five-year migration matrices

• Foreign imports and exports of U.S. states
  – Foreign exports by origin of movement (OM) state 1999-2015
  – Foreign imports by state of destination (SD) 1999-2015
Outline

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- Conclusions
Income Convergence 1963-80

Note: Slope coefficient: -0.0236; standard error: 0.0038; R-squared: 0.4758.
Income Convergence 1980-2000

Note: Slope coefficient: -0.0135; standard error: 0.0039; R-squared: 0.2092.
Income Convergence 2000-2017

Note: Slope coefficient: 0.0095; standard error: 0.0061; R-squared: 0.0712.
Importance of Initial Conditions

- Much of the decline in the speed of convergence in income per capita can be explained by initial conditions
• Capital adjustment important for dynamics of income per capita
• Migration important for dynamics of population
Spectral Analysis
Half-lifes

Note: Half-life Corresponds to the time in years for the state variables to converge half of the way towards steady-state for a shock to productivity and amenities for which its initial impact on the state variables ($R\tilde{f}$) corresponds to an eigenvector ($u_k$) of the transition matrix ($P$); figure shows mean and maximum half-life across eigenvectors of the transition matrix in each year from 1965-2015.
• **Eigen-shock**: shock to productivity and amenities ($\tilde{f}_k$) for which the initial impact of these shocks on the state variables coincides with a real eigenvector of the transition matrix: $u_k = R\tilde{f}_k$
Correlation Steady-State Gaps Over Time
• **Eigen-shock**: shock to productivity and amenities ($\tilde{f}_k$) for which the initial impact of these shocks on the state variables coincides with a real eigenvector of the transition matrix: $\tilde{f}_k = R^{-1}u_k$
Correlation Shocks Over Time (Shocks)
Parameters and Speed of Convergence
Non-Linear Solution and Linearization

• Invert non-linear model (prod., amenities, trade & migration costs)
• Start from steady-state implied by these 1990 fundamentals
• Shock by vector of productivity shocks 1990-2000
• Compare transition paths in our linearization and non-linear model
Conclusions

• How does the spatial distribution of economic activity respond to local shocks? (e.g. productivity, transport infrastructure, trade)

• A key challenge is modelling forward-looking capital investments in quantitative spatial models with population mobility – Interaction investment and migration in all locations and time periods

• We make four main contributions:
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  4. Applications: US state data 1965-2015; state-industry data 1999-2015 – Decline in rate of income convergence over time ($\beta$-convergence) – Slow convergence and heterogeneous impact of shocks – Heterogeneity explained by the interaction of capital and labor dynamics
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Thank You
Population Gap from Steady-State

Predictive Power Initial Steady-State

Note: Slope coefficient: 0.8709; standard error: 0.1081; R-squared: 0.5035.

- Robust to controlling for initial log population and capital stock and initial log population growth
Non-linear Model Inversion

- Parameters: $\beta = 0.95$, $\theta = 5$, $\rho = 3\beta$, $\lambda = 0.65$, $(1 - \delta) = 0.95$
- Recover unobserved fundamentals from the non-linear model
  - Economy can be anywhere on transition path / in steady-state
  - Assume perfect foresight but allow any expected path fundamentals

\[
\frac{S_{nit}S_{int}}{S_{nnt}S_{iit}} = \left( \frac{\tau_{nit}\tau_{int}}{\tau_{nnt}\tau_{iit}} \right)^{-\theta} = (\tau_{nit})^{-2\theta}, \quad \frac{D_{igt}D_{git}}{D_{ggt}D_{iit}} = \left( \frac{\kappa_{git}\kappa_{igt}}{\kappa_{ggt}\kappa_{iit}} \right)^{-1/\rho} = (\kappa_{git})^{-2/\rho}
\]

\[
\ell_{it} = \sum_{n=1}^{N} \frac{\left( w_{it} \left( \ell_{it} / k_{it} \right)^{1-\lambda} \tau_{nit} / z_{it} \right)^{-\theta}}{\sum_{m=1}^{N} \left( w_{mt} \left( \ell_{mt} / k_{mt} \right)^{1-\lambda} \tau_{nmt} / z_{mt} \right)^{-\theta}} \ell_{nt} + \ln b_{it} = (\nu_{it} - \nu_{it+1}) + (1 - \beta) \nu_{it+1} - \ln \left( \frac{S_{iit}^{-1}}{(D_{iit})^{1/\rho}} \right) - \ln z_{it}
\]

- Intuition: migration flows capture expectations
Steady-state Comparative Statics

• To begin with, start at steady-state (relax later): \{w^*_i, v^*_i, \ell^*_i, k^*_i\}

• Consider \(d \ln z \neq 0, d \ln b \neq 0, \) and \(d \ln \tau = d \ln \kappa = d \ln \bar{\ell} = 0\)

\[
\begin{align*}
    d \ln k^* &= \underbrace{d \ln \ell^*}_{\text{change in population}} + \underbrace{d \ln w^*}_{\text{change in wages}} - \underbrace{d \ln p^*}_{\text{change in the price index}} \\
    d \ln p^* &= S \left[ d \ln w^* - (1 - \lambda) \left( d \ln k^* - d \ln \ell^* \right) - d \ln z \right] \\
    d \ln w^* + d \ln \ell^* &= T \left( d \ln w^* + d \ln \ell^* \right) + \theta (TS - I) \left[ d \ln w^* - (1 - \lambda) \left( d \ln k^* - d \ln \ell^* \right) - d \ln z \right] \\
    d \ln \ell^* &= E d \ln \ell^* + \frac{\beta}{\rho} (I - ED) d v^* \\
    d v^* &= d \ln b + d \ln w^* - d \ln p^* + \beta D d v^*
\end{align*}
\]
Steady-state Comparative Statics

• Totally differentiating the general equilibrium conditions of the model and stacking them in matrix form

Proposition

The steady-state response of the endogenous variables to productivity and amenity shocks satisfies the linear system:

\[
\begin{bmatrix}
\frac{d \ln \ell^*}{d \ln z_n} \\
\frac{d \ln k^*}{d \ln z_n} \\
\frac{d \ln w^*}{d \ln z_n} \\
\frac{d \ln v^*}{d \ln z_n}
\end{bmatrix}
= \begin{bmatrix}
L^{z*} \\
K^{z*} \\
W^{z*} \\
V^{z*}
\end{bmatrix}
\frac{d \ln z}{d \ln z} + \begin{bmatrix}
L^{b*} \\
K^{b*} \\
W^{b*} \\
V^{b*}
\end{bmatrix}
\frac{d \ln b}{d \ln b}
\]

where the \( N \times N \) matrices \( \{L^{z*}, K^{z*}, W^{z*}, V^{z*}, L^{b*}, K^{b*}, W^{b*}, V^{b*}\} \) are functions of the four observed matrices of expenditure shares (\( S \)), income shares (\( T \)), outmigration shares (\( D \)) and immigration shares (\( E \)) and the structural parameters of the model \( \{\beta, \theta, \rho, \lambda, \delta\} \).

• Element \( [L^{z*}]_{in} = \frac{d \ln \ell^*_i}{d \ln z_n} \)
  - Elasticity of steady-state population in location \( i \) (\( \ell^*_i \)) with respect to an increase in productivity in location \( n \) (\( z_n \))
Approximation Quality (Steady-State)

- Start from steady-state implied by 1990 fundamentals
- Shock by vector of productivity shocks 1990-2000
- Compare steady-state changes in our linearization & non-linear model

![Graph showing approximated solution vs. relative change in population]
Approximation Quality (Transition)

- Start from steady-state implied by 1990 fundamentals
- Shock by vector of productivity shocks 1990-2000
- Compare transition paths in our linearization and non-linear model
CRRA Utility

- Landlords’ intertemporal utility

\[ \nu_{it}^k = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^{t+s} \frac{(c_{it+s}^k)^{1-1/\psi}}{1 - 1/\psi} \]

- Budget constraint

\[ r_{it} k_{it} = p_{it} \left( c_{it}^k + k_{it+1} - (1 - \delta) k_{it} \right) \]

- Gross return on capital: \( R_{it} \equiv 1 - \delta + r_{it} / p_{it} \)

- Optimal savings rate

\[ k_{it+1} = (1 - \zeta_{it}) R_{it} k_{it} \]

\[ \zeta_{it}^{-1} = 1 + \beta^\psi \left( \mathbb{E}_t \left[ R_{it+1}^\frac{\psi-1}{\psi} \zeta_{t+1}^{-1/\psi} \right] \right)^\psi \]

- (compare with log utility, where \( k_{it+1} = \beta R_{it} k_{it} \))
Intertemporal Consumption-Investment

- Intertemporal optimization problem

\[ \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln c_{it} - \mu_t \left[ p_{it} c_{it}^k + p_{it} (k_{it+1} - (1 - \delta) k_{it}) - r_{it} k_{it} \right] \]

- Euler equation

\[ \frac{c_{it+1}^k}{c_{it}^k} = \beta \left( \frac{r_{it+1}}{p_{it+1}} + (1 - \delta) \right) \]

- Conjecture policy functions

\[ p_{it} c_{it}^k = (1 - \beta) (r_{it} + p_{it} (1 - \delta)) k_{it} \]

\[ k_{it+1} = \beta \left( \frac{r_{it}}{p_{it}} + (1 - \delta) \right) k_{it} \]

- Confirm that this conjecture satisfies the Euler equation
Constant Perceived Return to Capital

• Profit maximization and zero profits

\[ w_{it} = (1 - \lambda) p_{it} z_{it} \left( \frac{k_{it}}{\ell_{it}} \right)^\lambda \]

\[ r_{it} = \lambda p_{it} z_{it} \left( \frac{k_{it}}{\ell_{it}} \right)^{\lambda - 1} \]

• Landlord income

\[ \Pi_{it} = r_{it} k_{it} = p_{it} z_{it} k_{it}^\lambda \ell_{it}^{1-\lambda} - w_{it} \ell_{it} \]

• Using profit maximization and zero profits, landlord income is

\[ \Pi_{it} = \lambda \left( p_{it} z_{it} \right)^{\frac{1}{\lambda}} \left( \frac{1 - \lambda}{w_{it}} \right)^{\frac{1-\lambda}{\lambda}} k_{it} \]
Investment Other Locations

- Realized rate of return to a landlord in location $n$ from allocating one unit of capital to location $i$ is:
  \[ \nu_{nit} = \frac{\alpha_{nit} r_{it}}{\phi_{nit}} \]

- Marginal efficiency of capital in $i$ drawn from Fréchet distribution
  \[ F_{nit}(\alpha) = e^{-\left(\frac{\alpha}{a_{it}}\right)^{-\epsilon}}, \quad a_{it} > 0, \quad \epsilon > 1 \]

- Capital from $n$ allocated to $i$
  \[ b_{nit} = \frac{k_{nit}}{k_{nt}} = \frac{(a_{it} r_{it} / \phi_{nit})^\epsilon}{\sum_{h=1}^{N} (a_{ht} r_{ht} / \phi_{nht})^\epsilon} \]

- Realized rate of return on capital owned by source location $n$ at time $t$ is the same across all host locations $i$ and given by
  \[ \nu_{nit} = \nu_{nt} = \Gamma \left( \frac{\epsilon - 1}{\epsilon} \right) \left[ \sum_{h=1}^{N} (a_{ht} r_{ht} / \phi_{nht})^\epsilon \right]^{\frac{1}{\epsilon}} \]
**Definition**

Given the state variables \( \{ \ell_{i0}, k_{i0} \} \) in each location in an initial period \( t = 0 \), a *sequential equilibrium* of the economy is a set of wages, expected values, mass of workers and stock of capital in each location in all subsequent time periods \( \{ w_{it}, v_{it}, \ell_{it}, k_{it} \}_{t=0}^{\infty} \) that solves the value function, the labor market clearing condition, the goods market clearing condition, and the capital market clearing and accumulation condition.

**Definition**

A *steady-state* of the economy is an equilibrium in which all location-specific variables (wages, expected values, mass of workers and stock of capital in each location) are time invariant: \( \{ w_i^*, v_i^*, \ell_i^*, k_i^* \} \).
Existence and Uniqueness

• The steady-state equilibrium \( \{ p_i^*, w_i^*, \ell_i^*, \phi_i^* \} \) solves:

\[
(p_i^*)^{-\theta} = \sum_{n=1}^{N} \psi \tilde{\tau}_{in} (p_n^*)^{-\theta(1-\lambda)} (w_n^*)^{-\theta \lambda},
\]

\[
(p_i^*)^{\theta(1-\lambda)} (w_i^*)^{1+\theta \lambda} \ell_i^* = \sum_{n=1}^{N} \psi \tilde{\tau}_{ni} (p_n^*)^{\theta} w_n^* \ell_n^*,
\]

\[
(p_i^*)^{\beta/\rho} (w_i^*)^{-\beta/\rho} \ell_i^* (\phi_i^*)^{-\beta} = \sum_{n=1}^{N} \tilde{\kappa}_{in} \ell_n^* (\phi_n^*)^{-1},
\]

\[
\phi_i^* = \sum_{n=1}^{N} \tilde{\kappa}_{ni} (p_n^*)^{-\beta/\rho} (w_n^*)^{\beta/\rho} (\phi_n^*)^{\beta},
\]

where

\[
\psi \equiv \left( \frac{1-\beta (1-\delta)}{\beta} \right)^{-\theta(1-\lambda)}, \quad \tilde{\tau}_{ni} \equiv (\tau_{ni}/z_i)^{-\theta},
\]

\[
\phi_i^* \equiv \sum_{n=1}^{N} \tilde{\kappa}_{ni} \exp \left( \frac{\beta}{\rho} v_n^{w*} \right), \quad \tilde{\kappa}_{in} \equiv \left( \kappa_{in}/b_n^\beta \right)^{-1/\rho}.
\]
Existence and Uniqueness

• This system of equations falls within the class for which Theorem 1 of Allen, Arkolakis and Li (2020) applies:

\[
\Lambda = \begin{bmatrix}
-\theta & 0 & 0 & 0 \\
\theta (1 - \lambda) & (1 + \theta \lambda) & 1 & 0 \\
\beta / \rho & -\beta / \rho & 1 & -\beta \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

\[
\Gamma = \begin{bmatrix}
-\theta (1 - \lambda) & -\theta \lambda & 0 & 0 \\
\theta & 1 & 1 & 0 \\
0 & 0 & 1 & -1 \\
-\beta / \rho & \beta / \rho & 0 & \beta \\
\end{bmatrix}.
\]
Dynamic Exact Hat Algebra

• Given an observed initial allocation
  \( \{ l_{i0} \}_{i=1}^{N}, \{ k_{i0} \}_{i=1}^{N}, \{ k_{i1} \}_{i=1}^{N}, \{ S_{ni0} \}_{n,i=1}^{N}, \{ D_{ni,-1} \}_{n,i=1}^{N} \)  

  \[ \hat{D}_{igt+1} = \frac{D_{igt} \left( \hat{u}_{gt+2} / \hat{\kappa}_{git+1} \right)^{1/\rho}}{\sum_{m=1}^{N} D_{imt} \left( \hat{u}_{mt+2} / \hat{\kappa}_{mit+1} \right)^{1/\rho}} \]

  \[ \hat{u}_{it+1} = \left( \frac{\hat{b}_{it+1}}{\hat{p}_{it+1}} \right)^{\beta} \left( \sum_{g=1}^{N} \left( \hat{u}_{gt+2} / \hat{\kappa}_{git+1} \right)^{1/\rho} \right)^{\beta\rho} \]

  \[ \hat{p}_{it+1} = \left( \sum_{m=1}^{N} S_{imt} \left( \hat{\tau}_{imt+1} \hat{w}_{mt+1} \left( \hat{l}_{mt+1} / \hat{\kappa}_{mt+1} \right)^{1-\mu} / \hat{z}_{mt+1} \right)^{-\theta} \right)^{-1/\theta} \]

  \[ \ell_{gt+1} = \sum_{i=1}^{N} D_{igt} \ell_{it} \]

• where \( u_{it} = \exp(v_{it}) \) and \( \hat{u}_{it+1} = u_{it+1} / u_{it} \)
Dynamic Exact Hat Algebra

\[ \hat{w}_{it+1} \hat{l}_{it+1} = \sum_{n=1}^{N} \frac{S_{nit+1} w_{nt} \ell_{nt}}{\sum_{k=1}^{N} S_{kit} w_{kt} \ell_{kt}} \hat{w}_{nt+1} \hat{l}_{nt+1} \]

\[ \hat{S}_{nit+1} \equiv \frac{S_{nit+1} \left( \hat{\tau}_{nit+1} \hat{w}_{it+1} \left( \hat{l}_{it+1} / \hat{k}_{it+1} \right)^{1-\mu} / \hat{z}_{it+1} \right)^{-\theta}}{\sum_{k=1}^{N} S_{nkt+1} \left( \hat{\tau}_{nkt+1} \hat{w}_{kt+1} \left( \hat{l}_{kt+1} / \hat{k}_{kt+1} \right)^{1-\mu} / \hat{z}_{kt+1} \right)^{-\theta}} \]

\[ \zeta_{it+1} = \beta R_{it+1}^{\psi-1} \frac{\zeta_{it}}{1 - \zeta_{it}} \]

\[ k_{it+1} = (1 - \zeta_{it}) R_{it} k_{it} \]

\[ (R_{it} - (1 - \delta)) = \frac{\hat{p}_{it+1} \hat{k}_{it+1}}{\hat{w}_{it+1} \hat{l}_{it+1}} (R_{it+1} - (1 - \delta)) \]
Transition Dynamics Decomposition

- Transition dynamics decomposition

\[ \tilde{x}_t = \tilde{P} \tilde{x}_{t-1} + \tilde{R} \tilde{f} \]
\[ \tilde{x}_{t-1} = \tilde{P} \tilde{x}_{t-2} + \tilde{R} \tilde{f} \]
\[ \vdots \]
\[ \tilde{x}_1 = \tilde{P} \tilde{x}_0 + \tilde{R} \tilde{f} \]
\[ \tilde{x}_0 = \tilde{P} \tilde{x}_{-1} \]

- Taking the difference between time \( t \) and \( t-1 \)

\[ \ln x_t - \ln x_{t-1} = P (\ln x_{t-1} - \ln x_{t-2}) \]
\[ \vdots \]
\[ = P^{t-1} (\ln x_1 - \ln x_0) \]
\[ = P^t (\ln x_0 - \ln x_{-1}) + P^{t-1} \tilde{R} \tilde{f} \]
We thus obtain:

\[
\ln x_t - \ln x_{t-1} = \left[ \ln x_t - \ln x_{t-1} \right] + \left[ \ln x_{t-1} - \ln x_{t-2} \right] + \cdots + \left[ \ln x_1 - \ln x_0 \right] + \left[ \ln x_0 - \ln x_{t-1} \right]
\]

\[
= \left[ P^t (\ln x_0 - \ln x_{t-1}) + P^{t-1} \tilde{Rf} \right] + \left[ P^{t-1} (\ln x_0 - \ln x_{t-1}) + P^{t-2} \tilde{Rf} \right] + \cdots + \left[ P (\ln x_0 - \ln x_{t-1}) + \tilde{Rf} \right] + \left[ \ln x_0 - \ln x_{t-1} \right]
\]

\[
= \sum_{s=0}^{t} P^s (\ln x_0 - \ln x_{t-1}) + \sum_{s=0}^{t-1} P^s \tilde{Rf}
\]
Any Convergent Sequence

Proposition

Consider an economy that is initially in steady-state at time $t = 0$ when agents learn about a convergent sequence of future shocks to productivity and amenities

$$\left\{ \tilde{f}_s \right\}_{s \geq 1} = \left\{ \begin{bmatrix} \tilde{z}_s \\ \tilde{b}_s \end{bmatrix} \right\}_{s \geq 1}$$

that is revealed under perfect foresight from time $t = 1$ onwards.

There exists a $2N \times 2N$ transition matrix ($P$) and a $2N \times 2N$ impact matrix ($R$) such that the dynamic path of state variables relative to the initial steady-state follows:

$$\tilde{x}_t = \sum_{s=t+1}^{\infty} (\Psi^{-1} \Gamma - P)^{-(s-t)} R \left( \tilde{f}_s - \tilde{f}_{s-1} \right) + R \tilde{f}_t + P \tilde{x}_{t-1} \quad \text{for all } t \geq 1,$$

with initial condition $\tilde{x}_0 = 0$ and where $\Psi$, $\Gamma$ are matrices from our solution to the second-order difference equation
Stochastic Fundamentals

• Productivity and amenities evolve stochastically over time according to the following AR(1) structure:

\[
\ln z_{it+1} - \ln z_{it} = \rho^z (\ln z_{it} - \ln z_{it-1}) + \omega^z_{it}, \quad |\rho^z| < 1,
\]

\[
\ln b_{it-1} - \ln b_{it} = \rho^b (\ln b_{it} - \ln b_{it-1}) + \omega^b_{it}, \quad |\rho^b| < 1,
\]

• Agents expect future shocks to fundamentals to decay to zero:

\[
\mathbb{E}_t [\tilde{z}_{it+s} - \tilde{z}_{it+s-1}] = (\rho^z)^s (\tilde{z}_{it} - \tilde{z}_{it-1}),
\]

\[
\mathbb{E}_t [\tilde{b}_{it+s} - \tilde{b}_{it+s-1}] = (\rho^b)^s (\tilde{b}_{it} - \tilde{b}_{it-1}),
\]

• Closed-form solution for the economy’s transition path

\[
\mathbb{E}_1 [\tilde{x}_t] = \sum_{s=t+1}^{\infty} (\Psi^{-1} \Gamma - P)^{-(s-t)} R \left( \mathbb{E}_1 [\tilde{f}_s - \tilde{f}_{s-1}] \right) + R \mathbb{E}_1 [\tilde{f}_t] + P \mathbb{E}_1 [\tilde{x}_{t-1}]
\]
Eigendecomposition

- Eigendecomposition of transition dynamics

\[ P = U\Lambda V, \quad \text{and hence} \quad P^s = \sum_{k=1}^{2N} \lambda_k^s u_k v_k' \]

\[ \tilde{x}_t = \sum_{s=0}^{t-1} P^s \tilde{R}f \]
\[ = \sum_{s=0}^{t-1} \left( \sum_{k=1}^{2N} \lambda_k^s u_k v_k' \right) \tilde{R}f \]
\[ = \sum_{k=1}^{2N} \left( \sum_{s=0}^{t-1} \lambda_k^s \right) u_k v_k' \tilde{R}f \]
\[ = \sum_{k=1}^{2N} \left( \frac{1 - \lambda_k^t}{1 - \lambda_k} \right) u_k v_k' \tilde{R}f \]
Eigendecomposition

- Eigendecomposition of transition dynamics

\[ P = U \Lambda V, \quad \text{and hence} \quad P^s = \sum_{k=1}^{2N} \lambda_k^s u_k v_k' \]

\[
\tilde{x}_t = \sum_{s=0}^{t-1} P^s \tilde{Rf} \\
= \sum_{s=0}^{t-1} \left( \sum_{k=1}^{2N} \lambda_k^s u_k v_k' \right) \tilde{Rf} \\
= \sum_{k=1}^{2N} \left( \sum_{s=0}^{t-1} \lambda_k^s \right) u_k v_k' \tilde{Rf} \\
= \sum_{k=1}^{2N} \left( \frac{1-\lambda_k^t}{1-\lambda_k} \right) u_k v_k' \tilde{Rf}
\]

\[
v_k' \tilde{Rf} = v_k' \sum_{i=1}^{2N} a_i \tilde{Rf}_i = \sum_{i=1}^{2N} a_i v_k' u_i = a_k
\]

\[
a = VR\tilde{f} = U^{-1} \tilde{Rf} = (R^{-1}U)^{-1} \tilde{f} \\
= (R^{-1}U)^{-1} \left( (R^{-1}U)^T \right)^{-1} \left( (R^{-1}U)^T \right) \tilde{f} \\
= \left( (R^{-1}U)^T (R^{-1}U) \right)^{-1} (R^{-1}U)^T \tilde{f}
\]
Speed of Convergence

- Suppose that $\widetilde{Rf}$ coincides with a real eigenvector: $\widetilde{Rf} = u_k$

$$\tilde{x}_t = \sum_{j=1}^{2N} \left( \frac{1 - \lambda_j^t}{1 - \lambda_j} \right) u_j v'_j \widetilde{Rf} = \sum_{j=1}^{2N} \frac{1 - \lambda_j^t}{1 - \lambda_j} u_j v'_j u_k = \frac{1 - \lambda_k^t}{1 - \lambda_k} u_k$$

- where we have used $UV' = UU^{-1} = I$
- Taking differences between periods $t + 1$ and $t$, we have:

$$\tilde{x}_{t+1} - \tilde{x}_t = \frac{1 - \lambda_{k+1}^t}{1 - \lambda_k} u_k - \frac{1 - \lambda_k^t}{1 - \lambda_k} u_k$$

which simplifies to:

$$(1 - \lambda_k) (\tilde{x}_{t+1} - \tilde{x}_t) = (1 - \lambda_k) \lambda_k^t u_k$$

and hence:

$$\tilde{x}_{t+1} - \tilde{x}_t = \lambda_k^t u_k$$
Speed of Convergence

- Noting that $\tilde{x}_t = \ln x_t - \ln x^\text{initial}$, we have:

$$\ln x_{t+1} - \ln x_t = \lambda^t_k u_k$$

- which implies exponential convergence to steady-state, such that for each location $i$:

$$\frac{x_{it+1}}{x_{it}} = \exp (\lambda^t_k u_{ik})$$

- We can solve for the half-life as:

$$\frac{1 - \lambda^t_k}{1 - \lambda_k} u_k = \frac{1}{2}$$

- which simplifies to:

$$\lambda^t_k = \frac{1}{2}$$

- and hence:

$$\ln \frac{1}{2} = t \ln \lambda_k, t = -\frac{\ln 2}{\ln \lambda_k}$$
## Predictive Power Initial Steady-State

<table>
<thead>
<tr>
<th>Outcome: 1965-2015 Pop. Log Growth</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1965-2015 Pop. Predicted Log Growth</td>
<td>0.871*** (0.108)</td>
<td>0.959*** (0.0780)</td>
<td>0.934*** (0.0674)</td>
<td>0.903*** (0.0846)</td>
</tr>
<tr>
<td>Log 1965 Population</td>
<td></td>
<td>-0.130*** (0.0326)</td>
<td>-0.124*** (0.0357)</td>
<td>-0.126*** (0.0381)</td>
</tr>
<tr>
<td>Log 1965 K-L Ratio</td>
<td></td>
<td></td>
<td>0.139 (0.175)</td>
<td>0.130 (0.185)</td>
</tr>
<tr>
<td>1965-1966 Growth Rate</td>
<td></td>
<td></td>
<td></td>
<td>2.417 (4.122)</td>
</tr>
</tbody>
</table>

| N | 49 | 49 | 49 | 49 |
| R² | 0.503 | 0.605 | 0.616 | 0.617 |