Dynamic Spatial General Equilibrium

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- How does the spatial distribution of economic activity respond to local shocks? (e.g. productivity, transport infrastructure, trade)
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4. Apply our framework to examine income convergence across U.S. states over time (both capital dynamics and labor mobility)

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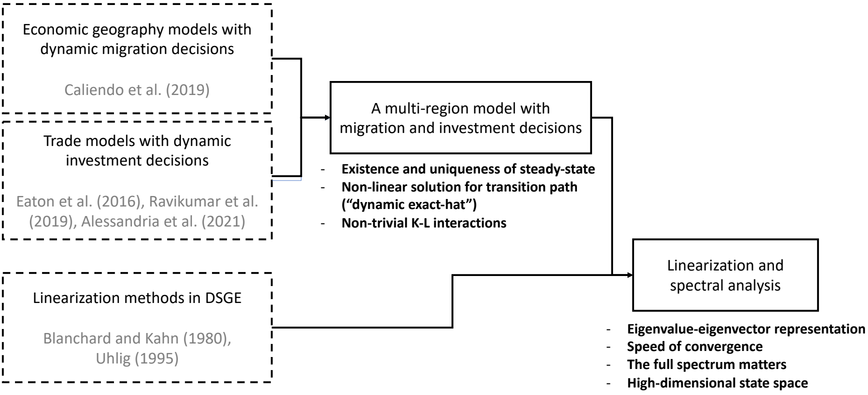
- Applications: US state data 1965-2015; state-industry data 1999-2015
  - Decline in rate of income convergence over time (β-convergence)
  - Slow convergence and heterogeneous impact of shocks
  - Heterogeneity explained by the interaction of capital and labor dynamics

Related Literature

- Theoretical work on economic geography
  - Krugman (1991, 1992), Helpman (1998), Fujita et al. (1999), Baldwin (2001)
- Static quantitative spatial trade models
  - Armington (1969), Eaton & Kortum (2002), Redding & Sturm (2008), Allen & Arkolakis (2014), Ramondo et al. (2016), Redding (2016), Donaldson (2018), Caliendo et al. (2018), Fajgelbaum et al. (2019), Fajgelbaum & Gaubert (2020)
- Dynamic models of capital accumulation in international trade
  - Anderson, Larch & Yotov 2015, Eaton, Kortum, Neiman & Romalis 2016, Alvarez 2017, Ravikumar, Santacreu & Sposi 2019, Alessandria, Choi & Ruhl 2021
- Dynamic models of trade and geography with labor mobility
  - Artuç et al. (2010), Desmet & Rossi-Hansberg (2014), Desmet et al. (2018), Caliendo et al. (2019), Caliendo & Parro (2020), Peters (2019), Peters & Walsh (2019), Walsh (2019), Allen & Donaldson (2020), Greaney (2020)
- Research on sufficient statistics for welfare in static trade models
  - Arkolakis et al. (2012), Adão et al. (2017), Adão et al. (2019), Baqaee & Farhi (2019), Huo et al. (2019), Barthelme et al. (2019), Kleinman et al. (2020), Bilal (2021)
- Regional income convergence and persistence of local shocks
  - Barro & Sala-i-Martin (1992), Blanchard & Katz (1992), Kim (1995), Mitchener & McLean (1999), Feyrer et al. (2007), Kovak (2013), Autor et al. (2013, 2020), Dix-Carneiro & Kovak (2017), Ganong & Shoag (2017), Alder et al. (2019)

Related Literature



Outline

- Dynamic Spatial Model
- Extensions
- Data
- Empirical Results
- Conclusions

Model Setup

- Multi-location, single-sector Armington model (extensions later)
- Economy consists of a set of locations $i \in \{1, \ldots, N\}$
- Locations differ in productivity, amenities, bilateral goods trade costs, and bilateral migration costs
- Two types of agents: workers and landlords
- Continuum of workers
  - Endowed with one unit of labor
  - Geographically mobile subject to migration costs
  - No savings-investment technology ("hand to mouth")
  - Make dynamic forward-looking migration decisions to maximize intertemporal utility
- Continuum of landlords in each location
  - Own the stock of local capital
  - Geographically immobile
  - Make dynamic forward-looking consumption-investment choices to maximize intertemporal utility

Worker Migration (CDP)

- At the beginning of period t, mass of workers ℓ_{it} in location i:
  - Produce and consume
  - Observe extreme value idiosyncratic mobility shocks $\{\epsilon_{gt}\}$
  - Choose optimal location for period $t+1$ given mobility costs κ_{git}
- Expected value of living in location i in period t depends on wage (w_{it}), cost of living (p_{it}), amenities (b_{it}) and the expected value of optimal location choice

$$v_{it} = \ln\left(\frac{w_{it}}{p_{it}}\right) + \ln b_{it} + \rho \ln \sum_{g=1}^{N} \left(\exp\left(\beta \mathbb{E}_t v_{gt+1}\right) / \kappa_{git}\right)^{1/\rho}$$

- Location choice probabilities

$$D_{igt} = \frac{\left(\exp\left(\beta \mathbb{E}_t v_{gt+1}\right) / \kappa_{git}\right)^{1/\rho}}{\sum_{k=1}^{N} \left(\exp\left(\beta \mathbb{E}_t v_{kt+1}\right) / \kappa_{kit}\right)^{1/\rho}}$$

- Population flow condition

$$\ell_{gt+1} = \sum_{i=1}^{N} D_{igt} \ell_{it}$$

Trade and Production

- Armington differentiation of goods by location of origin

$$p_{nt} = \left[\sum_{i=1}^{N} p_{nit}^{-\theta} \right]^{-1/\theta}, \qquad \theta = \sigma - 1, \qquad \sigma > 1$$

- Competitive production and iceberg trade costs $\tau_{nit} \geq 1$
- Cost in location n of sourcing a variety from location i is

$$p_{nit} = \frac{\tau_{nit} w_{it}^{\lambda} r_{it}^{1-\lambda}}{z_{it}}, \qquad 0 < \lambda < 1$$

- Using profit maximization to substitute for equilibrium labor input, landlord income is linear in capital

$$\Pi_{it} = \lambda \left(p_{it} z_{it} \right)^{\frac{1}{\lambda}} \left(\frac{1-\lambda}{w_{it}} \right)^{\frac{1-\lambda}{\lambda}} k_{it}$$

Landlord Investment

- Landlords optimal intertemporal consumption-investment decision

$$v_{it}^k = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^{t+s} \frac{\left(c_{it+s}^k\right)^{1-1/\psi}}{1-1/\psi}$$

- Landlords in a location can produce one unit of capital in that location using one unit of the local consumption index
- Local capital is geographically immobile once installed (buildings and structures) and depreciates at constant rate δ
- Intertemporal budget constraint

$$r_{it} k_{it} = p_{it} c_{it}^k + p_{it} \left(k_{it+1} - (1-\delta) k_{it}\right)$$

- CRRA preferences and linear income in capital imply linear saving rate (as Angeletos 2007 and Moll 2014) ▸ more ▸ rtrans ▸ grav

$$k_{it+1} = (1-\varsigma_{it}) R_{it} k_{it}, \qquad R_{it} \equiv 1 - \delta + r_{it}/p_{it}$$

$$\varsigma_{it}^{-1} = 1 + \beta^{\psi} \left(\mathbb{E}_t \left[R_{it+1}^{\frac{\psi-1}{\psi}} \varsigma_{t+1}^{-\frac{1}{\psi}} \right] \right)^{\psi}$$

General Equilibrium

- Value function

$$v_{it} = \ln\left(\frac{w_{it}}{p_{it}}\right) + \ln b_{it} + \rho \ln \sum_{g=1}^{N} \left(\exp\left(\beta \mathbb{E}_t v_{gt+1}\right) / \kappa_{git}\right)^{1/\rho}$$

$$p_{nt} = \left[\sum_{i=1}^{N} \left(w_{it} \left(\frac{1-\lambda}{\lambda}\right)^{1-\lambda} \left(\ell_{it} / k_{it}\right)^{1-\lambda} \tau_{nit} / z_{it}\right)^{-\theta}\right]^{-1/\theta}$$

- Goods market clearing

$$w_{it}\ell_{it} = \sum_{n=1}^{N} S_{nit} w_{nt}\ell_{nt}, \quad S_{nit} \equiv \frac{\left(w_{it}\left(\ell_{it}/k_{it}\right)^{1-\lambda} \tau_{nit}/z_{it}\right)^{-\theta}}{\sum_m \left(w_{mt}\left(\ell_{mt}/k_{mt}\right)^{1-\lambda} \tau_{nmt}/z_{mt}\right)^{-\theta}}, \quad T_{int} \equiv \frac{S_{nit} w_{nt}\ell_{nt}}{w_{it}\ell_{it}}$$

- Labor market clearing

$$\ell_{gt+1} = \sum_{i=1}^{N} D_{igt}\ell_{it}, \quad D_{igt} = \frac{\left(\exp\left(\beta \mathbb{E}_t v_{gt+1}\right) / \kappa_{git}\right)^{1/\rho}}{\sum_{m=1}^{N} \left(\exp\left(\beta \mathbb{E}_t v_{mt+1}\right) / \kappa_{mit}\right)^{1/\rho}}, \quad E_{git} \equiv \frac{\ell_{it} D_{igt}}{\ell_{gt+1}}$$

- Capital market clearing and accumulation

$$\frac{r_{it}}{p_{it}} = \frac{1-\lambda}{\lambda} \frac{w_{it}}{p_{it}} \frac{\ell_{it}}{k_{it}}, \quad k_{it+1} = \left(1 - \varsigma_{it}\right) R_{it} k_{it}, \quad \varsigma_{it}^{-1} = 1 + \beta^{\psi} \left(\mathbb{E}_t \left[R_{it+1}^{\frac{\psi-1}{\psi}} \varsigma_{t+1}^{-\frac{1}{\psi}}\right]\right)^{\psi}$$

Existence and Uniqueness

- Dynamic spatial model with many locations, rich geography of trade and migration costs, and two sources of dynamics

Proposition

A sufficient condition for the existence of a unique steady-state spatial distribution of economic activity $\{\ell_i^*, k_i^*, w_i^*, R_i^*, v_i^*\}$ *(up to a choice of units) given time-invariant locational fundamentals* $\{z_i^*, b_i^*, \tau_{ni}^*, \kappa_{ni}^*\}$ *is that the spectral radius of a coefficient matrix (**A**) of model parameters* $\{\psi, \theta, \beta, \rho, \mu, \delta\}$ *is less than or equal to one.* ▸ Proof

- When we introduce agglomeration forces
  - Analogous condition for the existence of unique equilibrium

Dynamic Exact Hat Algebra

Proposition

Given an initial observed allocation of the economy,
$\left(\{l_{i0}\}_{i=1}^{N}, \{k_{i0}\}_{i=1}^{N}, \{k_{i1}\}_{i=1}^{N}, \{S_{ni0}\}_{n,i=1}^{N}, \{D_{ni,-1}\}_{n,i=1}^{N}\right)$*, and an expected sequence of changes in fundamentals,*
$\left\{\{\hat{z}_{it}\}_{i=1}^{N}, \{\hat{b}_{it}\}_{i=1}^{N}, \{\hat{\tau}_{ijt}\}_{i,j=1}^{N}, \{\hat{\kappa}_{ijt}\}_{i,j=1}^{N}\right\}_{t=1}^{\infty}$*, the solution for the sequence of changes in the model's endogenous variables does not require information on the level of fundamentals,*
$\left\{\{z_{it}\}_{i=1}^{N}, \{b_{it}\}_{i=1}^{N}, \{\tau_{ijt}\}_{i,j=1}^{N}, \{\kappa_{ijt}\}_{i,j=1}^{N}\right\}_{t=0}^{\infty}$.

- Generalizes existing results for dynamic migration decisions to incorporate dynamic investment decisions ▸ more
- Can undertake counterfactuals in the model without having to solve for the initial level of fundamentals
- Can invert the non-linear model to recover the unobserved shocks $\left\{\{\hat{z}_{it}\}_{i=1}^{N}, \{\hat{b}_{it}\}_{i=1}^{N}, \{\hat{\tau}_{ijt}\}_{i,j=1}^{N}, \{\hat{\kappa}_{ijt}\}_{i,j=1}^{N}\right\}_{t=1}^{\infty}$ ▸ more

Linearization

- Linearize the model to characterize transition dynamics analytically
- Suppose that the economy at time $t = 0$ is on a convergence path towards an initial steady-state with constant fundamentals $(\boldsymbol{z}, \boldsymbol{b}, \boldsymbol{\kappa}, \boldsymbol{\tau})$

1. At time $t = 0$, agents learn about one-time, permanent shocks to fundamentals ($\widetilde{\boldsymbol{f}} = \begin{bmatrix} \widetilde{\boldsymbol{z}} \\ \widetilde{\boldsymbol{b}} \end{bmatrix}$) from time $t = 1$ onwards that are revealed under perfect foresight
2. At time $t = 0$, agents learn about a convergent sequence of shocks to fundamentals $\left\{\widetilde{\boldsymbol{f}}_s\right\}_{s\geq 1} = \left\{\begin{bmatrix} \widetilde{\boldsymbol{z}}_s \\ \widetilde{\boldsymbol{b}}_s \end{bmatrix}\right\}_{s\geq 1}$ from time $t = 1$ onwards that are revealed under perfect foresight
3. Given the initial value state variables at time $t = 0$ (x_0), suppose that productivity and amenities evolve stochastically according to an AR(1) process, and agents have rational expectations

- Transition path: 2nd-order difference equation in state variables ($\widetilde{\boldsymbol{\ell}}_t$, $\widetilde{\boldsymbol{k}}_t$) that solve with method of undetermined coefficients (Uhlig 1999)

Closed-form Transition Path

Proposition

Suppose that the economy at time $t = 0$ is on a convergence path towards an initial steady-state with constant fundamentals $(\boldsymbol{z}, \boldsymbol{b}, \boldsymbol{\kappa}, \boldsymbol{\tau})$. At time $t = 0$, agents learn about one-time, permanent shocks to productivity and amenities ($\widetilde{\boldsymbol{f}} = \begin{bmatrix} \widetilde{\boldsymbol{z}} \\ \widetilde{\boldsymbol{b}} \end{bmatrix}$) from time $t = 1$ onwards. There exists a $2N \times 2N$ transition matrix $(\boldsymbol{P})$ and a $2N \times 2N$ impact matrix $(\boldsymbol{R})$ such that the second-order difference equation system has a closed-form solution *of the form:*

$$\widetilde{\boldsymbol{x}}_{t+1} = \boldsymbol{P}\widetilde{\boldsymbol{x}}_t + \boldsymbol{R}\widetilde{\boldsymbol{f}} \quad \text{for } t \geq 1.$$

where $\widetilde{\boldsymbol{x}}_{\boldsymbol{t}} \equiv \begin{bmatrix} \widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}} \\ \widetilde{\boldsymbol{k}}_{\boldsymbol{t}} \end{bmatrix}$ and a tilde denotes a log deviation from the initial steady-state: $\tilde{\boldsymbol{\ell}}_{\boldsymbol{t}} \equiv \ln \boldsymbol{\ell}_{\boldsymbol{t}} - \ln \boldsymbol{\ell}^*_{initial}$ and $\{\boldsymbol{P}, \boldsymbol{R}\}$ can be recovered from the observed data $\{\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{D}, \boldsymbol{E}\}$ and the structural parameters of the model $\{\theta, \beta, \rho, \lambda, \psi, \delta\}$

Convergence Versus Fundamental Shocks

- Exact additive decomposition of the dynamics of the spatial distribution of economic activity: ▸ more

$$\ln \boldsymbol{x}_t - \ln \boldsymbol{x}_{-1} = \underbrace{\sum_{s=0}^{t} \boldsymbol{P}^s \left(\ln \boldsymbol{x}_0 - \ln \boldsymbol{x}_{-1}\right)}_{\substack{\text{convergence given} \\ \text{initial fundamentals}}} + \underbrace{\sum_{s=0}^{t-1} \boldsymbol{P}^s \boldsymbol{R} \widetilde{\boldsymbol{f}}}_{\substack{\text{dynamics from} \\ \text{fundamental shocks}}} \qquad \text{for all } t \geq 1,$$

- With no shocks to productivity and amenities ($\widetilde{\boldsymbol{f}} = \boldsymbol{0}$), we have:

$$\ln \boldsymbol{x}^*_{\text{initial}} = \lim_{t \to \infty} \ln \boldsymbol{x}_t = \ln \boldsymbol{x}_{-1} + \left(\boldsymbol{I} - \boldsymbol{P}\right)^{-1} \left(\ln \boldsymbol{x}_0 - \ln \boldsymbol{x}_{-1}\right),$$

- Using only initial state variables (for $t = 0$ and $t = -1$) and trade and migration matrices (and hence $\boldsymbol{P}$ and $\boldsymbol{R}$), we can compute implied steady-states with unchanged fundamentals
- Given counterfactual shocks to fundamentals ($\widetilde{\boldsymbol{f}}$), we can compute changes in steady-states, even without observing initial state variables

Spectral Analysis

- Use our linearization to characterize the economy's transition path in terms of lower-dimensional components
- Undertake an eigendecomposition of the transition matrix

$$\boldsymbol{P} \equiv \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V},$$

- where $\boldsymbol{\Lambda}$ is a diagonal matrix of eigenvalues arranged in decreasing order by absolute values, and $\boldsymbol{V} = \boldsymbol{U}^{-1}$
- For each eigenvalue λ_k, the left-eigenvectors ($\boldsymbol{u_k}$) and right-eigenvectors ($\boldsymbol{v'_k}$) satisfy

$$\lambda_k \boldsymbol{u}_k = \boldsymbol{P}\boldsymbol{u}_k, \qquad \lambda_k \boldsymbol{v}'_k = \boldsymbol{v}'_k \boldsymbol{P}$$

- Define an eigen-shock as a shock to productivity and amenities ($\widetilde{\boldsymbol{f}}_k$) for which the initial impact of these shocks on the state variables ($\boldsymbol{R}\widetilde{\boldsymbol{f}}_k$) coincides with a real eigenvector of the transition matrix ($\boldsymbol{u}_k$)

$$\widetilde{\boldsymbol{f}}_k = \boldsymbol{R}^{-1}\boldsymbol{u}_k$$

- Can recover these eigen-shocks from $\{\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{D}, \boldsymbol{E}\}$ and $\{\theta, \beta, \rho, \lambda, \psi, \delta\}$

Speed of Convergence

Proposition

Consider an economy that is initially in steady-state at $t = 0$ when agents learn about one-time, permanent shocks to productivity and amenities ($\widetilde{\boldsymbol{f}} = \begin{bmatrix} \widetilde{\boldsymbol{z}} \\ \widetilde{\boldsymbol{b}} \end{bmatrix}$) from $t = 1$ onwards. Suppose the initial impact of the shock to fundamentals on the state variables at $t = 1$ coincides with an eigenvector ($\boldsymbol{R}\widetilde{\boldsymbol{f}} = \boldsymbol{u}_k$) of the transition matrix ($\boldsymbol{P}$) (eigen-shock). The transition path of the state variables ($\widetilde{\boldsymbol{x}}_{\boldsymbol{t}} \equiv \begin{bmatrix} \widetilde{\boldsymbol{\ell}}_{\boldsymbol{t}} \\ \widetilde{\boldsymbol{k}}_{\boldsymbol{t}} \end{bmatrix}$) reduces to:

$$\widetilde{\boldsymbol{x}}_t = \frac{1 - \lambda_k^t}{1 - \lambda_k} \boldsymbol{u}_k,$$

and the half-life is given by:

$$t_i^{(1/2)} \left(\widetilde{\boldsymbol{f}} \right) = - \left\lceil \frac{\ln 2}{\ln \lambda_k} \right\rceil$$

for all state variables $i = 1, \cdots, 2N$, where $\lceil \cdot \rceil$ is the ceiling function.

more

Outline

- Dynamic Spatial Model
- Extensions
  - Trade deficits
  - Shocks to trade and migration costs
  - Agglomeration and dispersion forces
  - Housing capital
  - Multi-sector
  - Multi-sector and input-output linkages
- Data
- Empirical Results
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Data

- Two empirical implementations
  - State-time data from 1965-2015 (decline Rust Belt and rise Sun Belt)
  - State-industry-time data from 1999-2015
- U.S. State GDP, population and capital stock
  - Bureau of Economic Analysis (BEA) 1965-2015
- Bilateral value of shipments between U.S. states
  - Commodity Flow Survey (CFS)
  - Commodity Transportation Survey (CTS)
- Bilateral migration flows between U.S. states
  - Population census and American Community Survey (ACS) 1960-2010
  - Five-year migration matrices
- Foreign imports and exports of U.S. states
  - Foreign exports by origin of movement (OM) state 1999-2015
  - Foreign imports by state of destination (SD) 1999-2015

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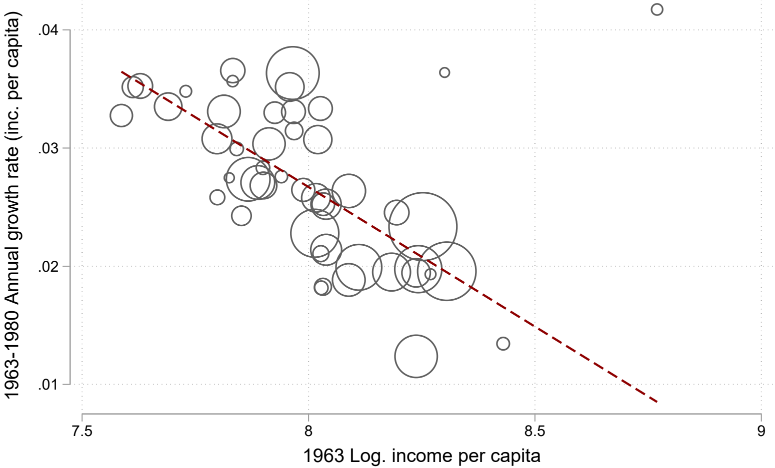
Income Convergence 1963-80

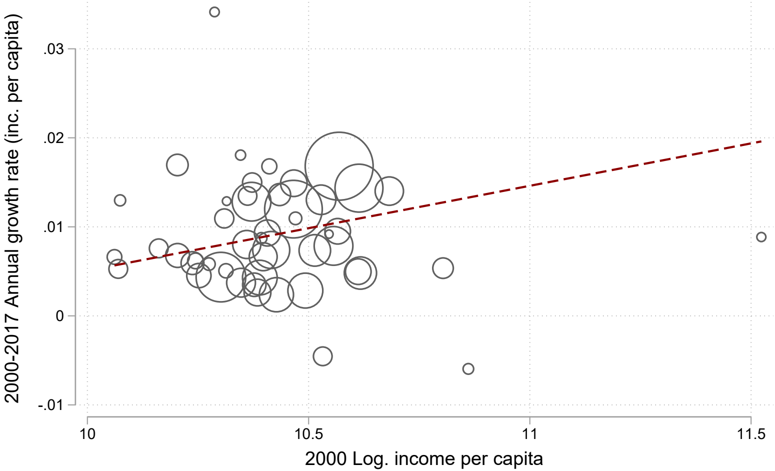


Note: Slope coefficient: -0.0236; standard error: 0.0038; R-squared: 0.4758.

Income Convergence 1980-2000

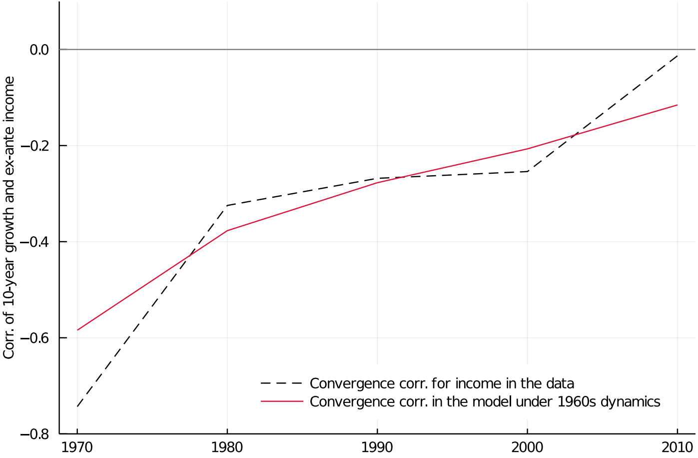
Note: Slope coefficient: -0.0135; standard error: 0.0039; R-squared: 0.2092.

Income Convergence 2000-2017



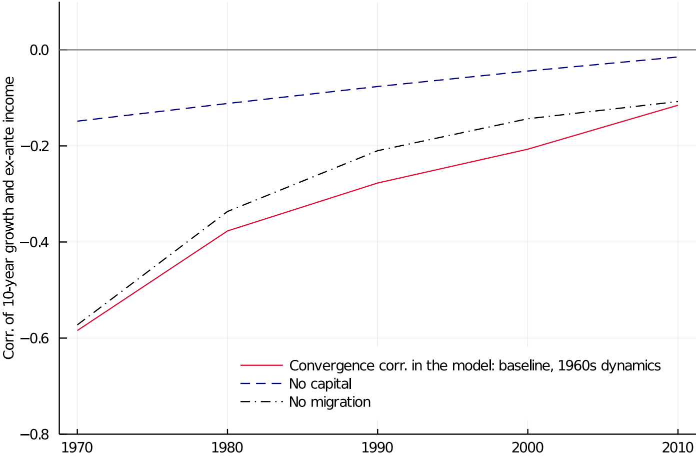
Note: Slope coefficient: 0.0095; standard error: 0.0061; R-squared: 0.0712.

Importance of Initial Conditions



- Much of the decline in the speed of convergence in income per capita can be explained by initial conditions

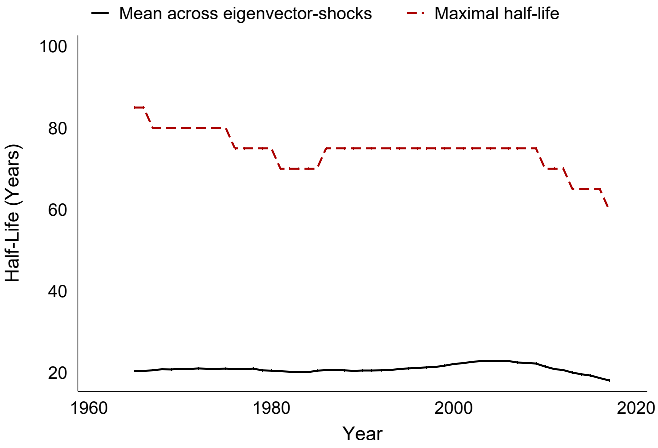
Capital Versus Labor Dynamics



- Capital adjustment important for dynamics of income per capita
- Migration important for dynamics of population

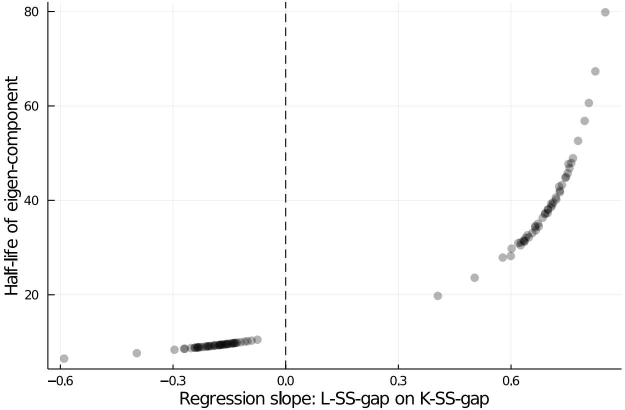
Spectral Analysis

Half-lifes



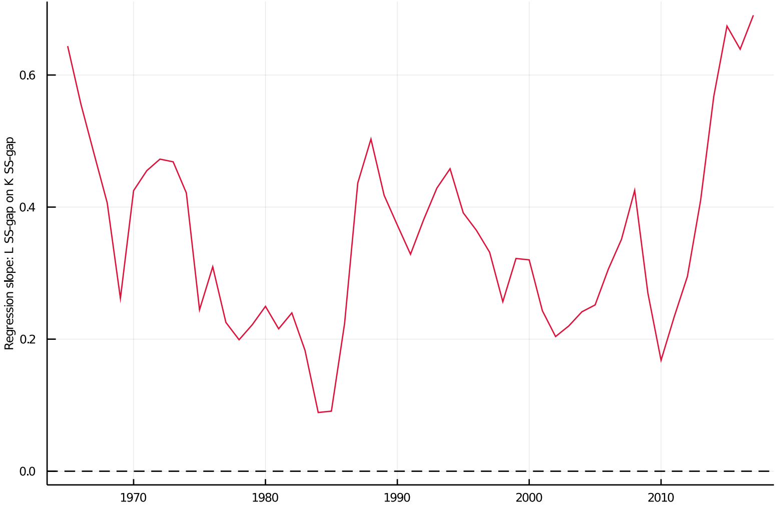
Note: Half-life Corresponds to the time in years for the state variables to converge half of the way towards steady-state for a shock to productivity and amenities for which its initial impact on the state variables ($\widetilde{\boldsymbol{Rf}}$) corresponds to an eigenvector ($\boldsymbol{u_k}$) of the transition matrix ($\boldsymbol{P}$); figure shows mean and maximum half-life across eigenvectors of the transition matrix in each year from 1965-2015.

Heterogeneity in Half Lives (SS gap)

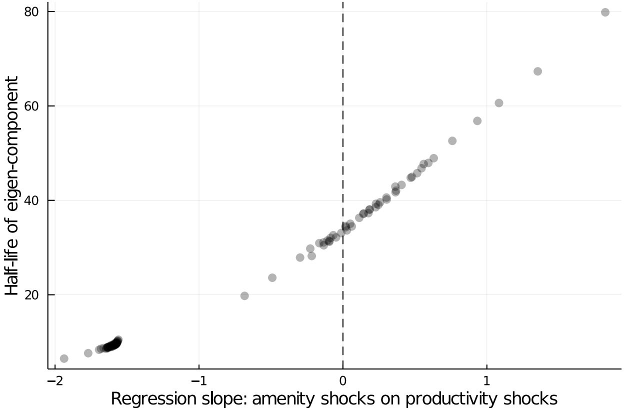


- Eigen-shock: shock to productivity and amenities ($\tilde{\boldsymbol{f}}_k$) for which the initial impact of these shocks on the state variables coincides with a real eigenvector of the transition matrix: $\boldsymbol{u}_k = \boldsymbol{R}\tilde{\boldsymbol{f}}_k$

Correlation Steady-State Gaps Over Time

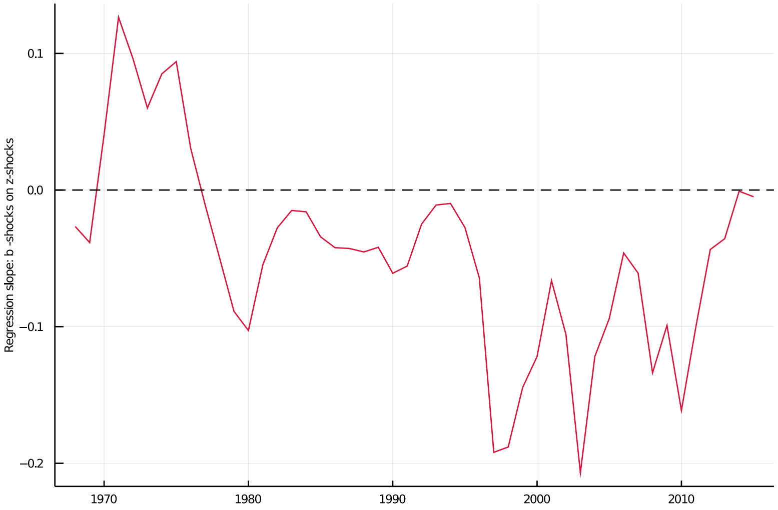


Heterogeneity in Half Lives (Shocks)

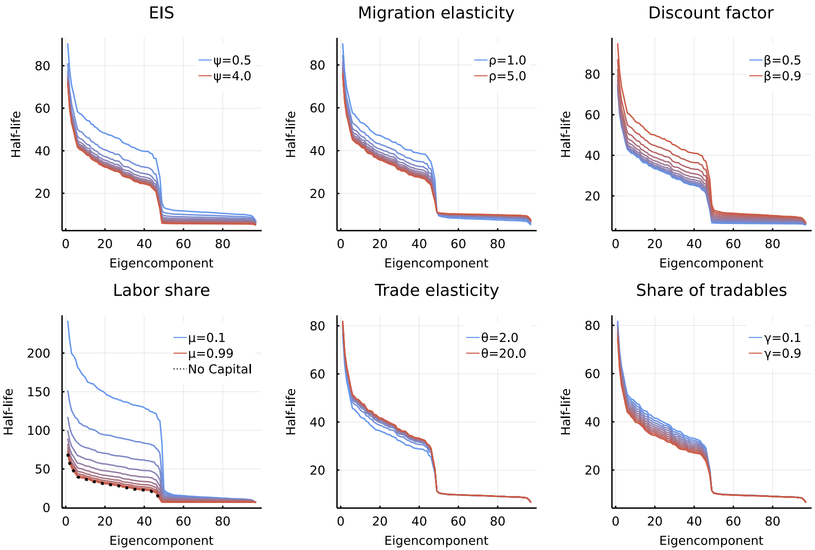


- Eigen-shock: shock to productivity and amenities ($\widetilde{\boldsymbol{f}}_k$) for which the initial impact of these shocks on the state variables coincides with a real eigenvector of the transition matrix: $\widetilde{\boldsymbol{f}}_k = \boldsymbol{R}^{-1}\boldsymbol{u}_k$

Correlation Shocks Over Time (Shocks)

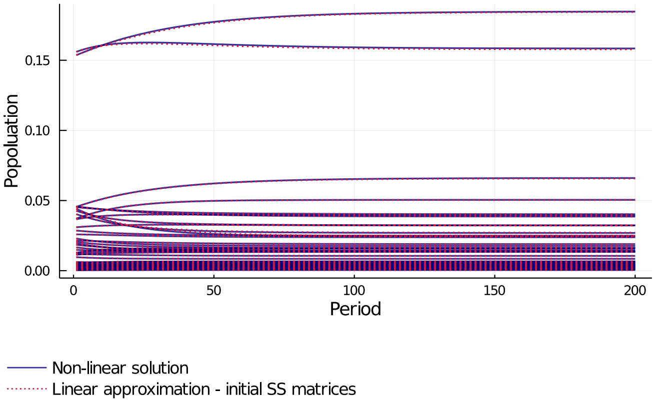


Parameters and Speed of Convergence



Non-Linear Solution and Linearization

- Invert non-linear model (prod., amenities, trade & migration costs)
- Start from steady-state implied by these 1990 fundamentals
- Shock by vector of productivity shocks 1990-2000
- Compare transition paths in our linearization and non-linear model



Conclusions

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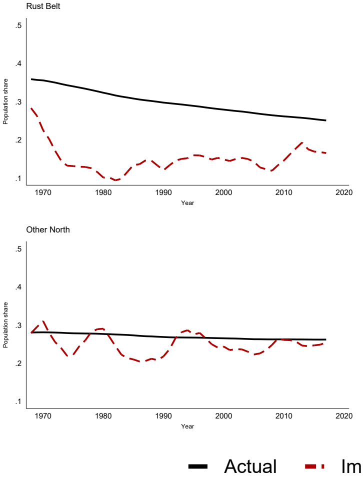
3. Linearize the model to characterize analytically determinants of speed of convergence (spectral analysis of transition matrix)
4. Applications: US state data 1965-2015; state-industry data 1999-2015
   - Decline in rate of income convergence over time (β-convergence)
   - Slow convergence and heterogeneous impact of shocks
   - Heterogeneity explained by the interaction of capital and labor dynamics

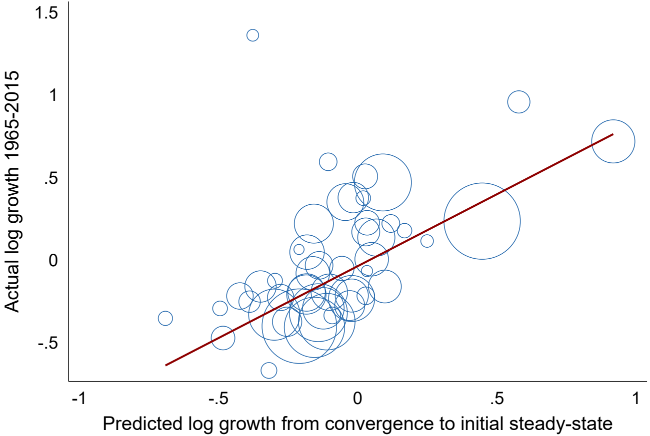
Thank You

Population Gap from Steady-State



Rust Belt: Illinois, Indiana, Michigan, New York, Ohio, Pennsylvania, West Virginia and Wisconsin. Sun Belt: Arizona, California, Florida, New Mexico and Nevada. North and South definitions based on Federal and Confederacy states

Predictive Power Initial Steady-State



Note: Slope coefficient: 0.8709; standard error: 0.1081; R-squared: 0.5035.

- Robust to controlling for initial log population and capital stock and initial log population growth ▸ more

Non-linear Model Inversion

- Parameters: $\beta = 0.95$, $\theta = 5$, $\rho = 3\beta$, $\lambda = 0.65$, $(1-\delta) = 0.95$
- Recover unobserved fundamentals from the non-linear model
  - Economy can be anywhere on transition path / in steady-state
  - Assume perfect foresight but allow any expected path fundamentals

$$\frac{S_{nit}S_{int}}{S_{nnt}S_{iit}} = \left(\frac{\tau_{nit}\tau_{int}}{\tau_{nnt}\tau_{iit}}\right)^{-\theta} = \left(\tau_{nit}\right)^{-2\theta}, \qquad \frac{D_{igt}D_{git}}{D_{ggt}D_{iit}} = \left(\frac{\kappa_{git}\kappa_{igt}}{\kappa_{ggt}\kappa_{iit}}\right)^{-1/\rho} = \left(\kappa_{git}\right)^{-2/\rho}$$

$$w_{it}\ell_{it} = \sum_{n=1}^{N} \frac{\left(w_{it}\left(\ell_{it}/k_{it}\right)^{1-\lambda}\tau_{nit}/z_{it}\right)^{-\theta}}{\sum_{m=1}^{N}\left(w_{mt}\left(\ell_{mt}/k_{mt}\right)^{1-\lambda}\tau_{nmt}/z_{mt}\right)^{-\theta}} w_{nt}\ell_{nt}$$

$$\ell_{gt+1} = \sum_{i=1}^{N} \frac{\left(\exp\left(\beta v_{gt+1}\right)/\kappa_{git}\right)^{1/\rho}}{\sum_{m=1}^{N}\left(\exp\left(\beta v_{mt+1}\right)/\kappa_{mit}\right)^{1/\rho}} \ell_{it}$$

$$\ln b_{it} = \left(v_{it} - v_{it+1}\right) + \left(1-\beta\right) v_{it+1} - \ln \frac{S_{iit}^{-\frac{1}{\theta}}}{\left(D_{iit}\right)^{\rho}} - \ln z_{it}$$

- Intuition: migration flows capture expectations ▸ backdynex

Steady-state Comparative Statics

- To begin with, start at steady-state (relax later): $\{w_i^*, v_i^*, \ell_i^*, k_i^*\}$
- Consider $\mathrm{d}\ln \boldsymbol{z} \neq 0$, $\mathrm{d}\ln \boldsymbol{b} \neq 0$, and $\mathrm{d}\ln \boldsymbol{\tau} = \mathrm{d}\ln \boldsymbol{\kappa} = \mathrm{d}\ln \bar{\ell} = 0$

$$\mathrm{d}\ln \boldsymbol{k}^* = \underbrace{\mathrm{d}\ln \boldsymbol{\ell}^*}_{\text{change in population}} + \underbrace{\mathrm{d}\ln \boldsymbol{w}^*}_{\text{change in wages}} - \underbrace{\mathrm{d}\ln \boldsymbol{p}^*}_{\text{change in the price index}}$$

$$\mathrm{d}\ln \boldsymbol{p}^* = \boldsymbol{S}\underbrace{\left[\mathrm{d}\ln \boldsymbol{w}^* - (1-\lambda)\left(\mathrm{d}\ln \boldsymbol{k}^* - \mathrm{d}\ln \boldsymbol{\ell}^*\right) - \mathrm{d}\ln \boldsymbol{z}\right]}_{\text{change in the production cost in each region}}$$

$$\mathrm{d}\ln \boldsymbol{w}^* + \mathrm{d}\ln \boldsymbol{\ell}^* = \underbrace{\boldsymbol{T}\left(\mathrm{d}\ln \boldsymbol{w}^* + \mathrm{d}\ln \boldsymbol{\ell}^*\right)}_{\text{market size}} + \underbrace{\theta\left(\boldsymbol{TS} - \boldsymbol{I}\right)\left[\mathrm{d}\ln \boldsymbol{w}^* - (1-\lambda)\left(\mathrm{d}\ln \boldsymbol{k}^* - \mathrm{d}\ln \boldsymbol{\ell}^*\right) - \mathrm{d}\ln \boldsymbol{z}\right]}_{\text{cross-substitution}}$$

$$\mathrm{d}\ln \boldsymbol{\ell}^* = \underbrace{\boldsymbol{E}\,\mathrm{d}\ln \boldsymbol{\ell}^*}_{\text{labor supply}} + \underbrace{\frac{\beta}{\rho}\left(\boldsymbol{I} - \boldsymbol{ED}\right)\mathrm{d}\boldsymbol{v}^*}_{\text{migration shares}}$$

$$\mathrm{d}\boldsymbol{v}^* = \underbrace{\mathrm{d}\ln \boldsymbol{b} + \mathrm{d}\ln \boldsymbol{w}^* - \mathrm{d}\ln \boldsymbol{p}^*}_{\text{flow utility}} + \underbrace{\beta \boldsymbol{D}\,\mathrm{d}\boldsymbol{v}^*}_{\text{continuation value}}$$

Steady-state Comparative Statics

- Totally differentiating the general equilibrium conditions of the model and stacking them in matrix form

Proposition

The steady-state response of the endogenous variables to productivity and amenity shocks satisfies the linear system:

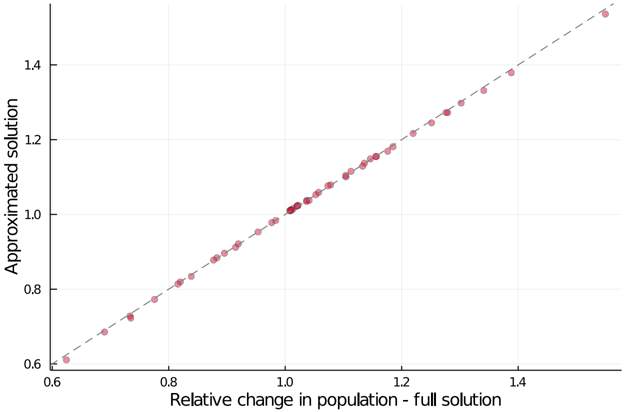
$$\begin{bmatrix} \mathrm{d}\ln \boldsymbol{\ell}^* \\ \mathrm{d}\ln \boldsymbol{k}^* \\ \mathrm{d}\ln \boldsymbol{w}^* \\ \mathrm{d}\ln \boldsymbol{v}^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}^{z*} \\ \boldsymbol{K}^{z*} \\ \boldsymbol{W}^{z*} \\ \boldsymbol{V}^{z*} \end{bmatrix} \mathrm{d}\ln \boldsymbol{z} + \begin{bmatrix} \boldsymbol{L}^{b*} \\ \boldsymbol{K}^{b*} \\ \boldsymbol{W}^{b*} \\ \boldsymbol{V}^{b*} \end{bmatrix} \mathrm{d}\ln \boldsymbol{b}$$

where the $N \times N$ matrices $\{\boldsymbol{L}^{z*}, \boldsymbol{K}^{z*}, \boldsymbol{W}^{z*}, \boldsymbol{V}^{z*}, \boldsymbol{L}^{b*}, \boldsymbol{K}^{b*}, \boldsymbol{W}^{b*}, \boldsymbol{V}^{b*}\}$ are functions of the four observed matrices of expenditure shares ($\boldsymbol{S}$), income shares ($\boldsymbol{T}$), outmigration shares ($\boldsymbol{D}$) and inmigration shares ($\boldsymbol{E}$) and the structural parameters of the model $\{\beta, \theta, \rho, \lambda, \delta\}$.

- Element $[\boldsymbol{L}^{z*}]_{in} = \mathrm{d}\ln \ell_i^* / \mathrm{d}\ln z_n$
  - Elasticity of steady-state population in location i (ℓ_i^*) with respect to an increase in productivity in location n (z_n)

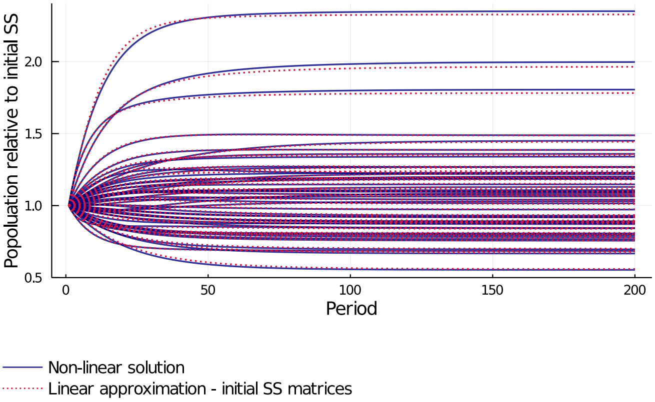
Approximation Quality (Steady-State)

- Start from steady-state implied by 1990 fundamentals
- Shock by vector of productivity shocks 1990-2000
- Compare steady-state changes in our linearization & non-linear model



Approximation Quality (Transition)

- Start from steady-state implied by 1990 fundamentals
- Shock by vector of productivity shocks 1990-2000
- Compare transition paths in our linearization and non-linear model



CRRA Utility

- Landlords' intertemporal utility

$$v_{it}^{k} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^{t+s} \frac{\left(c_{it+s}^{k}\right)^{1-1/\psi}}{1-1/\psi}$$

- Budget constraint

$$r_{it} k_{it} = p_{it} \left(c_{it}^{k} + k_{it+1} - (1-\delta)\, k_{it} \right)$$

- Gross return on capital: $R_{it} \equiv 1 - \delta + r_{it} / p_{it}$
- Optimal savings rate

$$k_{it+1} = (1 - \varsigma_{it})\, R_{it} k_{it}$$

$$\varsigma_{it}^{-1} = 1 + \beta^{\psi} \left(\mathbb{E}_t \left[R_{it+1}^{\frac{\psi-1}{\psi}} \varsigma_{t+1}^{-\frac{1}{\psi}} \right] \right)^{\psi}$$

- (compare with log utility, where $k_{it+1} = \beta R_{it} k_{it}$) back

Intertemporal Consumption-Investment

- Intertemporal optimization problem

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln c_{it}^k - \mu_t \left[p_{it} c_{it}^k + p_{it} \left(k_{it+1} - (1-\delta) k_{it} \right) - r_{it} k_{it} \right]$$

- Euler equation

$$\frac{c_{it+1}^k}{c_{it}^k} = \beta \left(r_{it+1} / p_{it+1} + (1-\delta) \right)$$

- Conjecture policy functions

$$p_{it} c_{it}^k = (1-\beta) \left(r_{it} + p_{it} (1-\delta) \right) k_{it}$$

$$k_{it+1} = \beta \left(r_{it} / p_{it} + (1-\delta) \right) k_{it}$$

- Confirm that this conjecture satisfies the Euler equation back

Constant Perceived Return to Capital

- Profit maximization and zero profits

$$w_{it} = (1 - \lambda)\, p_{it} z_{it} \left(\frac{k_{it}}{\ell_{it}}\right)^{\lambda}$$

$$r_{it} = \lambda p_{it} z_{it} \left(\frac{k_{it}}{\ell_{it}}\right)^{\lambda - 1}$$

- Landlord income

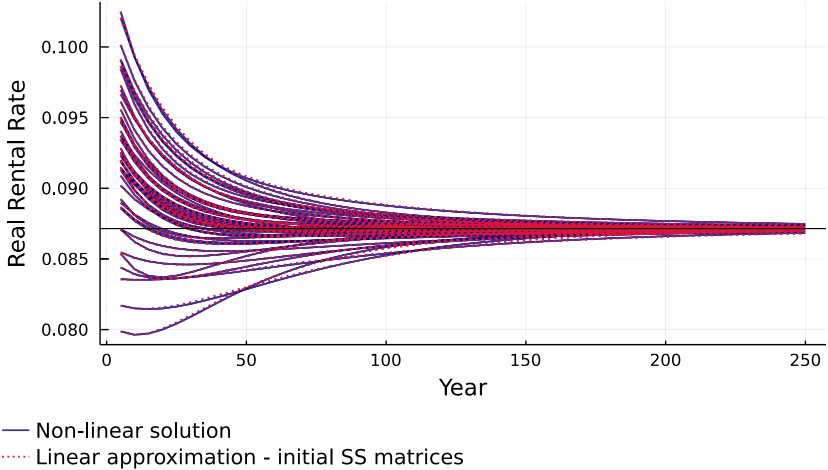
$$\Pi_{it} = r_{it} k_{it} = p_{it} z_{it} k_{it}^{\lambda} \ell_{it}^{1-\lambda} - w_{it} \ell_{it}$$

- Using profit maximization and zero profits, landlord income is

$$\Pi_{it} = \lambda \left(p_{it} z_{it}\right)^{\frac{1}{\lambda}} \left(\frac{1-\lambda}{w_{it}}\right)^{\frac{1-\lambda}{\lambda}} k_{it}$$

back

Rental Rate Transition



back

Investment Other Locations

- Realized rate of return to a landlord in location n from allocating one unit of capital to location i is:

$$v_{nit} = \frac{\alpha_{nit} r_{it}}{\phi_{nit}}$$

- Marginal efficiency of capital in i drawn from Fréchet distribution

$$F_{nit}(\alpha) = e^{-(\alpha / a_{it})^{-\epsilon}}, \qquad a_{it} > 0, \qquad \epsilon > 1$$

- Capital from n allocated to i

$$b_{nit} = \frac{k_{nit}}{k_{nt}} = \frac{(a_{it} r_{it} / \phi_{nit})^{\epsilon}}{\sum_{h=1}^{N} (a_{ht} r_{ht} / \phi_{nht})^{\epsilon}}$$

- Realized rate of return on capital owned by source location n at time t is the same across all host locations i and given by

$$v_{nit} = v_{nt} = \Gamma\left(\frac{\epsilon - 1}{\epsilon}\right) \left[\sum_{h=1}^{N} (a_{ht} r_{ht} / \phi_{nht})^{\epsilon}\right]^{\frac{1}{\epsilon}}$$

back

Sequential Equilibrium

Definition

Given the state variables $\{\ell_{i0}, k_{i0}\}$ in each location in an initial period $t = 0$, a *sequential equilibrium* of the economy is a set of wages, expected values, mass of workers and stock of capital in each location in all subsequent time periods $\{w_{it}, v_{it}, \ell_{it}, k_{it}\}_{t=0}^{\infty}$ that solves the value function, the labor market clearing condition, the goods market clearing condition, and the capital market clearing and accumulation condition.

Definition

A *steady-state* of the economy is an equilibrium in which all location-specific variables (wages, expected values, mass of workers and stock of capital in each location) are time invariant: $\{w_i^*, v_i^*, \ell_i^*, k_i^*\}$.

Existence and Uniqueness

- The steady-state equilibrium $\{p_i^*, w_i^*, \ell_i^*, \phi_i^*\}$ solves: ▸ back

$$\left(p_i^*\right)^{-\theta} = \sum_{n=1}^{N} \psi \widetilde{\tau}_{in} \left(p_n^*\right)^{-\theta(1-\lambda)} \left(w_n^*\right)^{-\theta\lambda},$$

$$\left(p_i^*\right)^{\theta(1-\lambda)} \left(w_i^*\right)^{1+\theta\lambda} \ell_i^* = \sum_{n=1}^{N} \psi \widetilde{\tau}_{ni} \left(p_n^*\right)^{\theta} w_n^* \ell_n^*,$$

$$\left(p_i^*\right)^{\beta/\rho} \left(w_i^*\right)^{-\beta/\rho} \ell_i^* \left(\phi_i^*\right)^{-\beta} = \sum_{n=1}^{N} \widetilde{\kappa}_{in} \ell_n^* \left(\phi_n^*\right)^{-1},$$

$$\phi_i^* = \sum_{n=1}^{N} \widetilde{\kappa}_{ni} \left(p_n^*\right)^{-\beta/\rho} \left(w_n^*\right)^{\beta/\rho} \left(\phi_n^*\right)^{\beta},$$

where $$\psi \equiv \left(\frac{1-\beta(1-\delta)}{\beta}\right)^{-\theta(1-\lambda)}, \qquad \widetilde{\tau}_{ni} \equiv \left(\tau_{ni}/z_i\right)^{-\theta},$$

$$\phi_i^* \equiv \sum_{n=1}^{N} \widetilde{\kappa}_{ni} \exp\left(\frac{\beta}{\rho} v_n^{w*}\right), \qquad \widetilde{\kappa}_{in} \equiv \left(\kappa_{in}/b_n^{\beta}\right)^{-1/\rho}.$$

Existence and Uniqueness

- This system of equations falls with the class for which Theorem 1 of Allen, Arkolakis and Li (2020) applies:

$$\mathbf{\Lambda} = \begin{bmatrix} -\theta & 0 & 0 & 0 \\ \theta(1-\lambda) & (1+\theta\lambda) & 1 & 0 \\ \beta/\rho & -\beta/\rho & 1 & -\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{\Gamma} = \begin{bmatrix} -\theta(1-\lambda) & -\theta\lambda & 0 & 0 \\ \theta & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -\beta/\rho & \beta/\rho & 0 & \beta \end{bmatrix}.$$

back

Dynamic Exact Hat Algebra

- Given an observed initial allocation $\left(\{l_{i0}\}_{i=1}^{N}, \{k_{i0}\}_{i=1}^{N}, \{k_{i1}\}_{i=1}^{N}, \{S_{ni0}\}_{n,i=1}^{N}, \{D_{ni,-1}\}_{n,i=1}^{N}\right)$

$$\hat{D}_{igt+1} = \frac{D_{igt}\left(\hat{u}_{gt+2}/\hat{\kappa}_{git+1}\right)^{1/\rho}}{\sum_{m=1}^{N} D_{imt}\left(\hat{u}_{mt+2}/\hat{\kappa}_{mit+1}\right)^{1/\rho}}$$

$$\hat{u}_{it+1} = \left(\hat{b}_{it+1}\frac{\hat{w}_{it+1}}{\hat{p}_{it+1}}\right)^{\beta}\left(\sum_{g=1}^{N}\left(\hat{u}_{gt+2}/\hat{\kappa}_{git+1}\right)^{1/\rho}\right)^{\beta\rho}$$

$$\hat{p}_{it+1} = \left(\sum_{m=1}^{N} S_{imt}\left(\hat{\tau}_{imt+1}\hat{w}_{mt+1}\left(\hat{l}_{mt+1}/\hat{k}_{mt+1}\right)^{1-\mu}/\hat{z}_{mt+1}\right)^{-\theta}\right)^{-1/\theta}$$

$$\ell_{gt+1} = \sum_{i=1}^{N} D_{igt}\ell_{it}$$

- where $u_{it} = \exp(v_{it})$ and $\hat{u}_{it+1} = u_{it+1}/u_{it}$

back

Dynamic Exact Hat Algebra

$$\hat{w}_{it+1}\hat{\ell}_{it+1} = \sum_{n=1}^{N} \frac{S_{nit+1} w_{nt} \ell_{nt}}{\sum_{k=1}^{N} S_{kit} w_{kt} \ell_{kt}} \hat{w}_{nt+1} \hat{\ell}_{nt+1}$$

$$\hat{S}_{nit+1} \equiv \frac{S_{nit+1} \left(\hat{\tau}_{nit+1} \hat{w}_{it+1} \left(\hat{l}_{it+1} / \hat{k}_{it+1} \right)^{1-\mu} / \hat{z}_{it+1} \right)^{-\theta}}{\sum_{k=1}^{N} S_{nkt+1} \left(\hat{\tau}_{nkt+1} \hat{w}_{kt+1} \left(\hat{l}_{kt+1} / \hat{k}_{kt+1} \right)^{1-\mu} / \hat{z}_{kt+1} \right)^{-\theta}}$$

$$\varsigma_{it+1} = \beta R_{it+1}^{\psi-1} \frac{\varsigma_{it}}{1 - \varsigma_{it}}$$

$$k_{it+1} = \left(1 - \varsigma_{it}\right) R_{it} k_{it}$$

$$\left(R_{it} - (1-\delta)\right) = \frac{\hat{p}_{it+1} \hat{k}_{it+1}}{\hat{w}_{it+1} \hat{l}_{it+1}} \left(R_{it+1} - (1-\delta)\right)$$

back

Transition Dynamics Decomposition

- Transition dynamics decomposition

$$
\begin{aligned}
\widetilde{x}_t &= \boldsymbol{P}\widetilde{x}_{t-1} + \boldsymbol{R}\tilde{\boldsymbol{f}} \\
\widetilde{x}_{t-1} &= \boldsymbol{P}\widetilde{x}_{t-2} + \boldsymbol{R}\tilde{\boldsymbol{f}} \\
\vdots & \quad \vdots \\
\widetilde{x}_1 &= P\widetilde{x}_0 + R\widetilde{f} \\
\widetilde{x}_0 &= P\widetilde{x}_{-1}
\end{aligned}
$$

- Taking the difference between time t and $t-1$

$$
\begin{aligned}
\ln x_t - \ln x_{t-1} &= P\left(\ln x_{t-1} - \ln x_{t-2}\right) \\
\vdots & \quad \vdots \\
&= P^{t-1}\left(\ln x_1 - \ln x_0\right) \\
&= P^t\left(\ln x_0 - \ln x_{-1}\right) + P^{t-1}R\widetilde{f}
\end{aligned}
$$

back

Transition Dynamics Decomposition

- We thus obtain:

$$\begin{aligned}
\ln x_t - \ln x_{-1} &= [\ln x_t - \ln x_{t-1}] + [\ln x_{t-1} - \ln x_{t-2}] + \cdots + [\ln x_1 - \ln x_0] + [\ln x_0 - \ln x_{-1}] \\
&= \left[P^t (\ln x_0 - \ln x_{-1}) + P^{t-1} R\widetilde{f}\right] + \left[P^{t-1} (\ln x_0 - \ln x_{-1}) + P^{t-2} R\widetilde{f}\right] \\
&+ \cdots + \left[P (\ln x_0 - \ln x_{-1}) + R\widetilde{f}\right] + [\ln x_0 - \ln x_{-1}] \\
&= \sum_{s=0}^{t} P^s (\ln x_0 - \ln x_{-1}) + \sum_{s=0}^{t-1} P^s R\widetilde{f}
\end{aligned}$$

back

Any Convergent Sequence

Proposition

Consider an economy that is initially in steady-state at time $t = 0$ when agents learn about a convergent sequence of future shocks to productivity and amenities $\left\{\widetilde{\boldsymbol{f}}_s\right\}_{s\geq 1} = \left\{\begin{bmatrix} \widetilde{\boldsymbol{z}}_s \\ \widetilde{\boldsymbol{b}}_s \end{bmatrix}\right\}_{s\geq 1}$ that is revealed under perfect foresight from time $t = 1$ onwards.

There exists a $2N \times 2N$ transition matrix ($\boldsymbol{P}$) and a $2N \times 2N$ impact matrix ($\boldsymbol{R}$) such that the dynamic path of state variables relative to the initial steady-state follows:

$$\widetilde{\boldsymbol{x}}_t = \sum_{s=t+1}^{\infty} \left(\boldsymbol{\Psi}^{-1}\boldsymbol{\Gamma} - \boldsymbol{P}\right)^{-(s-t)} \boldsymbol{R}\left(\widetilde{\boldsymbol{f}}_s - \widetilde{\boldsymbol{f}}_{s-1}\right) + \boldsymbol{R}\widetilde{\boldsymbol{f}}_t + \boldsymbol{P}\widetilde{\boldsymbol{x}}_{t-1} \qquad \text{for all } t \geq 1,$$

with initial condition $\widetilde{\boldsymbol{x}}_0 = \boldsymbol{0}$ and where $\boldsymbol{\Psi}, \boldsymbol{\Gamma}$ are matrices from our solution to the second-order difference equation

Stochastic Fundamentals

- Productivity and amenities evolve stochastically over time according to the following AR(1) structure:

$$\ln z_{it+1} - \ln z_{it} = \rho^z \left(\ln z_{it} - \ln z_{it-1}\right) + \varpi^z_{it}, \qquad |\rho^z| < 1,$$

$$\ln b_{it-1} - \ln b_{it} = \rho^b \left(\ln b_{it} - \ln b_{it-1}\right) + \varpi^b_{it}, \qquad \left|\rho^b\right| < 1,$$

- Agents expect future shocks to fundamentals to decay to zero:

$$\mathbb{E}_t \left[\widetilde{z}_{it+s} - \widetilde{z}_{it+s-1}\right] = \left(\rho^z\right)^s \left(\widetilde{z}_{it} - \widetilde{z}_{it-1}\right),$$

$$\mathbb{E}_t \left[\widetilde{b}_{it+s} - \widetilde{b}_{it+s-1}\right] = \left(\rho^b\right)^s \left(\widetilde{b}_{it} - \widetilde{b}_{it-1}\right),$$

- Closed-form solution for the economy's transition path

$$\mathbb{E}_1 \left[\widetilde{\boldsymbol{x}}_t\right] = \sum_{s=t+1}^{\infty} \left(\boldsymbol{\Psi}^{-1}\boldsymbol{\Gamma} - \boldsymbol{P}\right)^{-(s-t)} \boldsymbol{R} \left(\mathbb{E}_1 \left[\widetilde{\boldsymbol{f}}_s - \widetilde{\boldsymbol{f}}_{s-1}\right]\right) + \boldsymbol{R}\mathbb{E}_1 \left[\widetilde{\boldsymbol{f}}_t\right] + \boldsymbol{P}\mathbb{E}_1 \left[\widetilde{\boldsymbol{x}}_{t-1}\right]$$

Eigendecomposition

- Eigendecomposition of transition dynamics

$$P = U\Lambda V, \quad \text{and hence} \quad P^s = \sum_{k=1}^{2N} \lambda_k^s u_k v_k'$$

$$\begin{aligned}
\widetilde{x}_t &= \textstyle\sum_{s=0}^{t-1} P^s R\widetilde{f} \\
&= \textstyle\sum_{s=0}^{t-1} \left(\sum_{k=1}^{2N} \lambda_k^s u_k v_k' \right) R\widetilde{f} \\
&= \textstyle\sum_{k=1}^{2N} \left(\sum_{s=0}^{t-1} \lambda_k^s \right) u_k v_k' R\widetilde{f} \\
&= \textstyle\sum_{k=1}^{2N} \left(\frac{1-\lambda_k^t}{1-\lambda_k} \right) u_k v_k' R\widetilde{f}
\end{aligned}$$

Eigendecomposition

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&= \sum_{k=1}^{2N} \left(\sum_{s=0}^{t-1} \lambda_k^s\right) u_k v_k' R\widetilde{f} \\
&= \sum_{k=1}^{2N} \left(\frac{1-\lambda_k^t}{1-\lambda_k}\right) u_k v_k' R\widetilde{f}
\end{aligned}$$

$$v_k' R\widetilde{f} = v_k' \sum_{i=1}^{2N} a_i R\widetilde{f}_i = \sum_{i=1}^{2N} a_i v_k' u_i = a_k$$

$$\begin{aligned}
a = VR\widetilde{f} = U^{-1}R\widetilde{f} &= \left(R^{-1}U\right)^{-1}\widetilde{f} \\
&= \left(R^{-1}U\right)^{-1}\left(\left(R^{-1}U\right)^T\right)^{-1}\left(\left(R^{-1}U\right)^T\right)\widetilde{f} \\
&= \left(\left(R^{-1}U\right)^T\left(R^{-1}U\right)\right)^{-1}\left(R^{-1}U\right)^T\widetilde{f}
\end{aligned}$$

Speed of Convergence

- Suppose that $\boldsymbol{R}\widetilde{\boldsymbol{f}}$ coincides with a real eigenvector: $\boldsymbol{R}\widetilde{\boldsymbol{f}} = \boldsymbol{u}_k$

$$\widetilde{\boldsymbol{x}}_t = \sum_{j=1}^{2N} \left(\frac{1-\lambda_j^t}{1-\lambda_j} \right) u_j v_j' R\widetilde{f} = \sum_{j=1}^{2N} \frac{1-\lambda_j^t}{1-\lambda_j} \boldsymbol{u}_j \boldsymbol{v}_j' \boldsymbol{u}_k = \frac{1-\lambda_k^t}{1-\lambda_k} \boldsymbol{u}_k$$

- where we have used $\boldsymbol{U}\boldsymbol{V}' = \boldsymbol{U}\boldsymbol{U}^{-1} = \boldsymbol{I}$
- Taking differences between periods $t+1$ and t, we have:

$$\widetilde{\boldsymbol{x}}_{t+1} - \widetilde{\boldsymbol{x}}_t = \frac{1-\lambda_k^{t+1}}{1-\lambda_k} \boldsymbol{u}_k - \frac{1-\lambda_k^t}{1-\lambda_k} \boldsymbol{u}_k$$

which simplifies to:

$$(1-\lambda_k)\left(\widetilde{\boldsymbol{x}}_{t+1} - \widetilde{\boldsymbol{x}}_t\right) = (1-\lambda_k)\,\lambda_k^t \boldsymbol{u}_k$$

and hence:

$$\left(\widetilde{\boldsymbol{x}}_{t+1} - \widetilde{\boldsymbol{x}}_t\right) = \lambda_k^t \boldsymbol{u}_k$$

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Speed of Convergence

- Noting that $\widetilde{\boldsymbol{x}}_t = \ln \boldsymbol{x}_t - \ln \boldsymbol{x}^*_{\text{initial}}$, we have:

$$\ln \boldsymbol{x}_{t+1} - \ln \boldsymbol{x}_t = \lambda_k^t \boldsymbol{u}_k$$

- which implies exponential convergence to steady-state, such that for each location i:

$$\frac{x_{it+1}}{x_{it}} = \exp\left(\lambda_k^t u_{ik}\right)$$

- We can solve for the half-life as:

$$\frac{\frac{1-\lambda_k^t}{1-\lambda_k} u_k}{\frac{1}{1-\lambda_k} u_k} = \frac{1}{2}$$

- which simplifies to:

$$\lambda_k^t = \frac{1}{2}$$

- and hence:

$$\ln \frac{1}{2} = t \ln \lambda_k, \, t = -\frac{\ln 2}{\ln \lambda_k}$$

back

Predictive Power Initial Steady-State

Outcome: 1965-2015 Pop. Log Growth	(1)	(2)	(3)	(4)
1965-2015 Pop. Predicted Log Growth	0.871***	0.959***	0.934***	0.903***
	(0.108)	(0.0780)	(0.0674)	(0.0846)
Log 1965 Population		-0.130***	-0.124***	-0.126***
		(0.0326)	(0.0357)	(0.0381)
Log 1965 K-L Ratio			0.139	0.130
			(0.175)	(0.185)
1965-1966 Growth Rate				2.417
				(4.122)
N	49	49	49	49
R^2	0.503	0.605	0.616	0.617

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