Technical Appendix to The Economics of Density: Evidence from the Berlin Wall*

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1 Introduction

This technical appendix contains a more detailed exposition of the model and its quantitative analysis. Section 2 introduces preferences, technology and endowments and characterizes general equilibrium. Section 3 outlines the quantitative analysis of the model.

2 Theoretical Model

To guide our empirical analysis, we develop a model in which the internal structure of the city is driven by a tension between production externalities (which favor the concentration of economic activity) and commuting costs and an inelastic supply of land (which favor the dispersion of economic activity). We focus on the canonical approach to modelling urban production externalities based on knowledge spillovers. Specifically, we build on the theory of equilibrium city structure of Lucas and Rossi-Hansberg (2002), henceforth LRH, which has the key advantages of modelling location in two spatial dimensions and not imposing a mono-centric city structure.[1] Both of these features are potentially relevant for our empirical analysis, as cities such as Berlin are arranged in latitude and longitude space and need not be mono-centric.

While LRH consider a perfectly symmetric city, in which the radius summarizes the spatial organization of economic activity, in reality cities need not be symmetric. Locations within cities can differ in terms of natural advantages for production (e.g. land gradient and natural supplies of water),

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[1] The classic urban agglomeration models of Alonso (1964), Mills (1967) and Muth (1969) impose a monocentric city structure. While Fujita and Ogawa (1982) and Fujita and Krugman (1995) allow for non-monocentricity, they model one-dimensional cities on the real line.

consumption amenities (e.g. proximity to parks and forests) and transport access (e.g. proximity to transport arteries). Instead of the concentric rings of economic activity found in a symmetric city, actual land rent gradients are typically asymmetric (e.g. higher rents in the West than the East) and there can be uneven clusters of economic activity throughout a city such as Berlin.

We depart from LRH by introducing such asymmetries and developing a tractable quantitative model that can be taken to observed data. We model the city as a set of discrete blocks, which can be irregular in their land area and geographic shape, and can vary in terms of their natural advantages for production and consumption amenities. This set of discrete blocks is connected by a transport network that determines bilateral travel times. This transport network can be itself irregular in geographic shape and can include multiple modes of transport, such as underground and suburban rail lines. To ensure that the model remains tractable despite its discreteness and asymmetries, we model heterogeneity in workers' commuting choices following Eaton and Kortum (2002). This heterogeneity ensures that the supply of commuters to each block is continuous in the relative wage paid by firms located there and gives rise to a gravity equation for bilateral commuting flows, which is consistent with empirical findings from studies of observed commuting flows.

Although the focus of our analysis is Berlin's division and reunification, the model provides a tractable quantitative framework that can be used to analyze alternative interventions in other contexts, such as the construction of new transport infrastructure or changes in urban taxation and spending policies.

2.1 Preferences and Endowments

We consider a city embedded within a larger economy: pre-war Germany before division, West Germany after division, and modern-day Germany after reunification. The city is composed of a set of discrete blocks indexed by $i = 1, ..., S$. The effective supply of land for each block ($L_i = \varphi_i K_i$) depends on geographical land area (K_i) and effective land services per unit of geographical area (φ_i). Effective land services (φ_i) capture, for example, the fraction of geographical land area that is developed and building density on developed land. The total effective supply of land for the city as a whole is therefore $\bar{L} = \sum_{i=1}^{S} \varphi_i K_i$. Land use is determined endogenously and we denote the fractions of land within each block allocated to commercial and residential use by θ_i and $1 - \theta_i$, respectively. We refer to any location with positive inhabitants as a "block of residence" and any location with positive employment as a "block of employment," where blocks can be incompletely specialized and hence serve as locations of both employment and residence. The land market is perfectly competitive and rent is accrued by absentee landlords and not spent within the city.[2]

[2] While the assumption of absentee landlords follows LRH and is standard, we could alternatively assume that land rent is redistributed lump sum to workers.

The city produces a single final good, which is sold to (or purchased from) the larger economy at a competitive price.[3] The city is populated by an endogenous measure of $\bar{H}$ agents, each of whom is endowed with one unit of labor that is supplied inelastically with zero disutility.[4] Workers are mobile across blocks within the city and between the city and the larger economy. City blocks are connected by a bilateral transport network, which workers can use to commute between their locations of residence and employment. Workers are risk neutral such that the utility of worker ω residing in block i and working in block j is linear in an aggregate consumption index ($C_{ij\omega}$):

$$U_{ij\omega} = C_{ij\omega}.$$

This aggregate consumption index is defined over consumption of a single final good ($c_{ij\omega}$) and residential land ($\ell_{ij\omega}$), and is assumed for simplicity to take the Cobb-Douglas form:[5]

$$C_i\left(c_{ij\omega}, \ell_{ij\omega}\right) = B_i c_{ij\omega}^{\beta} \ell_{ij\omega}^{1-\beta}, \qquad 0 < \beta < 1, \tag{1}$$

where the parameter B_i captures residential consumption amenities (such as green spaces and scenic views) that make a block a more or less pleasant place to live, as emphasized in Roback (1982).

The final good is traded at zero cost and is chosen as the numeraire, so that $p_i = p_j = 1$ for all i and j. Given this choice of numeraire, the first-order conditions for consumer equilibrium imply the following demands for the final good and residential land:

$$c_{ij\omega} = \beta v_{ij\omega}, \tag{2}$$

$$\ell_{ij\omega} = (1-\beta)\frac{v_{ij\omega}}{Q_i}, \tag{3}$$

where $v_{ij\omega}$ denotes income net of commuting costs for worker ω residing in block i and commuting to work at block j; Q_i is the residential rent in block i.[6]

Income net of commuting costs ($v_{ij\omega}$) depends on the wage per effective unit of labor at the block of employment j (w_j) the loss of labor time in commuting between blocks i and j (d_{ij}) and a stochastic shock to the productivity of worker ω residing in block i and employed in block j ($z_{ij\omega}$):

$$v_{ij\omega} = \frac{z_{ij\omega} w_j}{d_{ij}}. \tag{4}$$

[3] Even during division, there was substantial trade between West Berlin and West Germany. In 1963, the ratio of exports to GDP in West Berlin was around 70 percent, with West Germany the largest trade partner. Overall, industrial production accounted for around 50 percent of West Berlin's GDP in this year (American Embassy 1965).

[4] While for simplicity, we model agents and workers as being synonymous, it is straightforward to extend the analysis to introduce families, where each worker has a fixed number of dependents that consume but do not work, and where workers maximize family utility. Empirically, we find an approximately log linear relationship between total employment by residence and total population across the districts of Berlin, suggesting that a constant labor force participation rate within the city is a reasonable approximation. While we focus on labor income, the model can be also extended to allow agents to have a constant amount of non-labor income.

[5] For empirical evidence using US data in support of the constant housing expenditure share implied by the Cobb-Douglas functional form, see Davis and Ortalo-Magne (2008).

[6] To clarify the exposition, we use i to denote a worker's block of residence and j to denote the worker's block of employment, except where otherwise indicated.

Commuting costs are modeled here as forgone labor earnings and the labor time lost in commuting between blocks i and j depends on travel time, τ_{ij}:

$$d_{ij} = e^{\kappa \tau_{ij}} \geq 1,$$

where $\kappa > 0$ parameterizes the magnitude of commuting costs.

The stochastic shock $z_{ij\omega}$ is an idiosyncratic productivity draw that determines a worker's effective units of labor in a given employment location. This captures the idea that workers and employment locations may have heterogeneous characteristics that make the pairing of a worker and employment location more or less productive. For a given wage (w_j) and commuting cost (d_{ij}), workers with higher values of $z_{ij\omega}$ receive higher incomes net of commuting costs, because they have more effective units of labor. Workers choose their block of residence before observing their idiosyncratic productivity draws across alternative possible employment locations.[7] Idiosyncratic productivity is drawn from an independent Fréchet distribution with a common scale parameter T and shape parameter ϵ. The cumulative distribution function for worker productivity is therefore:

$$F_{ij}(z) = e^{-Tz^{-\epsilon}}, \qquad T > 0, \ \epsilon > 1, \tag{5}$$

where $\epsilon > 1$ is required for the distribution of worker productivity to have a finite mean.[8] The idiosyncratic productivity draw is independently distributed across both workers and employment locations. Therefore it varies for a given worker ω residing in a given block i across alternative possible blocks of employment j. Since workers with the same productivity draw and the same blocks of employment and residence behave symmetrically, we suppress ω and index workers from now on by i, j and z alone, except where otherwise indicated.

In general, workers residing within the same block have different idiosyncratic productivity draws, choose different blocks of employment, and receive different incomes net of commuting costs. Residential land market clearing within each block of residence implies:

$$\mathbb{E}\left[\ell_{ijz}\right] H_{Ri} = (1 - \theta_i)\, L_i, \tag{6}$$

where H_{Ri} denotes the measure of residents for block i and the presence of the expectations operator ($\mathbb{E}$) reflects the fact that workers with different idiosyncratic productivity draws choose different

[7] While we interpret z as an idiosyncratic productivity draw that determines effective units of labor, an equivalent interpretation is as a stochastic shock to commuting costs. In the specification here workers choose their block of residence before their block of employment, but it is straightforward to consider an alternative specification where workers choose their block of employment first, as long as income net of commuting costs in each block of residence varies stochastically (e.g. because of stochastic commuting costs).

[8] While it is straightforward to allow the Fréchet scale parameter, T, to vary across blocks of residence i or blocks of employment j, such variation plays a similar role in the model to differents in consumption amenities or natural advantages in production. Hence we assume that T is common across blocks.

amounts of residential land (ℓ_{ijz}). From the consumer's first order condition (3), expected residential land use, $\mathbb{E}\left[\ell_{ijz}\right]$, depends linearly on expected worker income, $\mathbb{E}\left[v_{ijz}\right] \equiv \bar{v}_i$.

Each worker chooses their block of residence to maximize their expected utility, taking as given goods and factor prices and the location decisions of firms and other workers. Substituting for equilibrium goods consumption (2) and residential land use (3) in preferences and taking expectations, population mobility requires that workers obtain the same expected utility across all blocks within the city populated in equilibrium:

$$\mathbb{E}\left[U_{ijz}\right] = \beta^{\beta}(1-\beta)^{1-\beta}\bar{v}_i B_i Q_i^{\beta-1} = \bar{U},$$

where $\bar{U}$ is workers' reservation level of expected utility in the larger economy and we choose units in which to measure expected utility such that $\bar{U} = 1$.

This population mobility condition pins down residential rents for each block populated in equilibrium as a function of expected worker income and residential consumption amenities:

$$Q_i^{1-\beta} = \bar{U}^{-1}\beta^{\beta}(1-\beta)^{1-\beta}\bar{v}_i B_i. \tag{7}$$

Expected worker income can be determined using the distribution of income net of commuting costs. From the ability distribution (5) and income net of commuting costs (4), the distribution of income net of commuting costs for a worker employed at block j and residing at block i is:

$$G_{ij}(v) = \Pr[V \leq v] = F_{ij}\left(\frac{v d_{ij}}{w_j}\right) = \Pr[Z \leq z],$$

$$G_{ij}(v) = e^{-Tv^{-\epsilon}d_{ij}^{-\epsilon}w_j^{\epsilon}}. \tag{8}$$

The distribution of income net of commuting costs across *all possible* blocks of employment j for a worker residing in block i is therefore:

$$1 - G_i(v) = 1 - \prod_{s=1}^{S} e^{-Tv^{-\epsilon}d_{is}^{-\epsilon}w_s^{\epsilon}},$$

where the left-hand side is the probability that a worker residing in block i has an income greater than v, and the right-hand side is one minus the probability that a worker has an income less than v for each potential block of employment. Therefore we have:

$$G_i(v) = e^{-\Phi_i v^{-\epsilon}}, \qquad \Phi_i = \sum_{s=1}^{S} T\left(\frac{w_s}{d_{is}}\right)^{\epsilon}. \tag{9}$$

From these distributions of income net of commuting costs, the probability that a worker residing at block i chooses to commute to work at block j is:

$$\pi_{ij} = \int_0^{\infty} \prod_{s \neq j} G_{is}(v) g_{ij}(v) dv,$$

where $G_{is}(v)$ is the probability that the worker draws an income net of commuting costs of less than v for another potential block of employment s, and $g_{ij}(v)$ is the probability that the worker draws an income net of commuting costs equal to v for block j. Evaluating the integral using the distribution of income net of commuting costs, we obtain:

$$\pi_{ij} = \frac{T\left(w_j/d_{ij}\right)^{\epsilon}}{\Phi_i} = \frac{\left(w_j/d_{ij}\right)^{\epsilon}}{\sum_{s=1}^{S}\left(w_s/d_{is}\right)^{\epsilon}}, \tag{10}$$

as shown in the appendix at the end of this document.

The stochastic shock to idiosyncratic productivity introduces heterogeneity in workers' commuting choices (10), such that even when the wage net of commuting costs is higher in block j than in all other blocks s ($w_j/d_{ij} > w_s/d_{is}$), not all of the residents of block i choose to commute to block j. This heterogeneity in workers' commuting choices ensures that each block of employment faces an upward-sloping supply curve for commuters that is continuous in the wage paid. Commuting flows (10) exhibit a gravity-type relationship, where the probability of commuting between blocks i and j depends on wages at the block of employment and travel time between the two blocks, as well as wages and travel time to all other potential blocks of employment. The total volume of commuters between blocks i and j equals the probability of commuting times the measure of workers residing at block i. Therefore the higher the population of a block of residence, the higher the wage at a block of employment, and the shorter the travel time between the two blocks, the greater the volume of bilateral commuters.[9]

With a Fréchet distribution of income net of commuting costs, the expected income net of commuting costs in a block of residence i can be evaluated as:

$$\bar{v}_i = \mathbb{E}\left[v_{ijz}\right] = \int_0^{\infty} v g_i(v) dv$$

$$\bar{v}_i = \Phi_i^{1/\epsilon}\Gamma\left(\frac{\epsilon-1}{\epsilon}\right) = \left[\sum_{s=1}^{S} T\left(w_s/d_{is}\right)^{\epsilon}\right]^{1/\epsilon}\Gamma\left(\frac{\epsilon-1}{\epsilon}\right), \tag{11}$$

where Γ is the Gamma function:

$$\Gamma\left(t\right) \equiv \int_0^{\infty} y^{t-1}e^{-y}dy,$$

and the derivation is again in the appendix at the end of this document. This expression has an intuitive interpretation: expected income net of commuting costs is higher in blocks of residence that are proximate (have favorable commuting access) to blocks of employment that pay high wages.

[9] An empirical literature finds that commuting flows are well described by such a gravity equation relationship, including for example Erlander and Stewart (1990) and Sen and Smith (1995). For empirical evidence for pre-war Berlin on the role of distance in dampening commuting flows, see Feder (1939).

To determine the equilibrium measure of residents in a block, we use residential land market clearing (6) and the first-order condition for residential land use (3), which together imply:

$$H_{Ri} = \frac{1-\theta_i}{1-\beta} \frac{L_i Q_i}{\bar{v}_i}. \tag{12}$$

Substituting for residential rents (7), we obtain:

$$H_{Ri} = \beta^{\frac{\beta}{1-\beta}} \bar{U}^{-\frac{1}{1-\beta}} \bar{v}_i^{\frac{\beta}{1-\beta}} B_i^{\frac{1}{1-\beta}} (1-\theta_i) L_i, \tag{13}$$

which determines a block's measure of residents (H_{Ri}) as a function of its expected worker income ($\bar{v}_i$), residential consumption amenities (B_i), its effective supply of land (L_i), and the fraction of land allocated to residential use ($1-\theta_i$).

Therefore both the measure of residents (13) and residential rents (7) are increasing in a block's expected income net of commuting costs (11), which itself depends on the transport network and wages in all potential blocks of employment. The consumer side of the model is completely characterized by these equations for the measure of residents (13), commuting probabilities (10), residential rents (7) and expected worker income (11), together with the equilibrium allocation of land for residential use ($1-\theta_i$). As commuting probabilities, expected worker income and the equilibrium allocation of land for residential use also depend on production decisions, we now turn to examine the production side of the model.

2.2 Production

The final good is produced under conditions of perfect competition and according to a constant returns to scale technology. For simplicity, we assume that the production technology is Cobb-Douglas, and hence the aggregate amount of final goods output produced in block j is:

$$X_j = A_j \left(\widetilde{H_{Mj}}\right)^{\alpha} (\theta_j L_j)^{1-\alpha},$$

where $\widetilde{H_{Mj}}$ denotes effective employment, adjusted for worker productivity and labor time lost in commuting, as characterized below.

The productivity of final goods production (A_j) depends on a parameter capturing natural advantages (a_j), such as the gradient of the land or the presence of a natural supply of water, and on agglomeration forces (Υ_j). Following LRH and a long literature in economics, we assume that these agglomeration forces take the form of knowledge spillovers that are increasing in the surrounding density of economic activity:[10]

$$A_j = \Upsilon_j^{\gamma} a_j, \qquad \gamma \geq 0, \tag{14}$$

[10] See also Alonso (1964), Fujita and Ogawa (1982), Lucas (2000), Muth (1969), Mills (1969) and Sveikauskas (1975). While we follow this long literature in modeling agglomeration forces as knowledge spillovers, other formalizations are possible, as discussed for example in Duranton and Puga (2004).

$$\Upsilon_j \equiv \sum_{s=1}^{S} e^{-\delta \tau_{js}} \left(\frac{\widetilde{H_{Ms}}}{K_s} \right), \qquad \delta \geq 0. \tag{15}$$

In this specification, knowledge spillovers depend on effective employment ($\widetilde{H_{Ms}}$) per unit of geographical land area (K_s). Effective employment densities in all blocks contribute towards knowledge spillovers, with weights depending on bilateral travel times (τ_{js}). The parameter $\gamma \geq 0$ determines the relative importance of knowledge spillovers for productivity, while $\delta \geq 0$ parameterizes the rate of decay of knowledge spillovers with travel time, where $0 \leq e^{-\delta \tau_{js}} \leq 1$.

Firms choose their block of production, effective employment and commercial land use to maximize their profits taking as given goods and factor prices, productivity and the location decisions of other firms and workers. The first-order conditions for profit maximization imply that firms equate labor's value marginal product with the wage. As a result, effective employment in block j is increasing in productivity (A_j), decreasing in the wage per effective unit of labor (w_j), increasing in the effective supply of land (L_j), and increasing in the fraction of land allocated to commercial use (θ_j):

$$\widetilde{H_{Mj}} = \left(\frac{\alpha A_j}{w_j} \right)^{\frac{1}{1-\alpha}} \theta_j L_j. \tag{16}$$

To determine the equilibrium commercial rent, q_j, often referred to as the bid rent, we use the requirement that profits are zero if the final good is produced:

$$A_j \left(\widetilde{H_{Mj}} \right)^{\alpha} (\theta_j L_j)^{1-\alpha} - w_j \widetilde{H_{Mj}} - q_j \theta_j L_j = 0,$$

which together with effective employment (16) yields the following expression for the equilibrium commercial rent:

$$q_j = (1-\alpha) \left(\frac{\alpha}{w_j} \right)^{\frac{\alpha}{1-\alpha}} A_j^{\frac{1}{1-\alpha}}. \tag{17}$$

Higher productivity (A_j) and lower wages (w_j) make a block a more attractive location for production and are therefore reflected in higher effective employment (16) and commercial rent (17).

Equilibrium wages are determined by labor market clearing, which requires that payments for effective labor input equal income net of commuting costs:

$$w_j \widetilde{H_{Mj}} = \sum_{i=1}^{S} \int_0^{\infty} \prod_{s \neq i} G_{is}(v) v g_{ij}(v) dv H_{Ri}$$

where $G_{is}(v)$ is the probability that the worker draws an income net of commuting costs of less than v for another potential block of employment s, and $g_{ij}(v)$ is the probability that the worker draws an income net of commuting costs equal to v for block j. Evaluating the integral using the distribution of income net of commuting costs, this labor market clearing condition can be written as:

$$w_j \widetilde{H_{Mj}} = \sum_{i=1}^{S} \frac{(w_j/d_{ij})^{\epsilon}}{\left[\sum_{s=1}^{S} (w_s/d_{is})^{\epsilon} \right]} \bar{v}_i H_{Ri},$$

where the left-hand side equals total payments to effective units of labor at the block of employment j after adjusting for worker productivity and the labor time lost in commuting; the right-hand side equals the sum across all blocks of residence i of the probability of commuting to block j times total labor income net of commuting costs in each block i.

Substituting for effective labor input using (16) and expected income net of commuting costs using (11), the labor market clearing condition becomes:

$$w_j^{-\frac{\alpha}{1-\alpha}} \left(\alpha A_j\right)^{\frac{1}{1-\alpha}} \theta_j L_j = \sum_{i=1}^{S} \frac{\left(w_j/d_{ij}\right)^{\epsilon}}{\left[\sum_{s=1}^{S} \left(w_s/d_{is}\right)^{\epsilon}\right]^{1-\frac{1}{\epsilon}}} \Gamma\left(\frac{\epsilon-1}{\epsilon}\right) H_{Ri}, \tag{18}$$

which provides a system of equations for each block of employment j that determines wages (w_j) as a function of productivity (A_j) land use (θ_j) the transport network (d_{ij}) and the measure of inhabitants of each block of residence (H_{Ri}).

Equilibrium wages in (18) depend on both demand-side considerations, as captured by the left-hand side, and supply-side considerations, as captured by the right-hand side. On the demand-side, blocks of employment with higher productivity, greater effective supplies of land and greater fractions of land allocated to commercial use have higher value marginal products of labor and hence a higher demand for effective labor. To attract a supply of effective labor sufficient to meet this demand, these locations pay higher equilibrium wages. On the supply-side, blocks of employment that are remote from blocks of residence with many inhabitants have to pay higher wages to achieve a given supply of effective labor.

Having determined effective employment ($\widetilde{H_{Mj}}$), the measure of employed workers (H_{Mj}) follows immediately from the commuting probabilities (10) and the measure of inhabitants in each block of residence (H_{Ri}):

$$H_{Mj} = \sum_{i=1}^{S} \pi_{ij} H_{Ri},$$

$$H_{Mj} = \sum_{i=1}^{S} \frac{\left(w_j/d_{ij}\right)^{\epsilon}}{\sum_{s=1}^{S} \left(w_s/d_{is}\right)^{\epsilon}} H_{Ri}. \tag{19}$$

Finally, the equilibrium allocation of land between residential and commercial use is determined by no-arbitrage between residential and commercial rents. Land is allocated to whichever use offers the highest rent net of the tax equivalent of land use regulations. As only the relative tax equivalent of land use regulations matters for the allocation of land, we normalize the tax equivalent of land use regulations for residential land to equal one, and allow the tax equivalent of land use regulations for commercial land to vary across blocks ($\Delta_i = 1 + \kappa_i$). Using equilibrium residential rents (7) and commercial rents (17), land use can be characterized as:

$$\theta_i = \begin{cases} 1 & \text{if } q_i > \Delta_i Q_i \\ \widehat{\theta}_i & \text{if } q_i = \Delta_i Q_i \\ 0 & \text{if } q_i < \Delta_i Q_i \end{cases}, \tag{20}$$

where positive fractions of land are only allocated to both commercial and residential use within the same block if residential rents and commercial rents net of the tax equivalent of land use regulations are equalized within that block.

The equilibrium fraction of land allocated to commercial use in blocks that are incompletely specialized ($\widehat{\theta}_i$) is implicitly defined by the no-arbitrage condition:

$$(1-\alpha)\left(\frac{\widetilde{H_{Mi}}}{\widehat{\theta}_i L_i}\right)^{\alpha} A_i = q_i = \Delta_i Q_i = \Delta_i \frac{1-\beta}{1-\widehat{\theta}_i} \bar{v}_i \frac{H_{Ri}}{L_i},$$

which can be equivalently written as:

$$(1-\alpha)\left(\frac{\alpha}{w_i}\right)^{\frac{\alpha}{1-\alpha}} A_i^{\frac{1}{1-\alpha}} = q_i = \Delta_i Q_i = \Delta_i (1-\beta) \beta^{\frac{\beta}{1-\beta}} \bar{U}^{-\frac{1}{1-\beta}} \bar{v}_i^{\frac{1}{1-\beta}} B_i^{\frac{1}{1-\beta}}.$$

where we have used firm profit maximization and zero profits, utility maximization and population mobility, and residential and land market clearing.

From the above no-arbitrage condition, variation in the tax equivalent of land use regulations (Δ_i) across blocks has similar effects on the equilibrium allocation of land as variation in consumption amenities (B_i) relative to natural advantages (a_i) and hence productivity (A_i). As discussed further below, we impose the normalization $\Delta_i = 1$, in which case variation in the tax equivalent of land use regulations is captured in the calibrated values of consumption amenities and natural advantages.

2.3 General Equilibrium

General equilibrium can be referenced by the following seven variables: effective workplace employment ($\widetilde{H_{Mi}}$), residence employment (H_{Ri}), commercial rent (Q_i), residential rent (q_i), wages (w_i), land use (θ_i), and expected worker income net of commuting costs ($\bar{v}_i$). These seven variables are determined by the following seven equations: effective workplace employment (16), residence employment (13), commercial rents (17), residential rents (7), wages (18), land use (20), and expected worker income (11).

3 Quantitative Analysis of the Model

The quantitative analysis of the model makes use of data on five observed variables: land rents ($\mathbb{Q}_i$), employment by workplace (H_{Mi}), employment by residence (H_{Ri}), bilateral travel times (τ_{ij}) and geographical land area (K_i). A single observed land rent ($\mathbb{Q}_i$) is reported in the data for each block, which in the model corresponds to either residential or commercial rent depending on the pattern of specialization: $\mathbb{Q}_i = I_i^C q_i + \left(1 - I_i^C\right) Q_i$, where I_i^C is an indicator variable, which is equal to one if a positive fraction of land is allocated to commercial use and zero otherwise.

The model's properties depend on eight parameters $\{\alpha, \beta, \gamma, T, \epsilon, \kappa, \delta, \bar{U}\}$ and four unobservables that vary across blocks: residential consumption amenities (B_i), natural advantage (a_i), effective land services (φ_i), and the tax equivalent of land use regulations (Δ_i), where we impose the normalization $\Delta_i = 1$. To determine the value of these parameters and unobservables, our analysis proceeds as follows. First, for a given value of the model's parameters $\{\alpha, \beta, \gamma, T, \epsilon, \kappa, \delta, \bar{U}\}$ we calibrate the unobserved values of natural production advantages (a_i), residential consumption amenities (B_i), and effective land services (φ_i) such that the observed distribution of rents, workplace employment and residence employment across blocks is an equilibrium of the model. These unobservables are determined as residuals using the structure of the model and we undertake this calibration for each year separately (1936, 1986, 2006).

Second, we repeat the calibration for alternative parameter values and search for the parameter values that minimize the change in the spatial pattern of the residuals as a result of division or reunification. That is, we search for the parameter values for which the impact of division or reunification on the organization of economic activity within Berlin is largely explained by endogenous changes in agglomeration and dispersion forces rather than by changes in natural advantages, consumption amenities and effective land services. We undertake this search over parameter values separately for the comparisons of the pre-war and division periods and the division and reunification periods to allow parameters, such as the strength of agglomeration and dispersion forces, to change over time.

One key advantage of our approach to determining agglomeration and dispersion forces is that we allow for arbitrary variation in unobserved natural production advantages, residential consumption amenities and effective land services across blocks. We identify the model's parameters by minimizing the change in the spatial pattern of these residuals. Another key advantage of our approach is that it is robust to multiple equilibria, which are a generic feature of models of agglomeration forces. We show below that natural production advantages, residential consumption amenities and effective land services can be uniquely determined in the model from the observed distribution of rents, workplace employment and residence employment across blocks, irrespective of whether another possible equilibrium distribution of these variables exists.[11]

[11] Another approach is to calibrate the model to the pre-war (or re-unification) period and simulate the impact of division, holding natural advantages, consumption amenities and effective land services constant at their values in the base period. Depending on the strength of agglomeration forces relative to the asymmetries in these characteristics across locations, the model can exhibit a unique equilibrium or multiple equilibria for the simulated impact of division. Both approaches can be implemented using our data and the results compared.

3.1 Model Calibration

Throughout this section, we take the model's parameter values $\{\alpha,\ \beta,\ \gamma,\ T,\ \epsilon,\ \kappa,\ \delta,\ \bar{U}\}$ as given, before discussing the search over alternative possible values for parameters in the next section. As the model has a recursive structure, the determination of unobserved natural production advantages, residential consumption amenities and effective land services is straightforward. We first use a system of equations for labor market equilibrium to uniquely determine wages in each location based on observed workplace and residence employment. Having solved for wages, we next use a second bloc of equations for consumer equilibrium to determine expected worker income and residential consumption amenities. We then use a third bloc of equations for producer equilibrium to determine productivity, effective labor input and natural production advantages. Finally, combining consumer and producer equilibrium, we solve for effective land services and the equilibrium fraction of land allocated to commercial and residential use.

3.1.1 Labor Market Clearing

Labor market clearing requires that workplace employment equals the sum of commuting flows from all blocks with positive residence employment:

$$H_{Mj} = \sum_{i=1}^{S} \pi_{ij} H_{Ri} = \sum_{i=1}^{S} \frac{(w_j/d_{ij})^{\epsilon}}{\sum_{s=1}^{S} (w_s/d_{is})^{\epsilon}} H_{Ri}, \qquad d_{ij} = e^{\kappa \tau_{ij}}, \tag{21}$$

Since the π_{ij} sum to one across blocks of employment j for each block of residence i, we require that total workplace employment, $\sum_{j=1}^{S} H_{Mj}$, equals total residence employment, $\sum_{i=1}^{S} H_{Ri}$, for Berlin as a whole. Empirically, the difference between total workplace and residence employment is small for Greater Berlin in the pre-war period and is essentially non-existent for West Berlin during the division period.[12] To ensure the equality between the left and right-hand side in the above expression for commuting flows, we rescale workplace employment in each block by a constant such that total workplace employment across all blocks equals total residence employment.

The above labor market clearing condition can be written as the following excess demand system:

$$D_j(\mathbf{w}) = H_{Mj} - \sum_{i=1}^{S} \frac{(w_j/d_{ij})^{\epsilon}}{\sum_{s=1}^{S} (w_s/d_{is})^{\epsilon}} H_{Ri} = 0, \tag{22}$$

where $\mathbf{H}_M \in \Re_+^S$ is the observed non-negative vector of workplace employment given in the data; $\mathbf{H}_R \in \Re_+^S$ is the observed non-negative vector of residence employment given in the data; and $d_{ij} =$

[12] While in principle total workplace and residence employment could differ because of commuting across the boundary of Greater Berlin, such net commuting was relatively small prior to the Second World War, partly because of the large geographical area of Greater Berlin, and was non-existent during division when West Berlin was isolated from its East German hinterland. In 1933, total workplace and residence employment in Greater Berlin were 1,628,622 and 1,591,723, respectively, implying net inward commuting of 36,899.

$e^{\kappa\tau_{ij}}$, where τ_{ij} denotes the observed travel time between blocks i and j; and $\mathbf{w} \in \Re_{+}^{S}$ is the unknown non-negative wage vector.

Lemma 1 *The excess demand system (22) exhibits the following properties:*

Property (i): *$D(\mathbf{w})$ is continuous.*

Property (ii): *$D(\mathbf{w})$ is homogenous of degree zero.*

Property (iii): *$\sum_{j\in S} D_j(\mathbf{w}) = 0$ for all $\mathbf{w} \in \Re_{+}^{S}$.*

Property (iv): *$D(\mathbf{w})$ exhibits gross substitution:*

$$\frac{\partial D_j(\mathbf{w})}{\partial w_k} > 0 \quad \text{for all } j,k, j \neq k \quad \text{for all } \mathbf{w} \in \Re_{+}^{S},$$
$$\frac{\partial D_j(\mathbf{w})}{\partial w_j} < 0 \quad \text{for all } j \quad \text{for all } \mathbf{w} \in \Re_{+}^{S}.$$

Proof. See the appendix at the end of this document. ■

Proposition 1 *Given the model parameters $\{\alpha, \beta, \gamma, T, \epsilon, \kappa, \delta, \bar{U}\}$, observed workplace employment ($\mathbf{H}_M \in \Re_{+}^{S}$), observed residence employment ($\mathbf{H}_R \in \Re_{+}^{S}$) and observed travel times between all blocks i and j (τ_{ij}), there exists a unique wage vector $\mathbf{w}^* \in \Re_{+}^{S}$ such that $D(\mathbf{w}^*) = 0$.*

Proof. See the appendix at the end of this document. ■

Intuitively, observed workplace and residence employment in each location, together with the gravity structure of commuting flows in the model, uniquely determine the values that wages in each location must take in order for the data to be consistent with an equilibrium of the model. Having solved for the unique equilibrium wage vector, all the other endogenous variables of the model and the three unobservables of interest $\{a_i, B_i, \varphi_i\}$ can be uniquely determined using the recursive structure of the model, as shown below. By inspection of (22), the solution to the labor market clearing condition implies that blocks with zero workplace employment have a zero equilibrium wage. In these blocks, either the entire effective supply of land can be more profitably employed residentially or there is neither commercial nor residential land use (e.g. parks and lakes with both zero workplace employment and zero residence employment).[13]

[13] Parks and forests account for around 25 percent of the area of West Berlin. Empty blocks with neither commercial nor residential land use can arise in the model as a result of zero consumption amenities, zero natural advantages and/or prohibitive land use regulations. While these empty blocks do not themselves contain economic activity, they influence the general equilibrium of the model to the extent that they influence bilateral travel times between other blocks containing positive workplace and/or residence employment.

3.1.2 Consumer Equilibrium

Having solved for equilibrium wages, expected worker income ($\bar{v}_i$) follows immediately from the Fréchet distribution for worker productivity in (11), which is reproduced for clarity below:

$$\bar{v}_i = \left[\sum_{s=1}^{S} T\left(w_s/d_{is}\right)^{\epsilon}\right]^{1/\epsilon} \Gamma\left(\frac{\epsilon-1}{\epsilon}\right), \qquad d_{ij} = e^{\kappa\tau_{ij}}. \tag{23}$$

Given expected worker income, consumption amenities (B_i) for each block with positive residents follow immediately from observed rents ($\mathbb{Q}$) and utility maximization and population mobility (7), as reproduced below:

$$B_i = \frac{\bar{U}\mathbb{Q}_i^{1-\beta}}{\beta^{\beta}(1-\beta)^{1-\beta}\bar{v}_i}. \tag{24}$$

In blocks with zero residents, either the block's entire effective supply of land can be more profitably employed commercially or there is neither commercial nor residential land use. For these blocks, we set consumption amenities equal to zero, which implies zero equilibrium residents from utility maximization and population mobility: $H_{Ri} = \beta^{\frac{\beta}{1-\beta}} \bar{U}^{-\frac{1}{1-\beta}} \bar{v}_i^{\frac{\beta}{1-\beta}} B_i^{\frac{1}{1-\beta}} (1-\theta_i) L_i$.[14]

From (21), (23) and (24), any change in the Fréchet scale parameter (T) results in an immediate and offsetting change in the calibrated value of consumption amenities (B_i). Therefore, without loss of generality, we impose the normalization $T = 1$. Additionally, from (24), any change in the reservation level of utility in the larger economy ($\bar{U}$) leads to an immediate and offsetting change in the calibrated value of consumption amenities (B_i). The choice of $\bar{U}$ is therefore equivalent to a choice of units in which to measure consumption amenities. Without loss of generality, we impose the normalization $\bar{U} = 1$.

3.1.3 Producer Equilibrium

Given observed rents ($\mathbb{Q}_j$) and the solution for wages (w_j) from the first bloc of equations, profit maximization and zero profits determine productivity for each block of employment (A_j) in (17), which is again reproduced for clarity below:

$$A_j = \left(\frac{w_j}{\alpha}\right)^{\alpha} \left(\frac{\mathbb{Q}_j}{1-\alpha}\right)^{1-\alpha}. \tag{25}$$

Intuitively, wages and observed rents together determine the value that unobserved productivity must take in a zero-profit equilibrium with positive production. As discussed above, the solution to the labor market clearing condition implies zero equilibrium wages for blocks with zero workplace employment.

[14] For blocks with zero residence employment but positive workplace employment, consumption amenities must satisfy an inequality constraint. Consumption amenities lie in between zero and the value at which residential rents are equal to commercial rents net of the tax equivalent of land use regulations. As in the case where blocks have both zero residence employment and zero workplace employment, we set consumption amenities equal to zero.

For these blocks, we set natural advantages (a_j) equal to zero, which implies zero productivity (A_j) and hence zero equilibrium wages.[15]

With the solutions for wages (w_j) and expected worker income ($\bar{v}_i$) from the previous blocs of equations in hand, observed residence employment (H_{Ri}) and travel times (τ_{ij}) can be used together with labor market clearing to solve for effective labor input for each block of employment ($\widetilde{H_{Mj}}$):

$$w_j \widetilde{H_{Mj}} = \sum_{i=1}^{S} \frac{(w_j/d_{ij})^{\epsilon}}{\sum_{s=1}^{S} (w_s/d_{is})^{\epsilon}} \bar{v}_i H_{Ri}, \qquad d_{ij} = e^{\kappa \tau_{ij}}, \tag{26}$$

where blocks with zero workplace employment have a zero wage and hence zero effective labor input.

Having solved for productivity (A_j) and effective labor input ($\widetilde{H_{Mj}}$), natural production advantages (a_j) can be determined from observed geographical land area (K_j) and travel times (τ_{ij}) using the specification of knowledge spillovers in (15), as again reproduced below:

$$a_j = A_j \left[\sum_{s=1}^{S} e^{-\delta \tau_{js}} \left(\frac{\widetilde{H_{Ms}}}{K_s} \right) \right]^{-\gamma}, \qquad d_{ij} = e^{\kappa \tau_{ij}}. \tag{27}$$

Therefore observed workplace employment, residence employment and rents, together with the recursive structure of the model, allow us to uniquely determine productivity and the contribution of knowledge spillovers to productivity irrespective of whether the model has a unique equilibrium or multiple equilibria. Intuitively, the observed variables and the structure of the model allow us to determine the values that these unobservables must take in order for the observed variables to be an equilibrium of the model, irrespective of whether another possible equilibrium exists.

3.1.4 Land Market Equilibrium

Given the solutions for consumption amenities (B_i) and expected worker income ($\bar{v}_i$) from consumer equilibrium, and using observed residence employment (H_{Ri}), we immediately obtain total demand for residential land in each block with positive residence employment:

$$(1-\theta_i) L_i = \frac{H_{Ri} \bar{U}^{\frac{1}{1-\beta}}}{\beta^{\frac{\beta}{1-\beta}} B_i^{\frac{1}{1-\beta}} \bar{v}_i^{\frac{\beta}{1-\beta}}}, \tag{28}$$

Given the solutions for wages (w_i), productivity (A_i) and effective labor input ($\widetilde{H_{Mi}}$) from producer equilibrium, we immediately obtain total demand for commercial land in each block with positive workplace employment:

$$\theta_i L_i = \widetilde{H_{Mi}} \left(\frac{w_i}{\alpha A_i} \right)^{\frac{1}{1-\alpha}}, \tag{29}$$

[15] For blocks with zero workplace employment but positive residence employment, natural advantages must satisfy an inequality constraint. Natural advantages lie in between zero and the value at which commercial rents net of the tax equivalent of land use regulations are equal to residential rents. As in the case where blocks have both zero residence employment and zero workplace employment, we set natural advantages equal to zero.

Combining the demands for residential land (28) and commercial land (29) with observed geographical land area (K_i), we can solve for effective land services (φ_i):

$$\varphi_i = \frac{L_i}{K_i} = \frac{\theta_i L_i + (1-\theta_i) L_i}{K_i}, \tag{30}$$

where effective land services captures the fraction of the geographical land area that is developed and the ratio of building floor space to developed land area, as noted above. Having determined φ_i, and hence $L_i = \varphi_i K_i$, we can recover θ_i and $(1-\theta_i)$ immediately from (28) and (29).

3.1.5 Complete Versus Incomplete Specialization

In blocks with positive workplace employment and zero residence employment, we have $\theta_i = 1$, $\mathbb{Q}_i = q_i$ and the calibrated values of natural advantage (a_i) are implicitly defined by:

$$A_i = a_i \left[\sum_{s=1}^{S} e^{-\delta \tau_{is}} \left(\frac{\widetilde{H_{Ms}}}{K_s} \right) \right]^{-\gamma} = \left(\frac{w_i}{\alpha} \right)^{\alpha} \left(\frac{\mathbb{Q}_i}{1-\alpha} \right)^{1-\alpha} = \left(\frac{w_i}{\alpha} \right)^{\alpha} \left(\frac{q_i}{1-\alpha} \right)^{1-\alpha} \tag{31}$$

In blocks with positive residence employment and zero workplace employment, we have $\theta_i = 0$, $\mathbb{Q}_i = Q_i$ and the calibrated values of consumption amenities (B_i) are defined by:

$$B_i = \frac{\bar{U} \mathbb{Q}_i^{1-\beta}}{\beta^{\beta} (1-\beta)^{1-\beta} \bar{v}_i} = \frac{\bar{U} Q_i^{1-\beta}}{\beta^{\beta} (1-\beta)^{1-\beta} \bar{v}_i}, \tag{32}$$

In blocks with both positive workplace employment and positive residence employment, we have $0 < \theta_i < 1$, $\mathbb{Q}_i = q_i = \Delta_i Q_i$ and the calibrated values of natural advantage and residential consumption amenities are determined by:

$$B_i = \frac{\bar{U} \mathbb{Q}_i^{1-\beta}}{\beta^{\beta} (1-\beta)^{1-\beta} \bar{v}_i} = \frac{\bar{U} \left(\Delta_i Q_i \right)^{1-\beta}}{\beta^{\beta} (1-\beta)^{1-\beta} \bar{v}_i} \tag{33}$$

$$A_i = a_i \left[\sum_{s=1}^{S} e^{-\delta \tau_{is}} \left(\frac{\widetilde{H_{Ms}}}{K_s} \right) \right]^{-\gamma} = \left(\frac{w_i}{\alpha} \right)^{\alpha} \left(\frac{\mathbb{Q}_i}{1-\alpha} \right)^{1-\alpha} = \left(\frac{w_i}{\alpha} \right)^{\alpha} \left(\frac{q_i}{1-\alpha} \right)^{1-\alpha}, \tag{34}$$

Since we calibrate B_i to ensure that (33) holds, and our observed rents ($\mathbb{Q}_i$) do not separate out residential rents (Q_i) and commercial rents (q_i), the calibrated values of B_i for these incompletely specialized blocks implicitly capture the impact of the tax equivalent of land use regulations (Δ_i). For a given unobserved value of Q_i, any change in Δ_i in (33) results in an immediate and offsetting change in B_i. We impose the normalization $\Delta_i = 1$, which implies that the impact of the tax equivalent of land use regulations (Δ_i) is captured in the calibrated values of B_i.

This completes the determination of unobserved natural production advantages (a_i), residential consumption amenities (B_i) and effective land services (φ_i), and all other endogenous variables of the model, for given parameter values $\{\alpha, \beta, \gamma, T, \epsilon, \kappa, \delta, \bar{U}\}$ and for given observed values of rents ($\mathbb{Q}_i$), workplace employment (H_{Mi}) and residence employment (H_{Ri}).

3.2 Determination of Parameter Values

To determine the model's parameters, we search over possible parameter values to minimize the change in the spatial pattern of natural production advantages, residential consumption amenities and effective land services between the pre-war and division periods or between the division and reunification periods. We undertake the analysis separately for division and reunification to allow model parameters, such as the strength of agglomeration and dispersion forces, to change over time.

[Work in progress]

A Appendix

A.1 Derivation of Commuting Probabilities

$$
\begin{aligned}
\pi_{ij} &= \Pr\left[v_{ij} \geq \max\{v_{is}\}; s \neq j\right], \\
&= \int_0^\infty \prod_{s\neq j} G_{is}(v) g_{ij}(v) dv, \\
&= \int_0^\infty \prod_{s\neq j} e^{-Tv^{-\epsilon} d_{is}^{-\epsilon} w_s^{\epsilon}} \left(\epsilon v^{-(\epsilon+1)} T d_{ij}^{-\epsilon} w_j^{\epsilon} e^{-Tv^{-\epsilon} d_{ij}^{-\epsilon} w_j^{\epsilon}}\right) dv, \\
&= \frac{T\left(w_j/d_{ij}\right)^{\epsilon}}{\Phi_i}.
\end{aligned}
$$

A.2 Derivation of Expected Income Net of Commuting Costs

$$
\mathbb{E}\left[v_i\right] = \int_0^\infty v g_i(v) dv,
$$

$$
\mathbb{E}\left[v_i\right] = \int_0^\infty v \epsilon v^{-(\epsilon+1)} \Phi_i e^{-\Phi_i v^{-\epsilon}} dv,
$$

Using the change of variable, $y_i \equiv \Phi_i v^{-\epsilon}$, this expression can be re-written as:

$$
\mathbb{E}\left[v_i\right] = \int_0^\infty v e^{-y_i} dy_i,
$$

$$
\mathbb{E}\left[v_i\right] = \Phi_i^{1/\epsilon} \int_0^\infty y_i^{-1/\epsilon} e^{-y_i} dy_i,
$$

$$
\mathbb{E}\left[v_i\right] = \Phi_i^{1/\epsilon} \Gamma\left(\frac{\epsilon - 1}{\epsilon}\right),
$$

where Γ is the Gamma function:

$$
\Gamma\left(t\right) \equiv \int_0^\infty y^{t-1} e^{-y} dy.
$$

A.3 Proof of Lemma 1

Property (i) follows immediately by inspection of (22).

Property (ii) follows immediately by inspection of (22).

Property (iii) can be established by noting:

$$\begin{aligned}\sum_{j=1}^{S} D_j(\mathbf{w}) &= \sum_{j=1}^{S} H_{Mj} - \sum_{i=1}^{S} \frac{\sum_{j=1}^{S}(w_j/d_{ij})^{\epsilon}}{\sum_{s=1}^{S}(w_s/d_{is})^{\epsilon}} H_{Ri} \\ &= \sum_{j=1}^{S} H_{Mj} - \sum_{i=1}^{S} H_{Ri} \\ &= 0.\end{aligned}$$

Property (iv) can be established by noting:

$$\frac{\partial D_j(\mathbf{w})}{\partial w_k} = \sum_{i \in S} \frac{(w_j/d_{ij})^{\varepsilon}\,\varepsilon\,(w_k/d_{ik})^{\varepsilon}\,w_k^{-1}}{\left[\sum_{s\in S}(w_s/d_{is})^{\varepsilon}\right]^2} H_{Ri} > 0.$$

and using homogeneity of degree zero, which implies:

$$\nabla D(\mathbf{w})\,\mathbf{w} = 0,$$

and hence:

$$\frac{\partial D_j(\mathbf{w})}{\partial w_j} < 0 \qquad \text{for all } j \qquad \text{for all } \mathbf{w} \in \Re_+^S.$$

Therefore we have established gross substitution.

A.4 Proof of Proposition 1

We first show that the excess demand system $D(\mathbf{w})$ has at most one (normalized) solution. Gross substitution implies that $D(\mathbf{w}) = D(\mathbf{w}')$ cannot occur whenever $\mathbf{w}$ and $\mathbf{w}'$ are two wage vectors that are not colinear. By homogeneity of degree zero, we can assume $\mathbf{w}' \geq \mathbf{w}$ and $w_j = w_j'$ for some j. Now consider altering the wage vector $\mathbf{w}'$ to obtain the wage vector $\mathbf{w}$ in $S-1$ steps, lowering (or keeping unaltered) the wage of all the other $S-1$ locations $k \neq j$ one at a time. By gross substitution, the excess demand of good j cannot decrease in any step, and because $\mathbf{w} \neq \mathbf{w}'$, it will actually increase in at least one step. Hence $D_j(\mathbf{w}) > D_j(\mathbf{w}')$ and we have a contradiction.

We next establish that there exists a wage vector $\mathbf{w}^* \in \Re_+^S$ such that $D(\mathbf{w}^*) = 0$. By homogeneity of degree zero, we can restrict our search for an equilibrium wage vector to the unit simplex $\Delta = \left\{\mathbf{w} \in \Re_+^S : \sum_{j=1}^{S} w_j = 1\right\}$. Define on Δ the function $D^+(\cdot)$ by $D_j^+(\mathbf{w}) = \max\{D_j(\mathbf{w}), 0\}$. Note that $D^+(\cdot)$ is continuous. Denote $\alpha(\mathbf{w}) = \sum_{j=1}^{S}\left[w_j + D_j^+(w)\right]$. We have $\alpha(\mathbf{w}) \geq 1$ for all $\mathbf{w}$.

Define a continuous function $f(\cdot)$ from the closed convex set Δ into itself by:

$$f(\mathbf{w}) = [1/\alpha(\mathbf{w})]\left[\mathbf{w} + D^+(\mathbf{w})\right].$$

Note that this fixed-point function tends to increase the wages of locations with excess demand for labor. By Brouwer's Fixed-point Theorem, there exists $\mathbf{w}^* \in \Delta$ such that $\mathbf{w}^* = f(\mathbf{w}^*)$.

Since $\sum_{j=1}^{S} D_j(\mathbf{w}) = 0$, it cannot be the case that $D_j(\mathbf{w}) > 0$ for all $j = 1, \ldots, S$ or $D_j(\mathbf{w}) < 0$ for all $j = 1, \ldots, S$. Additionally, if $D_j(\mathbf{w}) > 0$ for some j and $D_k(\mathbf{w}) < 0$ for some $k \neq j$, $\mathbf{w} \neq f(\mathbf{w})$. It follows that at the fixed point for wages, $\mathbf{w}^* = f(\mathbf{w}^*)$, $D_j(\mathbf{w}) = 0$ for all j.

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