Web Appendix for Goods Trade, Factor Mobility and Welfare (Not for Publication)*

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1 Introduction

This appendix contains the technical derivations of expressions for each section of the paper, the proofs of propositions and additional supplementary material. Sections 2-6 have the same title as the corresponding section of the paper. Section 7 considers an extension of the model to allow land to be used commercially and to incorporate intermediate inputs in production.

2 Theoretical Framework

We consider an economy consisting of many (potentially asymmetric) locations indexed by $i, n \in N$. Locations can differ from one another in terms of land supply, productivity, amenities and their geographical location relative to one another. Bilateral trade costs for goods are assumed to take the iceberg form, such that d_{ni} units of a good must be shipped from location i for one unit to arrive in location n, where $d_{ni} > 1$ for $n \neq i$ and $d_{nn} = 1$. Land and labor are the two factors of production. Workers are mobile across locations but have idiosyncratic draws for preferences for each location.[1]

2.1 Consumer Preferences

Preferences for worker ω residing in location n depend on goods consumption (C_n), residential land use (H_{Un}) and idiosyncratic amenity shocks to the utility from residing in each location n:[2]

$$U_n(\omega) = b_n(\omega) \left(\frac{C_n(\omega)}{\alpha} \right)^{\alpha} \left(\frac{H_{Un}(\omega)}{1-\alpha} \right)^{1-\alpha}, \qquad 0 < \alpha < 1. \tag{1}$$

The goods consumption index (C_n) is defined over consumption of a fixed continuum of goods $j \in [0,1]$:

$$C_n = \left[\int_0^1 c_n(j)^{\rho} dj \right]^{\frac{1}{\rho}}, \tag{2}$$

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[1] While we interpret the idiosyncratic draws in terms of worker preferences, an alternative formulation is possible in terms of idiosyncratic draws for worker productivity for each location. These two formulations have similar predictions for expected utility but different predictions for expected real wages across locations.

[2] For empirical evidence using U.S. data in support of the constant housing expenditure share implied by the Cobb-Douglas functional form, see Davis and Ortalo-Magné (2011).

where the CES parameter (ρ) determines the elasticity of substitution between goods ($\sigma = 1/(1-\rho)$). The corresponding dual price index for goods consumption (P_n) is:

$$P_n = \left[\int_0^1 p_n(j)^{1-\sigma} dj \right]^{\frac{1}{1-\sigma}}, \qquad \sigma = \frac{1}{1-\rho}. \tag{3}$$

The idiosyncratic amenity shocks ($b_n(\omega)$) capture the idea that workers have heterogeneous preferences for living in each location (e.g. different preferences for climate, proximity to the coast etc). We assume that these amenity shocks are drawn independently across locations and workers from a Fréchet distribution:

$$G_n(b) = e^{-B_n b^{-\epsilon}}, \tag{4}$$

where the scale parameter B_n determines average amenities for location n and the shape parameter ϵ controls the dispersion of amenities across workers for each location. Each worker is endowed with one unit of labor that is supplied inelasticity with zero disutility.

2.2 Production

Each location draws an idiosyncratic productivity $z(j)$ for each good j. Productivity is independently drawn across goods and locations from a Fréchet distribution:

$$F_i(z) = e^{-A_i z^{-\theta}}, \tag{5}$$

where the scale parameter A_i determines average productivity for location i and the shape parameter θ controls the dispersion of productivity across goods.

Goods are homogeneous in the sense one unit of a given good is the same as any other unit of that good. Each good is produced with labor under conditions of perfect competition according to a linear technology.[3] The cost to a consumer in location n of purchasing one unit of good j from location i is therefore:

$$p_{ni}(j) = \frac{d_{ni} w_i}{z_i(j)}, \tag{6}$$

where w_i denotes the wage in location i.

2.3 Expenditure Shares and Price Indices

We begin by characterizing expenditure shares and price indices. The representative consumer in location n sources each good from the lowest-cost supplier to that location:

$$p_n(j) = \min\{p_i(j); i \in N\}. \tag{7}$$

Using equilibrium prices (6) and the properties of the Fréchet distribution following Eaton and Kortum (2002), the share of expenditure of location n on goods produced by location i is:

$$\pi_{ni} = \frac{A_i \left(d_{ni} w_i\right)^{-\theta}}{\sum_{s \in N} A_s \left(d_{ns} w_s\right)^{-\theta}}, \tag{8}$$

where the elasticity of trade with respect to trade costs is determined by the Fréchet shape parameter for productivity θ. The price index dual to (2) can be expressed as:

$$P_n = \gamma \left[\sum_{i \in N} A_i \left(d_{ni} w_i\right)^{-\theta} \right]^{-1/\theta}, \tag{9}$$

[3] While to simplify the exposition we assume that land is only used residentially, it is straightforward to also allow land to be used commercially, as shown in a later section of this appendix.

where $\gamma = \left[\Gamma\left(\frac{\theta+1-\sigma}{\theta}\right)\right]^{\frac{1}{1-\sigma}}$ and $\Gamma(\cdot)$ denotes the Gamma function. To ensure a finite value for the price index, we require $\theta > \sigma - 1$. Using the trade share (8) and $d_{nn} = 1$, the goods price index can be equivalently written as:

$$P_n^{-\theta} = \frac{\gamma^{-\theta} A_n w_n^{-\theta}}{\pi_{nn}}. \tag{10}$$

2.4 Residential Choices and Income

We next examine worker location choices. Given the specification of consumer preferences (1), the corresponding indirect utility function is:

$$U_n(\omega) = \frac{b_n(\omega) v_n(\omega)}{P_n^{\alpha} r_n^{1-\alpha}}, \tag{11}$$

where r_n is the land rent for location n and v_n is the income of worker ω in location n, which differs from the wage w_n because income from land rents is redistributed to the residents of each location, as discussed below. Since indirect utility is a monotonic function of the amenity draw, it too has a Fréchet distribution:

$$G_n(U) = e^{-\psi_n U^{-\epsilon}}, \qquad \psi_n = B_n \left(v_n / P_n^{\alpha} r_n^{1-\alpha}\right)^{\epsilon}.$$

Each worker chooses the location that offers her the highest utility after taking into account her idiosyncratic preferences. Using the above distribution of indirect utility, the probability that a worker chooses to live in location $n \in N$ is:

$$\frac{L_n}{\bar{L}} = \frac{B_n \left(v_n / P_n^{\alpha} r_n^{1-\alpha}\right)^{\epsilon}}{\sum_{k \in N} B_k \left(v_k / P_k^{\alpha} r_k^{1-\alpha}\right)^{\epsilon}}. \tag{12}$$

Each location faces a labor supply curve that is upward sloping in real income. Therefore, real income in general differs across locations, because higher real income has to be paid to attract workers with lower idiosyncratic tastes for a location. Expected utility for a worker across locations is:

$$\bar{U} = \delta \left[\sum_{k \in N} B_k \left(v_k / P_k^{\alpha} r_k^{1-\alpha}\right)^{\epsilon}\right]^{\frac{1}{\epsilon}}, \tag{13}$$

where $\delta = \Gamma\left((\epsilon - 1)/\epsilon\right)$ and $\Gamma\left(\cdot\right)$ is the Gamma function. To ensure a finite value for expected utility, we require $\epsilon > 1$.

An implication of the Fréchet distribution of utility is that expected utility conditional on living in location n is the same across all locations n and equal to expected utility for the economy as a whole. On the one hand, more attractive location characteristics directly raise the utility of a worker with a given idiosyncratic utility draw, which increases expected utility. On the other hand, more attractive location characteristics attract workers with lower idiosyncratic utility draws, which reduces expected utility. With a Fréchet distribution of utility, these two effects exactly offset one another. Therefore, although real incomes in general differ across locations for the reasons discussed above, expected utility (taking into account idiosyncratic shocks) is the same across locations. Hence this common value for expected utility provides a sufficient statistic for capturing the welfare gains from trade for all locations.

Expenditure on land in each location is redistributed lump sum to the workers residing in that location. Therefore total income in each location (v_n) equals labor income plus expenditure on residential land:

$$v_n L_n = w_n L_n + (1 - \alpha) v_n L_n = \frac{w_n L_n}{\alpha}. \tag{14}$$

Labor income in each location equals expenditure on goods produced in that location:

$$w_i L_i = \sum_{n \in N} \pi_{ni} w_n L_n. \tag{15}$$

Land market clearing implies that the equilibrium land rent can be determined from the equality of land income and expenditure:

$$r_n = \frac{(1-\alpha)v_n L_n}{H_n} = \frac{1-\alpha}{\alpha}\frac{w_n L_n}{H_n}. \tag{16}$$

2.5 General Equilibrium

The general equilibrium of the model can be represented by the measure of workers (L_n) in each location $n \in N$, the share of each location's expenditure on goods produced in other locations (π_{ni}) and the wage in each location (w_n). Using labor income (15), the trade share (8), the price index (10), residential choice probabilities (12) and land market clearing (16), this equilibrium triple $\{L_n, \pi_{ni}, w_n\}$ solves the following system of equations for all $i, n \in N$. First, each location's income must equal expenditure on the goods produced in that location:

$$w_i L_i = \sum_{n \in N} \pi_{ni} w_n L_n. \tag{17}$$

Second, location expenditure shares are:

$$\pi_{ni} = \frac{A_i \left(d_{ni} w_i\right)^{-\theta}}{\sum_{k \in N} A_k \left(d_{nk} w_k\right)^{-\theta}}. \tag{18}$$

Third, residential choice probabilities imply:

$$\frac{L_n}{\bar{L}} = \frac{B_n \left(\frac{A_n}{\pi_{nn}}\right)^{\frac{\alpha\epsilon}{\theta}} \left(\frac{L_n}{H_n}\right)^{-\epsilon(1-\alpha)}}{\sum_{k \in N} B_k \left(\frac{A_k}{\pi_{kk}}\right)^{\frac{\alpha\epsilon}{\theta}} \left(\frac{L_k}{H_k}\right)^{-\epsilon(1-\alpha)}}. \tag{19}$$

2.6 Existence and Uniqueness

We now show that there exists a unique general equilibrium that solves the system of equations (17)-(19). Using the trade share (8), the requirement that income equals expenditure for each location $n \in N$ can be re-written as:

$$\frac{w_i^{1+\theta} L_i}{A_i} = \sum_{n \in N} d_{ni}^{-\theta} \pi_{nn} \frac{w_n^{1+\theta} L_n}{A_n}. \tag{20}$$

Additionally, using labor income (15), the residential choice probabilities (12), expected utility (13) and land market clearing (16), we obtain the following alternative expression for the goods price index (P_n):

$$P_n^{-\theta} = \left(\frac{w_n}{\bar{W}}\right)^{-\theta} B_n^{-\frac{\theta}{\alpha\epsilon}} H_n^{-\theta\left(\frac{1-\alpha}{\alpha}\right)} L_n^{\theta\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right)}, \tag{21}$$

where $\bar{W} = \left[\alpha^{\epsilon}\left(\frac{1-\alpha}{\alpha}\right)^{\epsilon(1-\alpha)}\left(\bar{U}/\delta\right)^{\epsilon}\left(\bar{L}\right)^{-1}\right]^{1/\alpha\epsilon}$ captures the expected utility ($\bar{U}$) across locations and parameters. Combining this expression for the goods price index with (10), the domestic trade share (π_{nn}) can be written as:

$$\pi_{nn} = \bar{W}^{-\theta}\gamma^{-\theta} A_n B_n^{\frac{\theta}{\alpha\epsilon}} H_n^{\theta\left(\frac{1-\alpha}{\alpha}\right)} L_n^{-\theta\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right)}. \tag{22}$$

Using this expression for the domestic trade share in the equality of income and expenditure (20), we obtain a first system of equations linking the wages and populations of locations $n \in N$ as a function of parameters and expected utility:

$$\bar{W}^{-\theta} = \frac{w_n^{1+\theta} L_n / A_n}{\gamma^{-\theta}\left[\sum_{k \in N} d_{kn}^{-\theta} B_k^{\frac{\theta}{\alpha\epsilon}} H_k^{\theta\left(\frac{1-\alpha}{\alpha}\right)} w_k^{1+\theta} L_k^{1-\theta\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right)}\right]}. \tag{23}$$

Returning to (21) and using the expression for the price index (9), we obtain a second system of equations linking the wages and populations of locations $n \in N$ as a function of parameters and expected utility:

$$\bar{W}^{-\theta} = \frac{w_n^{-\theta} B_n^{-\frac{\theta}{\alpha\epsilon}} H_n^{-\theta\left(\frac{1-\alpha}{\alpha}\right)} L_n^{\theta\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right)}}{\gamma^{-\theta}\left[\sum_{k\in N} A_k \left(d_{nk} w_k\right)^{-\theta}\right]}. \tag{24}$$

We characterize the properties of the general equilibrium of the model under the assumption that transport costs (d_{ni}) are "quasi-symmetric," which implies that they can be partitioned into an importer component (D_n), an exporter component (D_i) and a symmetric bilateral component ($D_{ni} = D_{in}$):

$$d_{ni} = \begin{cases} 1 & \text{if } n = i \\ D_n D_i D_{ni} & \text{if } n \neq i \end{cases}, \tag{25}$$

where $D_n > 1$, $D_i > 1$ and $D_{ni} = D_{in} > 1$. Under this assumption, the two wage systems (23) and (24) imply the following closed form solution linking the endogenous variables for each location $n \in N$:

$$w_n^{1+2\theta} A_n^{-1} B_n^{\frac{\theta}{\alpha\epsilon}} H_n^{\theta\left(\frac{1-\alpha}{\alpha}\right)} L_n^{1-\theta\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right)} = \kappa, \tag{26}$$

where κ is a scalar. If equation (26) holds, then any functions w_n and L_n that satisfy the system of equations (23) will also satisfy the system of equations (24) (and vice versa). In the proposition below, we prove below that equation (26) is the unique relationship between w_n and L_n that satisfies both systems. Substituting this relationship (26) into (24), we obtain the following system of equations for equilibrium populations.

$$L_n^{\tilde{\theta}\gamma_1} A_n^{-\tilde{\theta}} B_n^{-\frac{\tilde{\theta}(1+\theta)}{\alpha\epsilon}} H_n^{-\frac{\tilde{\theta}(1+\theta)(1-\alpha)}{\alpha}} = \bar{W}^{-\theta}\gamma^{-\theta}\left[\sum_{k\in N} d_{nk}^{-\theta} A_k^{\frac{\tilde{\theta}(1+\theta)}{\theta}} B_k^{\frac{\tilde{\theta}\theta}{\alpha\epsilon}} H_k^{\frac{\tilde{\theta}\theta(1-\alpha)}{\alpha}} \left(L_k^{\tilde{\theta}\gamma_1}\right)^{\frac{\gamma_2}{\gamma_1}}\right], \tag{27}$$

$$\tilde{\theta} \equiv \frac{\theta}{1+2\theta},$$

$$\gamma_1 \equiv 1 + (1+\theta)\left(\frac{1}{\alpha\epsilon} + \frac{1-\alpha}{\alpha}\right),$$

$$\gamma_2 \equiv 1 - \theta\left(\frac{1}{\alpha\epsilon} + \frac{1-\alpha}{\alpha}\right) < \gamma_1.$$

This system of equations (27) uniquely determines the equilibrium population of each location $n \in N$ up to a normalization for expected utility ($\bar{W}$) (a choice of units in which to measure expected utility).

Proposition 1 *Given the land area, productivity and amenity parameters $\{H_n, A_n, B_n\}$ and quasi-symmetric bilateral trade frictions $\{d_{ni}\}$ for each location $n \in N$, there exist unique equilibrium populations (L_n^*), trade shares (π_{ni}^*) and wages (w_n^*).*

Proof. The proof follows the same structure as in Allen and Arkolakis (2014). Given the land area, productivity and amenity parameters $\{H_n, A_n, B_n\}$ and bilateral trade frictions $\{d_{ni}\}$, there exists a unique fixed point in the system of equations (27) because $\gamma_2/\gamma_1 < 1$ (see for example Fujimoto and Krause 1985). This unique fixed point determines the unique equilibrium population (L_n^*) for each location $n \in N$ up to a normalization determined by the expected utility ($\bar{W}$). This normalization ($\bar{W}$) is determined by combining the system of equations (27) with the requirement that the labor market clear: $\sum_{n\in N} L_n = \bar{L}$. Having determined the unique equilibrium population (L_n^*) for each location, the closed-form relationship between the endogenous variables (26) immediately yields the unique equilibrium wage (w_n^*) for each location $n \in N$. Having solved for equilibrium wages and populations $\{L_n^*, w_n^*\}$, the expenditure shares (18) determine unique equilibrium trade shares (π_{ni}^*) for each pair of locations $n, i \in N$. ■

2.7 Comparative Statics

Although we allow for both trade costs and heterogeneity in worker preferences, and consider a large number of locations that can differ from one another in productivity, amenities, land supplies and bilateral trade costs, the model admits closed-form expressions for the comparative statics of the endogenous variables with respect to the relative value of these location characteristics. To characterize these comparative statics, we re-write the system of equations for equilibrium populations (27) as the following implicit function:

$$\begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_N \end{pmatrix} = \begin{pmatrix} \Omega_1^I \\ \vdots \\ \Omega_N^I \end{pmatrix} - \begin{pmatrix} \Omega_1^{II} \\ \vdots \\ \Omega_N^{II} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{28}$$

$$\begin{aligned}
\Omega_n^I &= L_n^{\tilde{\theta}\gamma_1} A_n^{-\theta} B_n^{-\frac{\tilde{\theta}(1+\theta)}{\alpha\epsilon}} H_n^{-\frac{\tilde{\theta}(1+\theta)(1-\alpha)}{\alpha}}, \\
\Omega_n^{II} &= \sum\nolimits_{k\in N} \Omega_{nk}^{II}, \\
\Omega_{nk}^{II} &= \bar{W}^{-\theta}\gamma^{-\theta} d_{nk}^{-\theta} A_k^{\frac{\tilde{\theta}(1+\theta)}{\theta}} B_k^{\frac{\tilde{\theta}\theta}{\alpha\epsilon}} H_k^{\frac{\tilde{\theta}\theta(1-\alpha)}{\alpha}} \left(L_k^{\tilde{\theta}\gamma_1}\right)^{\frac{\gamma_2}{\gamma_1}},
\end{aligned}$$

where Ω_n^{II} has an interpretation as a *market access* term that captures the goods market access of each location (depending on trade costs d_{nk}) to the characteristics of other locations.

We show below that the implicit function (Ω_n) is monotonically decreasing in the productivities, amenities and land supplies of all locations and monotonically increasing in the trade costs to other locations. An implication is that the equilibrium population of each location (L_n) depends solely on the *relative* rather than the absolute levels of these characteristics. We also show below that the implicit function (Ω_n) is monotonically increasing in own population (L_n) and monotonically decreasing in the population of other locations (L_k for $k \neq n$). Therefore the system of equations for equilibrium populations (27) satisfies gross substitution and yields unambiguous comparative static predictions.

We first show that the implicit function (28) is monotonically decreasing in the productivity (A_n), amenities (B_n) and land supply (H_n) of each location. Consider the derivative of the implicit function (Ω_n) for location n with respect to these characteristics of location n:

$$\frac{\partial \Omega_n}{\partial A_n} = -\left[\theta + \frac{\tilde{\theta}(1+\theta)}{\theta}\frac{\Omega_{nn}^{II}}{\Omega_n^{II}}\right]\frac{\Omega_n^I}{A_n} < 0, \tag{29}$$

$$\frac{\partial \Omega_n}{\partial B_n} = -\left[\frac{\tilde{\theta}(1+\theta)}{\alpha\epsilon} + \frac{\tilde{\theta}\theta}{\alpha\epsilon}\frac{\Omega_{nn}^{II}}{\Omega_n^{II}}\right]\frac{\Omega_n^I}{B_n} < 0, \tag{30}$$

$$\frac{\partial \Omega_n}{\partial H_n} = -\left[\frac{\tilde{\theta}(1+\theta)(1-\alpha)}{\alpha} + \frac{\tilde{\theta}\theta(1-\alpha)}{\alpha}\frac{\Omega_{nn}^{II}}{\Omega_n^{II}}\right]\frac{\Omega_n^I}{H_n} < 0, \tag{31}$$

where we have used $\Omega_n^{II} = \Omega_n^I$. Consider the derivative of the implicit function (Ω_k) for location $k \neq n$ with respect to these characteristics of location n:

$$\frac{\partial \Omega_k}{\partial A_n} = -\frac{\tilde{\theta}(1+\theta)}{\theta}\frac{\Omega_{kn}^{II}}{\Omega_k^{II}}\frac{\Omega_k^I}{A_n} < 0, \qquad \forall\, k \neq n, \tag{32}$$

$$\frac{\partial \Omega_k}{\partial B_n} = -\frac{\tilde{\theta}\theta}{\alpha\epsilon}\frac{\Omega_{kn}^{II}}{\Omega_k^{II}}\frac{\Omega_k^I}{B_n} < 0, \qquad \forall\, k \neq n, \tag{33}$$

$$\frac{\partial \Omega_k}{\partial H_n} = -\frac{\tilde{\theta}\theta(1-\alpha)}{\alpha}\frac{\Omega_{kn}^{II}}{\Omega_k^{II}}\frac{\Omega_k^I}{H_n} < 0, \qquad \forall\, k \neq n, \tag{34}$$

where we have used $\Omega_k^{II} = \Omega_k^I$.

We next show that the implicit function (28) is monotonically increasing in trade costs to other locations. Under our assumption of quasi-symmetric trade costs, we have:

$$d_{ni} = \begin{cases} 1 & \text{if } n = i \\ D_n D_i D_{ni} & \text{if } n \neq i \end{cases},$$

where $D_n > 1$, $D_i > 1$ and $D_{ni} = D_{in} > 1$. Consider the derivative of the implicit function (Ω_n) for location n with respect to the common component of trade costs (D_n) for location n:

$$\frac{\partial \Omega_n}{\partial D_n} = \theta \sum_{k \neq n} \frac{\Omega_{nk}^{II}}{D_n} > 0, \tag{35}$$

Now consider the derivative of the implicit function (Ω_k) for location $k \neq n$ with respect to the common component of trade costs (D_n) for location n:

$$\frac{\partial \Omega_k}{\partial D_n} = \theta \frac{\Omega_{kn}^{II}}{D_n} > 0, \qquad \forall\, k \neq n. \tag{36}$$

Finally, we show that the implicit function for each location is monotonically increasing in its own population and monotonically decreasing in the population of other locations:

$$\frac{\partial \Omega_n}{\partial L_n} = \tilde{\theta}\gamma_1 \left[1 - \frac{\gamma_2}{\gamma_1}\frac{\Omega_{nn}^{II}}{\Omega_n^{II}}\right]\frac{\Omega_n^{I}}{L_n} > 0, \tag{37}$$

$$\frac{\partial \Omega_k}{\partial L_n} = -\tilde{\theta}\gamma_1 \frac{\gamma_2}{\gamma_1}\frac{\Omega_{kn}^{II}}{\Omega_k^{II}}\frac{\Omega_k^{I}}{L_n} < 0, \qquad \forall\, k \neq n, \tag{38}$$

where we have again used $\Omega_n^{II} = \Omega_n^{I}$.

Proposition 2 *Assuming that bilateral trade frictions (d_{ni}) are quasi-symmetric, an increase in the productivity (A_n), amenities (B_n) or land supply (H_n) of a location n relative to all other locations increases the equilibrium population of that location relative to all other locations $k \neq n$, other things equal. An increase in location n's trade costs to all other locations $k \neq n$ (D_n) decreases the equilibrium population of that location relative to all other locations $k \neq n$, other things equal.*

Proof. Consider an increase in location n's productivity ($dA_n > 0$) and a decrease in the productivity of all other locations ($dA_k < 0$ for $k \neq n$) such that expected utility ($\bar{W}$) remains constant. Differentiating in (28), we have:

$$\boldsymbol{dL} = -\boldsymbol{\Omega}_L^{-1}\, \boldsymbol{\Omega}_A\, \boldsymbol{dA}, \tag{39}$$

$$\begin{pmatrix} dL_1 \\ \vdots \\ dL_n \\ \vdots \\ dL_N \end{pmatrix} = - \begin{pmatrix} \frac{\partial \Omega_1}{\partial L_1} & \cdots & \frac{\partial \Omega_1}{\partial L_n} & \cdots & \frac{\partial \Omega_1}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_n}{\partial L_1} & \cdots & \frac{\partial \Omega_n}{\partial L_n} & \cdots & \frac{\partial \Omega_n}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_N}{\partial L_1} & \cdots & \frac{\partial \Omega_N}{\partial L_n} & \cdots & \frac{\partial \Omega_N}{\partial L_N} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Omega_1}{\partial A_1} & \cdots & \frac{\partial \Omega_1}{\partial A_n} & \cdots & \frac{\partial \Omega_1}{\partial A_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_n}{\partial A_1} & \cdots & \frac{\partial \Omega_n}{\partial A_n} & \cdots & \frac{\partial \Omega_n}{\partial A_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_N}{\partial A_1} & \cdots & \frac{\partial \Omega_N}{\partial A_n} & \cdots & \frac{\partial \Omega_N}{\partial A_N} \end{pmatrix} \begin{pmatrix} dA_1 \\ \vdots \\ dA_n \\ \vdots \\ dA_N \end{pmatrix}.$$

From Cramer's rule (see for example Samuelson 1953), we have:

$$\frac{\partial L_k}{\partial A_n} = -\frac{\sum_i \frac{\partial \Omega_i}{\partial A_n} \triangle_{ik}}{\triangle},$$

where

$$\triangle = \begin{vmatrix} \frac{\partial \Omega_1}{\partial L_1} & \cdots & \frac{\partial \Omega_1}{\partial L_n} & \cdots & \frac{\partial \Omega_1}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_n}{\partial L_1} & \cdots & \frac{\partial \Omega_n}{\partial L_n} & \cdots & \frac{\partial \Omega_n}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_N}{\partial L_1} & \cdots & \frac{\partial \Omega_N}{\partial L_n} & \cdots & \frac{\partial \Omega_N}{\partial L_N} \end{vmatrix},$$

and $\triangle_{ik}$ indicates the cofactor of the element of the ith row and kth column.

We have shown above that $\partial \Omega_n / \partial L_k < 0 < \partial \Omega_n / \partial L_n$ (from (37) and (38)). Therefore the matrix $\mathbf{\Omega}_L$ satisfies gross substitution, with positive diagonal terms and negative off-diagonal terms. It follows that all the successive principal minors of $\mathbf{\Omega}_L$ are positive and the matrix $\mathbf{\Omega}_L$ is positive definite with a positive determinant $\triangle$ (see for example Theorem 4.C.5 in Takayama 1995). We have also shown above that $\partial \Omega_n / \partial A_n < \partial \Omega_n / \partial A_k < 0$ (from (29) and (32)). Therefore each element of $\mathbf{\Omega}_A$ is negative. It follows that the system of equations (39) satisfies the properties for unambiguous comparative statics in Proposition 17.G.3 of Section 17.G of Mas-Collel, Whinston, and Green (1995). Since $dA_n > 0$ and $dA_k < 0$ for all $k \neq n$, we have $dL_n > 0$ and $dL_k < 0$ for all $k \neq n$.

Consider an increase in location n's amenities ($dB_n > 0$) and a decrease in the amenities of all other locations ($dB_k < 0$ for $k \neq n$) such that expected utility ($\bar{W}$) remains constant. Differentiating in (28), we have:

$$\boldsymbol{dL} = -\mathbf{\Omega}_L{}^{-1}\, \mathbf{\Omega}_B\, \boldsymbol{d}B, \tag{40}$$

$$\begin{pmatrix} dL_1 \\ \vdots \\ dL_n \\ \vdots \\ dL_N \end{pmatrix} = - \begin{pmatrix} \frac{\partial \Omega_1}{\partial L_1} & \cdots & \frac{\partial \Omega_1}{\partial L_n} & \cdots & \frac{\partial \Omega_1}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_n}{\partial L_1} & \cdots & \frac{\partial \Omega_n}{\partial L_n} & \cdots & \frac{\partial \Omega_n}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_N}{\partial L_1} & \cdots & \frac{\partial \Omega_N}{\partial L_n} & \cdots & \frac{\partial \Omega_N}{\partial L_N} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Omega_1}{\partial B_1} & \cdots & \frac{\partial \Omega_1}{\partial B_n} & \cdots & \frac{\partial \Omega_1}{\partial B_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_n}{\partial B_1} & \cdots & \frac{\partial \Omega_n}{\partial B_n} & \cdots & \frac{\partial \Omega_n}{\partial B_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_N}{\partial B_1} & \cdots & \frac{\partial \Omega_N}{\partial B_n} & \cdots & \frac{\partial \Omega_N}{\partial B_N} \end{pmatrix} \begin{pmatrix} dB_1 \\ \vdots \\ dB_n \\ \vdots \\ dB_N \end{pmatrix}.$$

From Cramer's rule (see for example Samuelson 1953), we have:

$$\frac{\partial L_k}{\partial B_n} = -\frac{\sum_i \frac{\partial \Omega_i}{\partial B_n} \triangle_{ik}}{\triangle},$$

where $\triangle$ and $\triangle_{ik}$ are defined above.

We have shown above that $\partial \Omega_n / \partial L_k < 0 < \partial \Omega_n / \partial L_n$ (from (37) and (38)). Therefore the matrix $\mathbf{\Omega}_L$ satisfies gross substitution, with positive diagonal terms and negative off-diagonal terms. It follows that all the successive principal minors of $\mathbf{\Omega}_L$ are positive and the matrix $\mathbf{\Omega}_L$ is positive definite with a positive determinant $\triangle$ (see for example Theorem 4.C.5 in Takayama 1995). We have also shown above that $\partial \Omega_n / \partial B_n < \partial \Omega_n / \partial B_k < 0$ (from (30) and (33)). Therefore each element of $\mathbf{\Omega}_B$ is negative. It follows that the system of equations (40) satisfies the properties for unambiguous comparative statics in Proposition 17.G.3 of Section 17.G of Mas-Collel, Whinston, and Green (1995). Since $dB_n > 0$ and $dB_k < 0$ for all $k \neq n$, we have $dL_n > 0$ and $dL_k < 0$ for all $k \neq n$.

Consider an increase in location n's land supply ($dH_n > 0$) and a decrease in the land supply of all other locations ($dH_k < 0$ for $k \neq n$) such that expected utility ($\bar{W}$) remains constant. Differentiating in (28), we have:

$$\boldsymbol{dL} = -\mathbf{\Omega}_L{}^{-1}\, \mathbf{\Omega}_H\, \boldsymbol{d}H, \tag{41}$$

$$\begin{pmatrix} dL_1 \\ \vdots \\ dL_n \\ \vdots \\ dL_N \end{pmatrix} = - \begin{pmatrix} \frac{\partial \Omega_1}{\partial L_1} & \cdots & \frac{\partial \Omega_1}{\partial L_n} & \cdots \frac{\partial \Omega_1}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial L_1} & \cdots & \frac{\partial \Omega_n}{\partial L_n} & \cdots \frac{\partial \Omega_n}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial L_1} & \cdots & \frac{\partial \Omega_N}{\partial L_n} & \cdots \frac{\partial \Omega_N}{\partial L_N} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Omega_1}{\partial H_1} & \cdots & \frac{\partial \Omega_1}{\partial H_n} & \cdots \frac{\partial \Omega_1}{\partial H_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial H_1} & \cdots & \frac{\partial \Omega_n}{\partial H_n} & \cdots \frac{\partial \Omega_n}{\partial H_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial H_1} & \cdots & \frac{\partial \Omega_N}{\partial H_n} & \cdots \frac{\partial \Omega_N}{\partial H_N} \end{pmatrix} \begin{pmatrix} dH_1 \\ \vdots \\ dH_n \\ \vdots \\ dH_N \end{pmatrix}.$$

From Cramer's rule (see for example Samuelson 1953), we have:

$$\frac{\partial L_k}{\partial H_n} = -\frac{\sum_i \frac{\partial \Omega_i}{\partial H_n} \triangle_{ik}}{\triangle},$$

where $\triangle$ and $\triangle_{ik}$ are defined above.

We have shown above that $\partial \Omega_n / \partial L_k < 0 < \partial \Omega_n / \partial L_n$ (from (37) and (38)). Therefore the matrix $\boldsymbol{\Omega}_L$ satisfies gross substitution, with positive diagonal terms and negative off-diagonal terms. It follows that all the successive principal minors of $\boldsymbol{\Omega}_L$ are positive and the matrix $\boldsymbol{\Omega}_L$ is positive definite with a positive determinant $\triangle$ (see for example Theorem 4.C.5 in Takayama 1995). We have also shown above that $\partial \Omega_n / \partial H_n < \partial \Omega_n / \partial H_k < 0$ (from (31) and (34)). Therefore each element of $\boldsymbol{\Omega}_H$ is negative. It follows that the system of equations (41) satisfies the properties for unambiguous comparative statics in Proposition 17.G.3 of Section 17.G of Mas-Collel, Whinston, and Green (1995). Since $dH_n > 0$ and $dH_k < 0$ for all $k \neq n$, we have $dL_n > 0$ and $dL_k < 0$ for all $k \neq n$.

Consider a reduction in the common component of a location n's trade costs ($dD_n < 0$) and an increase in the common component of trade costs for all other locations ($dD_k > 0$ for $k \neq n$) such that expected utility ($\bar{W}$) remains constant. Differentiating in (28), we have:

$$\boldsymbol{d}L = -\boldsymbol{\Omega}_L{}^{-1}\, \boldsymbol{\Omega}_D\, \boldsymbol{d}D, \tag{42}$$

$$\begin{pmatrix} dL_1 \\ \vdots \\ dL_n \\ \vdots \\ dL_N \end{pmatrix} = - \begin{pmatrix} \frac{\partial \Omega_1}{\partial L_1} & \cdots & \frac{\partial \Omega_1}{\partial L_n} & \cdots \frac{\partial \Omega_1}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial L_1} & \cdots & \frac{\partial \Omega_n}{\partial L_n} & \cdots \frac{\partial \Omega_n}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial L_1} & \cdots & \frac{\partial \Omega_N}{\partial L_n} & \cdots \frac{\partial \Omega_N}{\partial L_N} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Omega_1}{\partial D_1} & \cdots & \frac{\partial \Omega_1}{\partial D_n} & \cdots \frac{\partial \Omega_1}{\partial D_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial D_1} & \cdots & \frac{\partial \Omega_n}{\partial D_n} & \cdots \frac{\partial \Omega_n}{\partial D_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial D_1} & \cdots & \frac{\partial \Omega_N}{\partial D_n} & \cdots \frac{\partial \Omega_N}{\partial D_N} \end{pmatrix} \begin{pmatrix} dD_1 \\ \vdots \\ dD_n \\ \vdots \\ dD_N \end{pmatrix}.$$

From Cramer's rule (see for example Samuelson 1953), we have:

$$\frac{\partial L_k}{\partial D_n} = -\frac{\sum_i \frac{\partial \Omega_i}{\partial D_n} \triangle_{ik}}{\triangle},$$

where $\triangle$ and $\triangle_{ik}$ are defined above.

We have shown above that $\partial \Omega_n / \partial L_k < 0 < \partial \Omega_n / \partial L_n$ (from (37) and (38)). Therefore the matrix $\boldsymbol{\Omega}_L$ satisfies gross substitution, with positive diagonal terms and negative off-diagonal terms. It follows that all the successive principal minors of $\boldsymbol{\Omega}_L$ are positive and the matrix $\boldsymbol{\Omega}_L$ is positive definite with a positive determinant $\triangle$ (see for example Theorem 4.C.5 in Takayama 1995). We have also shown above that $\partial \Omega_n / \partial D_n > 0$ and $\partial \Omega_n / \partial D_k > 0$ (from (35) and (36)). Therefore each element of $\boldsymbol{\Omega}_D$ is positive. It follows that the system of equations (42) satisfies the properties for unambiguous comparative statics in Proposition 17.G.3 of Section 17.G of Mas-Collel, Whinston, and Green (1995). Since $dD_n < 0$ and $dD_k > 0$ for all $k \neq n$, we have $dL_n > 0$ and $dL_k < 0$ for all $k \neq n$. ■

2.8 Recovering Location Fundamentals

Given values for the model's parameters $\{\alpha, \theta, \epsilon\}$, a parameterization of bilateral trade costs $\{d_{ni}\}$ and data on populations, wages and land supplies $\{L_n, w_n, H_n\}$, we now show that the solution to the general equilibrium of the model can be used to recover the unobserved location characteristics of amenities (B_n) and productivities (A_n).

Proposition 3 *Given the model parameters $\{\alpha, \theta, \epsilon\}$, a parameterization of bilateral trade costs $\{d_{ni}\}$ and data on populations, wages and land supplies $\{L_n, w_n, H_n\}$, there exist unique values of amenities (B_n) and productivities (A_n) that are consistent with the data up to a normalization that corresponds to a choice of units in which to measure productivity and amenities.*

Proof. Given the model's parameters $\{\alpha, \theta, \epsilon\}$ and a parameterization of trade costs $\{d_{ni}\}$, we first show how the population and wage data $\{L_n, w_n\}$ can be used to recover unique values of unobserved productivities $\{A_n\}$ up to a normalization that corresponds to a choice of units in which to measure productivity. The requirement that income in each location equals expenditure on goods produced by that location defines an excess demand system in unobserved productivities

$$D_i(\boldsymbol{A}) = w_i L_i - \sum_{n \in N} \frac{A_i \left(d_{ni} w_i\right)^{-\theta}}{\sum_{k \in N} A_k \left(d_{nk} w_k\right)^{-\theta}} w_n L_n = 0, \qquad i, n \in N, \tag{43}$$

where we use bold math font to denote a vector or matrix. This excess demand system exhibits the following properties in $\boldsymbol{A}$:

Property (i): $D(\boldsymbol{A})$ is continuous.
Property (ii): $D(\boldsymbol{A})$ is homogenous of degree zero.
Property (iii): $\sum_{i \in N} D_i\left(\boldsymbol{A}\right) = 0$.
Property (iv): $D(\boldsymbol{A})$ exhibits gross substitution:

$$\begin{aligned} \frac{\partial D_i\left(\boldsymbol{A}\right)}{\partial A_k} &> 0 \qquad \text{for all } i, k, i \neq k, \\ \frac{\partial D_i\left(\boldsymbol{A}\right)}{\partial A_i} &< 0 \qquad \text{for all } i. \end{aligned}$$

Property (i) follows immediately by inspection of (43).
Property (ii) follows immediately by inspection of (43).
Property (iii) can be established by noting:

$$\begin{aligned} \sum_{i \in N} D_i\left(\boldsymbol{A}\right) &= \sum_{i \in N} w_i L_i - \sum_{n \in N} \frac{\sum_{i \in N} A_i \left(d_{ni} w_i\right)^{-\theta}}{\sum_{k \in N} A_k \left(d_{nk} w_k\right)^{-\theta}} w_n L_n \\ &= \sum_{i \in N} w_i L_i - \sum_{n \in N} w_n L_n \\ &= 0. \end{aligned}$$

Property (iv) can be established by noting:

$$\frac{\partial D_i\left(\boldsymbol{A}\right)}{\partial A_k} = \sum_{n \in N} \frac{A_i \left(d_{ni} w_i\right)^{-\theta}}{\sum_{k \in N} A_k \left(d_{nk} w_k\right)^{-\theta}} \frac{\left(d_{nk} w_k\right)^{-\theta}}{\sum_{s \in N} A_s \left(d_{ns} w_s\right)^{-\theta}} w_n L_n > 0.$$

and using homogeneity of degree zero, which implies:

$$\nabla D\left(\boldsymbol{A}\right) \boldsymbol{A} = 0,$$

and hence:

$$\frac{\partial D_i(\boldsymbol{A})}{\partial A_i} < 0.$$

These properties imply that the system $D(\boldsymbol{A})$ has at most one (normalized) solution. Gross substitution implies that $D(\boldsymbol{A}) = D(\boldsymbol{A'})$ cannot occur whenever $\boldsymbol{A}$ and $\boldsymbol{A'}$ are not colinear. By homogeneity of degree zero, we can assume $\boldsymbol{A'} \geq \boldsymbol{A}$ and $A_i = A'_i$ for some i. Now consider altering the productivity vector $\boldsymbol{A'}$ to obtain the productivity vector $\boldsymbol{A}$ in $S-1$ steps, lowering (or keeping unaltered) the productivity of all the other $S-1$ locations $k \neq i$ one at a time. By gross substitution, the excess demand in location i cannot decrease in any step, and because $\boldsymbol{A} \neq \boldsymbol{A'}$, it will actually increase in at least one step. Hence $D_i(\boldsymbol{A}) > D_i(\boldsymbol{A'})$ and we have a contradiction.

The above properties also imply that there exists a unique vector of productivities $\boldsymbol{A}^*$ such that $D(\boldsymbol{A}^*) = 0$. By homogeneity of degree zero, we can restrict our search for the unique productivity vector to the unit simplex $\Delta = \left\{\boldsymbol{A} : \sum_{i \in N} A_i = 1\right\}$. Define on Δ the function $D^+(\cdot)$ by $D_i^+(\boldsymbol{A}) = \max\{D_i(\boldsymbol{A}), 0\}$. Note that $D^+(\cdot)$ is continuous. Denote $\xi(\boldsymbol{A}) = \sum_{i \in N}\left[A_i + D_i^+(A)\right]$. We have $\xi(\boldsymbol{A}) \geq 1$ for all $\boldsymbol{A}$. Define a continuous function $f(\cdot)$ from the closed convex set Δ into itself by:

$$f(\boldsymbol{A}) = [1/\xi(\boldsymbol{A})]\left[\boldsymbol{A} + D^+(\boldsymbol{A})\right].$$

By Brouwer's Fixed-point Theorem, there exists $\boldsymbol{A}^* \in \Delta$ such that $\boldsymbol{A}^* = f(\boldsymbol{A}^*)$. Since $\sum_{i \in N} D_i(\boldsymbol{A}) = 0$, it cannot be the case that $D_i(\boldsymbol{A}) > 0$ for all $i \in N$ or $D_i(\boldsymbol{A}) < 0$ for all $i \in N$. Additionally, if $D_i(\boldsymbol{A}) > 0$ for some i and $D_k(\boldsymbol{A}) < 0$ for some $k \neq i$, $\boldsymbol{A} \neq f(\boldsymbol{A})$. It follows that at the fixed point for wages, $\boldsymbol{A}^* = f(\boldsymbol{A}^*)$, and $D_i(\boldsymbol{A}) = 0$ for all i.

Using these solutions for productivities and the data on wages and populations $\{A_n, w_n, L_n\}$, we can solve for trade shares (π_{ni}) using (8), land rents (r_n) using (16), and price indices (P_n) using (10). Using these solutions for land rents and price indices and the data on wages $\{r_n, P_n, w_n\}$, we can recover unique values of unobserved amenities $\{B_n\}$ up to a normalization that corresponds to a choice of units in which to measure amenities. The residential choice probabilities imply the following excess demand system:

$$\mathbb{D}(\boldsymbol{B}) = \frac{L_n}{\bar{L}} - \frac{B_n\left(P_n^{\alpha} r_n^{1-\alpha}\right)^{-\epsilon}(w_n)^{\epsilon}}{\sum_{k \in N} B_k\left(P_k^{\alpha} r_k^{1-\alpha}\right)^{-\epsilon}(w_k)^{\epsilon}} = 0, \qquad n \in N, \tag{44}$$

which exhibits the same properties in amenities ($\boldsymbol{B}$) that the excess demand system (43) exhibits in productivities ($\boldsymbol{A}$). Therefore there exists a unique vector of amenities $\boldsymbol{B}^*$ such that $\mathbb{D}(\boldsymbol{B}^*) = 0$. ■

2.9 Counterfactuals

The system of equations for general equilibrium (17)-(19) provides an approach for undertaking model-based counterfactuals that uses only parameters and the values of endogenous variables in the initial equilibrium (as in Dekle, Eaton, and Kortum 2007). In contrast to standard trade models, these model-based counterfactuals now yield predictions for the reallocation of the mobile factor labor across locations.

The system of equations for general equilibrium (17)-(19) must hold both before and after a change in trade frictions, productivity or amenities. We denote the value of variables in the counterfactual equilibrium with a prime (x') and the relative value of variables in the counterfactual and initial equilibria by a hat ($\hat{x} = x'/x$). Using this notation, the system of equations for the counterfactual equilibrium (17)-(19) can be re-written as follows:

$$\hat{w}_i \hat{\lambda}_i Y_i = \sum_{n \in N} \pi'_{ni} \hat{w}_n \hat{\lambda}_n Y_n, \tag{45}$$

$$\hat{\pi}_{ni}\pi_{ni} = \frac{\pi_{ni}\hat{A}_i\left(\hat{d}_{ni}\hat{w}_i\right)^{-\theta}}{\sum_{k \in N}\pi_{nk}\hat{A}_k\left(\hat{d}_{nk}\hat{w}_k\right)^{-\theta}}, \tag{46}$$

$$\hat{\lambda}_n \lambda_n = \frac{\hat{B}_n \hat{A}_n^{\frac{\alpha\epsilon}{\theta}} \hat{\pi}_{nn}^{-\frac{\alpha\epsilon}{\theta}} \hat{\lambda}_n^{-\epsilon(1-\alpha)} \lambda_n}{\sum_{k \in N} \hat{B}_k \hat{A}_k^{\frac{\alpha\epsilon}{\theta}} \hat{\pi}_{kk}^{-\frac{\alpha\epsilon}{\theta}} \hat{\lambda}_k^{-\epsilon(1-\alpha)} \lambda_k}, \tag{47}$$

where $Y_i = w_i L_i$ denotes labor income and $\lambda_n = L_n/\bar{L}$ denotes the population share in the initial equilibrium. This system of equations can be solved for $\{\hat{\lambda}_n, \hat{w}_n, \hat{\pi}_{ni}\}$ given the observed variables in the initial equilibrium $\{\lambda_n, Y_n, \pi_{ni}\}$ and an assumed comparative static. For example, a reduction in trade costs holding productivity and amenities constant corresponds to $\hat{d}_{ni} < 1$, $\hat{A}_n = 1$ and $\hat{B}_n = 1$, while an increase in productivity corresponds to $\hat{A}_n > 1$.

2.10 Welfare Gains from Trade

We now examine the implications of worker mobility with heterogeneous preferences for the welfare gains from trade. We first show that the common change in welfare between an actual and a counterfactual equilibrium across locations ($\hat{\bar{U}} = \bar{U}'/\bar{U}$) can be written as a weighted average of the change in real income in each location. From expected utility (13) and the residential choice probabilities (12), we have:

$$\hat{\bar{U}} = \frac{\bar{U}'}{\bar{U}} = \left[\sum_{n \in N} \frac{L_n}{\bar{L}} \left(\frac{\hat{v}_k}{\hat{P}_k^{\alpha} \hat{r}_k^{1-\alpha}} \right)^{\epsilon} \right]^{\frac{1}{\epsilon}}, \tag{48}$$

where the weights depend on location population shares. Using expenditure equals income (14), the price index (10) and land market clearing (16), real income for each location can be written in terms of its domestic trade share (π_{nn}), population (L_n) and parameters:

$$\frac{v_k}{P_k^{\alpha} r_k^{1-\alpha}} = \frac{\left(\frac{A_n}{\pi_{nn}} \right)^{\frac{\alpha}{\theta}} \left(\frac{L_k}{H_k} \right)^{-(1-\alpha)}}{\alpha \left(\frac{1-\alpha}{\alpha} \right)^{1-\alpha} \gamma^{\alpha}}. \tag{49}$$

Combining (48) and (49), the common change in welfare between the two equilibria can be expressed as the weighted average of the changes in domestic trade shares and populations of each location:

$$\hat{\bar{U}} = \frac{\bar{U}'}{\bar{U}} = \left[\sum_{n \in N} \frac{L_n}{\bar{L}} \left(\hat{\pi}_{nn}^{-\alpha/\theta} \hat{L}_n^{-(1-\alpha)} \right)^{\epsilon} \right]^{\frac{1}{\epsilon}}. \tag{50}$$

While this expression features the changes in the domestic trade shares and populations of all locations, we now show that welfare also can be expressed in terms of the characteristics of any one individual location. From expected utility (13), the residential choice probabilities (12) and real income (49), the common level of utility across locations can be expressed as:

$$\bar{U}_n = \bar{U} = \frac{\delta B_n^{\frac{1}{\epsilon}} \left(\frac{A_n}{\pi_{nn}} \right)^{\frac{\alpha}{\theta}} H_n^{1-\alpha} L_n^{-\left(\frac{1}{\epsilon} + (1-\alpha)\right)}}{\alpha \left(\frac{1-\alpha}{\alpha} \right)^{1-\alpha} \gamma^{\alpha} \left(\bar{L}^j \right)^{-\frac{1}{\epsilon}}}, \qquad \forall\, n. \tag{51}$$

Population mobility implies that this relationship must hold for each location. Locations with higher productivity (A_n), better amenities (B_n), better goods market access to other locations (lower π_{nn}) and higher supplies of land (H_n) have higher populations, which bids up the price of land until expected utility conditional on living in each location is the same for all locations.

An implication of this result is that the domestic trade share in the open economy equilibrium (π_{nn}^T), populations in the closed and open economies (L_n^A and L_n^T), the trade elasticity (θ), the elasticity of labor supply with respect to real income (ϵ) and the consumption goods share (α) are sufficient statistics for the welfare gains from trade:

$$\frac{\bar{U}_n^T}{\bar{U}_n^A} = \frac{\bar{U}^T}{\bar{U}^A} = \left(\frac{1}{\pi_{nn}^T} \right)^{\frac{\alpha}{\theta}} \left(\frac{L_n^A}{L_n^T} \right)^{\frac{1}{\epsilon} + (1-\alpha)}, \qquad \forall\, n, \tag{52}$$

where we use the superscript T to denote the trade equilibrium and the superscript A to denote the autarky equilibrium; we have used $\pi_{nn}^{A}=1$; and in general $L_n^A \neq L_n^T$.

Intuitively, if some locations have better market access than others in the open economy (as reflected in a lower open economy domestic trade share π_{nn}^{T}), the opening of goods trade will lead to a larger reduction in the consumption price index in these locations. This larger reduction in the consumption price index in turn creates an incentive for migration from locations with worse market access to those with better market access. This labor mobility provides the mechanism that restores equilibrium, as the price of land is bid up in locations with better market access and bid down in those with worse market access, until expected utility is equalized across all locations. Therefore, computing the common value for the welfare gains from trade across all locations involves taking into account not only domestic trade shares (which affect consumption price indices) but also population redistributions (which affect the price of the immobile factor land).

Although labor mobility ensures the equalization of expected utility across all locations, real income is not equalized, because of the heterogeneity in workers' preferences for locations. Each location faces an upward sloping supply curve for workers, as higher real income has to be paid to attract workers with lower realizations for idiosyncratic tastes for that location. Only in the special case of no idiosyncratic heterogeneity in worker tastes ($\epsilon \to \infty$) is real income equalized across locations. In this special case, expected utility is given by:

$$\bar{U}_n = \bar{U} = \frac{\left(\frac{A_n}{\pi_{nn}}\right)^{\frac{\alpha}{\theta}} H_n^{1-\alpha} L_n^{-(1-\alpha)}}{\alpha\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}\gamma^{\alpha}}, \qquad \forall\, n, \tag{53}$$

and the welfare gains from trade are:

$$\frac{\bar{U}_n^T}{\bar{U}_n^A} = \frac{\bar{U}^T}{\bar{U}^A} = \left(\frac{1}{\pi_{nn}^T}\right)^{\frac{\alpha}{\theta}} \left(\frac{L_n^A}{L_n^T}\right)^{1-\alpha}, \qquad \forall\, n, \tag{54}$$

which corresponds to the limiting case of (52) in which $\epsilon \to \infty$.

In another special case of perfect labor immobility, expected utility takes the same form as in (53), except that expected utility in general differs across locations:

$$\bar{U}_n = \frac{\left(\frac{A_n}{\pi_{nn}}\right)^{\frac{\alpha}{\theta}} H_n^{1-\alpha} L_n^{-(1-\alpha)}}{\alpha\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}\gamma^{\alpha}} \neq \bar{U}_k, \qquad n \neq k. \tag{55}$$

Similarly, the welfare gains from trade in general differ across locations under perfect labor immobility:

$$\frac{\bar{U}_n^T}{\bar{U}_n^A} = \left(\frac{1}{\pi_{nn}^T}\right)^{\frac{\alpha}{\theta}} \neq \frac{\bar{U}_k^T}{\bar{U}_k^A}, \qquad n \neq k, \tag{56}$$

which corresponds to the limiting case of (52), in which $L_n^T = L_n^A$ because of labor immobility. Intuitively, in this limiting case, locations with better access to markets in the open economy experience larger welfare gains from trade, because labor mobility no longer provides a mechanism for utility equalization through changes in the price of the immobile factor land.

3 Agglomeration Forces

In this section, we examine the implications of introducing agglomeration forces in our setting with both trade costs and labor mobility with heterogeneous worker preferences. These agglomeration forces take the form of pecuniary externalities as a result of transport costs, increasing returns to scale and love of variety, as in the new economic geography literature following Krugman (1991), Krugman and Venables (1995) and Helpman (1998), and synthesized in Fujita, Krugman, and Venables (1999). This literature typically restricts attention to stylized settings with a small number of symmetric locations and assumes either perfect labor mobility, perfect labor immobility or a mechanical relationship between migration flows and relative wages. In contrast, we consider a rich geography with a large number of asymmetric locations, and allow for a positive finite elasticity of labor supply to each location.

3.1 Consumer Preferences

Preferences are again defined over goods consumption (C_n) and residential land use (H_{Un}) and take the same form as in (1). The goods consumption index (C_n), however, is now defined over the endogenous measures of horizontally differentiated varieties supplied by each location (M_i):

$$C_n = \left[\sum_{i \in N} \int_0^{M_i} c_{ni}\left(j\right)^{\rho} dj \right]^{\frac{1}{\rho}}, \tag{57}$$

where trade between locations i and n is again subject to iceberg variable trade costs of $d_{ni} \geq 1$.

3.2 Production

Varieties are produced under conditions of monopolistic competition and increasing returns to scale. To produce a variety, a firm must incur a fixed cost of F units of labor and a constant variable cost in terms of labor that depends on a location's productivity A_i. Therefore the total amount of labor ($l_i(j)$) required to produce $x_i(j)$ units of a variety j in country i is:

$$l_i(j) = F + \frac{x_i(j)}{A_i}. \tag{58}$$

Profit maximization and zero profits imply that equilibrium prices are a constant mark-up over marginal cost:

$$p_{ni}(j) = \left(\frac{\sigma}{\sigma - 1} \right) \frac{d_{ni} w_i}{A_i}, \tag{59}$$

and equilibrium employment for each variety is equal to a constant:

$$l_i(j) = \bar{l} = \sigma F. \tag{60}$$

Given this constant equilibrium employment for each variety, labor market clearing implies that the total measure of varieties supplied by each location is proportional to the endogenous supply of workers choosing to locate there:

$$M_i = \frac{L_i}{\sigma F}. \tag{61}$$

3.3 Expenditure Shares and Price Indices

Using the CES expenditure function, equilibrium prices (59) and labor market clearing (61), the share of location n's expenditure on goods produced in location i is:

$$\pi_{ni} = \frac{M_i p_{ni}^{1-\sigma}}{\sum_{k \in N} M_k p_{nk}^{1-\sigma}} = \frac{L_i \left(\frac{d_{ni} w_i}{A_i} \right)^{1-\sigma}}{\sum_{k \in N} L_k \left(\frac{d_{nk} w_k}{A_k} \right)^{1-\sigma}}, \tag{62}$$

where the elasticity of trade with respect to trade costs is now determined by the elasticity of substitution ($\sigma - 1$). Furthermore, trade shares now depend directly on population (L_i) because this determines the endogenous measure of varieties produced by a location through the labor market clearing condition (61).

Using equilibrium prices (59) and labor market clearing (61), the price index dual to the consumption index (57) can be expressed as:

$$P_n = \frac{\sigma}{\sigma - 1} \left(\frac{1}{\sigma F}\right)^{\frac{1}{1-\sigma}} \left[\sum_{i \in N} L_i \left(\frac{d_{ni} w_i}{A_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}. \tag{63}$$

Using the trade share (62) and $d_{nn} = 1$, the goods price index can be equivalently written as:

$$P_n^{1-\sigma} = \frac{\frac{L_n}{\sigma F}\left(\frac{\sigma}{\sigma-1}\frac{w_n}{A_n}\right)^{1-\sigma}}{\pi_{nn}}, \tag{64}$$

which again depends directly on population (L_n) through the endogenous measure of varieties.

3.4 Residential Choices and Income

Residential choices take a similar form as in section 2. Using the Fréchet distribution of idiosyncratic shocks to amenities, the probability that a worker chooses to live in location $n \in N$ is:

$$\frac{L_n}{\bar{L}} = \frac{B_n \left(v_n / P_n^{\alpha} r_n^{1-\alpha}\right)^{\epsilon}}{\sum_{k \in N} B_k \left(v_k / P_k^{\alpha} r_k^{1-\alpha}\right)^{\epsilon}}. \tag{65}$$

Expected worker utility is:

$$\bar{U} = \delta \left[\sum_{k \in N} B_k \left(v_k / P_k^{\alpha} r_k^{1-\alpha}\right)^{\epsilon}\right]^{\frac{1}{\epsilon}}, \tag{66}$$

where $\delta = \Gamma\left((\epsilon - 1)/\epsilon\right)$; $\Gamma(\cdot)$ is the Gamma function; and $\epsilon > 1$. The Fréchet distribution of utility again implies that expected utility conditional on residing in location n is the same across all locations n and equal to expected utility for the economy as a whole.

Expenditure on land in each location is redistributed lump sum to the workers residing in that location, which implies that total income (v_n) equals labor income plus expenditure on residential land (as in (14)). Land market clearing implies that the equilibrium land rent again can be determined from the equality of land income and expenditure (as in (16)).

3.5 General Equilibrium

The general equilibrium of the model again can be represented by the measure of workers (L_n) in each location $n \in N$, the share of each location's expenditure on goods produced by other locations (π_{ni}) and the wage in each location (w_n). Using labor income (15), the trade share (62), residential choice probabilities (65) and land market clearing (16), the equilibrium triple $\{L_n, \pi_{ni}, w_n\}$ solves the following system of equations for all $i, n \in N$. First, each location's income must equal expenditure on the goods produced in that location:

$$w_i L_i = \sum_{n \in N} \pi_{ni} w_n L_n. \tag{67}$$

Second, location expenditure shares are:

$$\pi_{ni} = \frac{L_i \left(\frac{d_{ni} w_i}{A_i}\right)^{1-\sigma}}{\sum_{k \in N} L_k \left(\frac{d_{nk} w_k}{A_k}\right)^{1-\sigma}}. \tag{68}$$

Third, residential choice probabilities imply:

$$\frac{L_n}{\bar{L}} = \frac{B_n A_n^{\alpha\epsilon} H_n^{\epsilon(1-\alpha)} \pi_{nn}^{-\frac{\alpha\epsilon}{\sigma-1}} L_n^{-\left(\epsilon(1-\alpha) - \frac{\alpha\epsilon}{\sigma-1}\right)}}{\sum_{k \in N} B_k A_k^{\alpha\epsilon} H_k^{\epsilon(1-\alpha)} \pi_{kk}^{-\frac{\alpha\epsilon}{\sigma-1}} L_k^{-\left(\epsilon(1-\alpha) - \frac{\alpha\epsilon}{\sigma-1}\right)}}. \tag{69}$$

3.6 Existence and Uniqueness

We now provide conditions for the existence of a unique general equilibrium that solves the system of equations (67)-(69). Using the trade share (62), the requirement that income equals expenditure for each location $n \in N$ can be re-written as:

$$w_i^{\sigma} A_i^{1-\sigma} = \sum_{n \in N} d_{ni}^{1-\sigma} \pi_{nn} w_n^{\sigma} A_n^{1-\sigma}. \tag{70}$$

Additionally, using labor income (15), the residential choice probabilities (65), expected utility (66) and land market clearing (16), we obtain the following alternative expression for the goods price index (P_n):

$$P_n^{1-\sigma} = \left(\frac{w_n}{\bar{W}}\right)^{1-\sigma} B_n^{\frac{1-\sigma}{\alpha\epsilon}} H_n^{(1-\sigma)\left(\frac{1-\alpha}{\alpha}\right)} L_n^{-(1-\sigma)\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right)}, \tag{71}$$

where $\bar{W} = \left[\alpha^{\epsilon}\left(\frac{1-\alpha}{\alpha}\right)^{\epsilon(1-\alpha)}\left(\bar{U}/\delta\right)^{\epsilon}\left(\bar{L}\right)^{-1}\right]^{1/\alpha\epsilon}$ captures the expected utility ($\bar{U}$) across locations and parameters. Combining this expression for the price index with (64), the domestic trade share (π_{nn}) can be written as:

$$\pi_{nn} = \bar{W}^{1-\sigma}\frac{1}{\sigma F}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} A_n^{\sigma-1} B_n^{\frac{\sigma-1}{\alpha\epsilon}} H_n^{\frac{(\sigma-1)(1-\alpha)}{\alpha}} L_n^{1-(\sigma-1)\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right)}. \tag{72}$$

Using this expression for the domestic trade share in the equality of income and expenditure (70), we obtain a first system of equations linking the wages and populations of locations $n \in N$ as a function of parameters and expected utility:

$$\bar{W}^{1-\sigma}\frac{1}{\sigma F}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} = \frac{w_n^{\sigma} A_n^{1-\sigma}}{\sum_{k\in N} d_{kn}^{1-\sigma} B_k^{\frac{\sigma-1}{\alpha\epsilon}} H_k^{\frac{(\sigma-1)(1-\alpha)}{\alpha}} w_k^{\sigma} L_k^{1-(\sigma-1)\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right)}}. \tag{73}$$

Returning to (71) and using the expression for the price index (63), we obtain a second system of equations linking the wages and populations of locations $n \in N$ as a function of parameters and expected utility:

$$\bar{W}^{1-\sigma}\frac{1}{\sigma F}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} = \frac{w_n^{1-\sigma} B_n^{\frac{1-\sigma}{\alpha\epsilon}} H_n^{\frac{(1-\sigma)(1-\alpha)}{\alpha}} L_n^{-(1-\sigma)\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right)}}{\left[\sum_{k\in N} L_k \left(\frac{d_{nk} w_k}{A_k}\right)^{1-\sigma}\right]}. \tag{74}$$

We again characterize the properties of the general equilibrium of the model under the assumption that transport costs (d_{ni}) are "quasi-symmetric," which implies that they can be partitioned into an importer component (D_n), an exporter component (D_i) and a symmetric bilateral component ($D_{ni} = D_{in}$):

$$d_{ni} = \begin{cases} 1 & \text{if } n = i \\ D_n D_i D_{ni} & \text{if } n \neq i \end{cases}, \tag{75}$$

where $D_n > 1$, $D_i > 1$ and $D_{ni} = D_{in} > 1$. Under this assumption, the two wage systems (73) and (74) imply the following closed-form solution linking the endogenous variables for each location $n \in N$:

$$w_n^{1-2\sigma} A_n^{\sigma-1} B_n^{-\frac{\sigma-1}{\alpha\epsilon}} H_n^{-\frac{(\sigma-1)(1-\alpha)}{\alpha}} L_n^{(\sigma-1)\left(\frac{1}{\alpha\epsilon}+\frac{1-\alpha}{\alpha}\right)} = \kappa, \tag{76}$$

where κ is a scalar. If equation (76) holds, then any functions w_n and L_n that satisfy the system of equations (73) will also satisfy the system of equations (74) (and vice versa). In the proposition below, we prove below that equation (76) is the unique relationship between w_n and L_n that satisfies both systems of equations. Substituting this relationship (76) into (74), we obtain the following system of equations in terms of equilibrium populations.

$$L_n^{\tilde{\sigma}\gamma_1} A_n^{-\tilde{\sigma}(\sigma-1)} B_n^{-\frac{\tilde{\sigma}\sigma}{\alpha\epsilon}} H_n^{-\frac{\tilde{\sigma}\sigma(1-\alpha)}{\alpha}} = \bar{W}^{1-\sigma}\left[\sum_{k\in N}\frac{1}{\sigma F}\left(\frac{\sigma d_{nk}}{\sigma-1}\right)^{1-\sigma} A_k^{\tilde{\sigma}\sigma} B_k^{\frac{\tilde{\sigma}(\sigma-1)}{\alpha\epsilon}} H_k^{\frac{\tilde{\sigma}(\sigma-1)(1-\alpha)}{\alpha}} \left(L_k^{\tilde{\sigma}\gamma_1}\right)^{\frac{\gamma_2}{\gamma_1}}\right], \tag{77}$$

$$\tilde{\sigma} \equiv \frac{\sigma - 1}{2\sigma - 1},$$

$$\frac{1-\tilde{\alpha}}{\tilde{\alpha}} \equiv \left(\frac{1}{\alpha\epsilon} + \frac{1-\alpha}{\alpha}\right), \qquad \tilde{\alpha} \equiv \frac{\alpha}{1+\frac{1}{\epsilon}},$$

$$\gamma_1 \equiv \sigma\left(\frac{1-\tilde{\alpha}}{\tilde{\alpha}}\right),$$

$$\gamma_2 \equiv 1 + \frac{\sigma}{\sigma-1} - (\sigma - 1)\left(\frac{1-\tilde{\alpha}}{\tilde{\alpha}}\right),$$

where expected utility $\bar{W}$ is implicitly defined by the requirement that the labor market clear across all locations: $\sum_{n \in N} L_n = \bar{L}$. The condition for there to exist a unique stable equilibrium is:

$$\sigma\left(1 - \tilde{\alpha}\right) > 1, \qquad \Leftrightarrow \qquad \frac{\gamma_2}{\gamma_1} < 1.$$

In the special case of the model in which there is no dispersion in idiosyncratic shocks to amenities ($\epsilon \to \infty$), this condition for a unique stable equilibrium reduces to the condition in the new economic geography model of Helpman (1998) for the case of two regions and perfect labor mobility of $\sigma\left(1-\alpha\right) > 1$.

Proposition 4 *Assume $\sigma\left(1-\tilde{\alpha}\right) > 1$. Given the land area, productivity and amenity parameters $\{H_n,\ A_n,\ B_n\}$ and quasi-symmetric bilateral trade frictions $\{d_{ni}\}$ for each location $n, i \in N$, there exist unique equilibrium populations (L_n^*), trade shares (π_{ni}^*) and wages (w_n^*).*

Proof. The proof follows the same structure as in Allen and Arkolakis (2014). Assume $\sigma\left(1-\tilde{\alpha}\right) > 1$. Given the land area, productivity and amenity parameters $\{H_n,\ A_n,\ B_n\}$ and bilateral trade frictions $\{d_{ni}\}$, there exists a unique fixed point in the system of equations (77) because $\gamma_2/\gamma_1 < 1$ (see for example Fujimoto and Krause 1985). This unique fixed point determines the unique equilibrium population (L_n^*) for each location $n \in N$ up to a normalization determined by expected utility $(\bar{W})$. This normalization $(\bar{W})$ is determined by combining the system of equations (77) with the requirement that the labor market clear: $\sum_{n \in N} L_n = \bar{L}$. Having determined the unique equilibrium population (L_n^*) for each location, the closed-form relationship between the endogenous variables (76) immediately yields the unique equilibrium wage (w_n^*) for each location $n \in N$. Having solved for equilibrium wages and populations $\{L_n^*,\ w_n^*\}$, the expenditure shares (68) determine unique equilibrium trade shares (π_{ni}^*) for each pair of locations $n, i \in N$. ∎

3.7 Comparative Statics

Despite the introduction of agglomeration forces in a setting with a large number of asymmetric locations, the model continues to admit closed-form expressions for the comparative statics of the endogenous variables with respect to the relative value of these location characteristics. To characterize these comparative statics, we re-write the system of equations for equilibrium populations (77) as the following implicit function:

$$\begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_N \end{pmatrix} = \begin{pmatrix} \Omega_1^I \\ \vdots \\ \Omega_N^I \end{pmatrix} - \begin{pmatrix} \Omega_1^{II} \\ \vdots \\ \Omega_N^{II} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{78}$$

$$\begin{aligned}
\Omega_n^I &= L_n^{\tilde{\sigma}\gamma_1} A_n^{-\tilde{\sigma}(\sigma-1)} B_n^{-\frac{\tilde{\sigma}\sigma}{\alpha\epsilon}} H_n^{-\frac{\tilde{\sigma}\sigma(1-\alpha)}{\alpha}}, \\
\Omega_n^{II} &= \sum\nolimits_{k \in N} \Omega_{nk}^{II}, \\
\Omega_{nk}^{II} &= \bar{W}^{1-\sigma} \frac{1}{\sigma F}\left(\frac{\sigma d_{nk}}{\sigma - 1}\right)^{1-\sigma} A_k^{\tilde{\sigma}\sigma} B_k^{\frac{\tilde{\sigma}(\sigma-1)}{\alpha\epsilon}} H_k^{\frac{\tilde{\sigma}(\sigma-1)(1-\alpha)}{\alpha}} \left(L_k^{\tilde{\sigma}\gamma_1}\right)^{\frac{\gamma_2}{\gamma_1}},
\end{aligned}$$

where Ω_n^{II} has an interpretation as a *market access* term that captures the goods market access of each location (depending on trade costs d_{nk}) to the characteristics of other locations.

We show below that the implicit function (Ω_n) is monotonically decreasing in the productivities, amenities and land supplies of all locations and monotonically increasing in the trade costs to other locations. An implication is that the equilibrium population of each location (L_n) depends solely on the *relative* rather than the absolute levels of these characteristics. We also show below that the implicit function (Ω_n) is monotonically increasing in own population (L_n) and monotonically decreasing in the population of other locations (L_k for $k \neq n$). Therefore the system of equations for equilibrium populations (78) satisfies gross substitution and yields unambiguous comparative static predictions.

We first show that the implicit function (78) is monotonically decreasing in the productivity (A_n), amenities (B_n) and land supply (H_n) of each location. Consider the derivative of the implicit function (Ω_n) for location n with respect to these characteristics of location n:

$$\frac{\partial \Omega_n}{\partial A_n} = -\left[\tilde{\sigma}(\sigma-1) + \tilde{\sigma}\sigma\frac{\Omega_{nn}^{II}}{\Omega_n^{II}}\right]\frac{\Omega_n^I}{A_n} < 0, \tag{79}$$

$$\frac{\partial \Omega_n}{\partial B_n} = -\left[\frac{\tilde{\sigma}\sigma}{\alpha\epsilon} + \frac{\tilde{\sigma}(\sigma-1)}{\alpha\epsilon}\frac{\Omega_{nn}^{II}}{\Omega_n^{II}}\right]\frac{\Omega_n^I}{B_n} < 0, \tag{80}$$

$$\frac{\partial \Omega_n}{\partial H_n} = -\left[\frac{\tilde{\sigma}\sigma(1-\alpha)}{\alpha} + \frac{\tilde{\sigma}(\sigma-1)(1-\alpha)}{\alpha}\frac{\Omega_{nn}^{II}}{\Omega_n^{II}}\right]\frac{\Omega_n^I}{H_n} < 0. \tag{81}$$

where we have used $\Omega_n^{II} = \Omega_n^I$. Consider the derivative of the implicit function (Ω_k) for location $k \neq n$ with respect to these characteristics of location n:

$$\frac{\partial \Omega_k}{\partial A_n} = -\tilde{\sigma}\sigma\frac{\Omega_{kn}^{II}}{\Omega_k^{II}}\frac{\Omega_k^I}{A_n} < 0 \qquad \forall\, k \neq n, \tag{82}$$

$$\frac{\partial \Omega_k}{\partial B_n} = -\frac{\tilde{\sigma}(\sigma-1)}{\alpha\epsilon}\frac{\Omega_{kn}^{II}}{\Omega_k^{II}}\frac{\Omega_k^I}{B_n} < 0 \qquad \forall\, k \neq n, \tag{83}$$

$$\frac{\partial \Omega_k}{\partial H_n} = -\frac{\tilde{\sigma}(\sigma-1)(1-\alpha)}{\alpha}\frac{\Omega_{kn}^{II}}{\Omega_k^{II}}\frac{\Omega_k^I}{H_n} < 0 \qquad \forall\, k \neq n. \tag{84}$$

where we have used $\Omega_k^{II} = \Omega_k^I$.

We next show that the implicit function (78) is monotonically increasing in trade costs to other locations. Under our assumption of quasi-symmetric trade costs, we have:

$$d_{ni} = \begin{cases} 1 & \text{if } n = i \\ D_n D_i D_{ni} & \text{if } n \neq i \end{cases},$$

where $D_n > 1$, $D_i > 1$ and $D_{ni} = D_{in} > 1$. Consider the derivative of the implicit function (Ω_n) for location n with respect to the common component of trade costs for location n:

$$\frac{\partial \Omega_n}{\partial D_n} = (\sigma-1)\sum_{k\neq n}\frac{\Omega_{nk}^{II}}{D_n} > 0. \tag{85}$$

Now consider the derivative of the implicit function (Ω_k) for location $k \neq n$ with respect to the common component of trade costs for location n:

$$\frac{\partial \Omega_k}{\partial D_n} = (\sigma-1)\frac{\Omega_{kn}^{II}}{D_n} > 0. \qquad \forall\, k \neq n, \tag{86}$$

Finally, we show that the implicit function for each location is monotonically increasing in its own population and monotonically decreasing in the population of other locations:

$$\frac{\partial \Omega_n}{\partial L_n} = \tilde{\sigma}\gamma_1 \left[1 - \frac{\gamma_2}{\gamma_1}\frac{\Omega_{nn}^{II}}{\Omega_n^{II}}\right]\frac{\Omega_n^I}{L_n} > 0, \tag{87}$$

$$\frac{\partial \Omega_k}{\partial L_n} = -\tilde{\sigma}\gamma_1 \frac{\gamma_2}{\gamma_1}\frac{\Omega_{kn}^{II}}{\Omega_k^{II}}\frac{\Omega_k^I}{L_n} < 0, \qquad \forall\, k \neq n, \tag{88}$$

where we have again used $\Omega_n^{II} = \Omega_n^I$.

Proposition 5 *Assuming $\sigma(1-\tilde{\alpha}) > 1$ and quasi-symmetric bilateral trade frictions (d_{ni}), an increase in the productivity (A_n), amenities (B_n) or land supply (H_n) of a location n relative to all other locations increases the equilibrium population of that location relative to all other locations $k \neq n$, other things equal. An increase in location n's trade costs to all other locations $k \neq n$ (D_n) decreases the equilibrium population of that location relative to all other locations $k \neq n$, other things equal.*

Proof. Consider an increase in location n's productivity ($dA_n > 0$) and a decrease in the productivity of all other locations ($dA_k < 0$ for $k \neq n$) such that expected utility ($\bar{W}$) remains constant. From the implicit function (78), we have:

$$\boldsymbol{dL} = -\boldsymbol{\Omega}_L^{-1}\,\boldsymbol{\Omega}_A\,\boldsymbol{dA},$$

$$\begin{pmatrix} dL_1 \\ \vdots \\ dL_n \\ \vdots \\ dL_N \end{pmatrix} = -\begin{pmatrix} \frac{\partial \Omega_1}{\partial L_1} & \cdots & \frac{\partial \Omega_1}{\partial L_n} & \cdots \frac{\partial \Omega_1}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial L_1} & \cdots & \frac{\partial \Omega_n}{\partial L_n} & \cdots \frac{\partial \Omega_n}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial L_1} & \cdots & \frac{\partial \Omega_N}{\partial L_n} & \cdots \frac{\partial \Omega_N}{\partial L_N} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Omega_1}{\partial A_1} & \cdots & \frac{\partial \Omega_1}{\partial A_n} & \cdots \frac{\partial \Omega_1}{\partial A_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial A_1} & \cdots & \frac{\partial \Omega_n}{\partial A_n} & \cdots \frac{\partial \Omega_n}{\partial A_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial A_1} & \cdots & \frac{\partial \Omega_N}{\partial A_n} & \cdots \frac{\partial \Omega_N}{\partial A_N} \end{pmatrix} \begin{pmatrix} dA_1 \\ \vdots \\ dA_n \\ \vdots \\ dA_N \end{pmatrix}.$$

From Cramer's rule (see for example Samuelson 1953), we have:

$$\frac{\partial L_k}{\partial A_n} = -\frac{\sum_i \frac{\partial \Omega_i}{\partial A_n}\triangle_{ik}}{\triangle},$$

where

$$\triangle = \begin{vmatrix} \frac{\partial \Omega_1}{\partial L_1} & \cdots & \frac{\partial \Omega_1}{\partial L_n} & \cdots \frac{\partial \Omega_1}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial L_1} & \cdots & \frac{\partial \Omega_n}{\partial L_n} & \cdots \frac{\partial \Omega_n}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial L_1} & \cdots & \frac{\partial \Omega_N}{\partial L_n} & \cdots \frac{\partial \Omega_N}{\partial L_N} \end{vmatrix},$$

and $\triangle_{ik}$ indicates the cofactor of the element of the ith row and kth column.

We have shown above that $\partial\Omega_n/\partial L_k < 0 < \partial\Omega_n/\partial L_n$ (from (87) and (88)). Therefore the matrix $\boldsymbol{\Omega}_L$ satisfies gross substitution, with positive diagonal terms and negative off-diagonal terms. It follows that all the successive principal minors of $\boldsymbol{\Omega}_L$ are positive and the matrix $\boldsymbol{\Omega}_L$ is positive definite with a positive determinant $\triangle$ (see for example Theorem 4.C.5 in Takayama 1995). We have also shown above that $\partial\Omega_n/\partial A_n < \partial\Omega_n/\partial A_k < 0$ (from (79) and (82)). Therefore each element of $\boldsymbol{\Omega}_A$ is negative. It follows that the system of equations (39) satisfies the properties for unambiguous comparative statics in Proposition 17.G.3 of Section 17.G of Mas-Collel, Whinston, and Green (1995). Since $dA_n > 0$ and $dA_k < 0$ for all $k \neq n$, we have $dL_n > 0$ and $dL_k < 0$ for all $k \neq n$.

Consider an increase in location n's amenities ($dB_n > 0$) and a decrease in the amenities of all other locations ($dB_k < 0$ for $k \neq n$) such that expected utility ($\bar{W}$) remains constant. From the implicit function (78), we have:

$$\boldsymbol{dL} = -\boldsymbol{\Omega_L}^{-1}\,\boldsymbol{\Omega}_B\,\boldsymbol{dB},$$

$$\begin{pmatrix} dL_1 \\ \vdots \\ dL_n \\ \vdots \\ dL_N \end{pmatrix} = -\begin{pmatrix} \frac{\partial \Omega_1}{\partial L_1} & \cdots & \frac{\partial \Omega_1}{\partial L_n} & \cdots \frac{\partial \Omega_1}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial L_1} & \cdots & \frac{\partial \Omega_n}{\partial L_n} & \cdots \frac{\partial \Omega_n}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial L_1} & \cdots & \frac{\partial \Omega_N}{\partial L_n} & \cdots \frac{\partial \Omega_N}{\partial L_N} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Omega_1}{\partial B_1} & \cdots & \frac{\partial \Omega_1}{\partial B_n} & \cdots \frac{\partial \Omega_1}{\partial B_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial B_1} & \cdots & \frac{\partial \Omega_n}{\partial B_n} & \cdots \frac{\partial \Omega_n}{\partial B_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial B_1} & \cdots & \frac{\partial \Omega_N}{\partial B_n} & \cdots \frac{\partial \Omega_N}{\partial B_N} \end{pmatrix} \begin{pmatrix} dB_1 \\ \vdots \\ dB_n \\ \vdots \\ dB_N \end{pmatrix}.$$

From Cramer's rule (see for example Samuelson 1953), we have:

$$\frac{\partial L_k}{\partial B_n} = -\frac{\sum_i \frac{\partial \Omega_i}{\partial B_n} \triangle_{ik}}{\triangle},$$

where $\triangle$ and $\triangle_{ik}$ are defined above.

We have shown above that $\partial \Omega_n / \partial L_k < 0 < \partial \Omega_n / \partial L_n$ (from (87) and (88)). Therefore the matrix $\boldsymbol{\Omega}_L$ satisfies gross substitution, with positive diagonal terms and negative off-diagonal terms. It follows that all the successive principal minors of $\boldsymbol{\Omega}_L$ are positive and the matrix $\boldsymbol{\Omega}_L$ is positive definite with a positive determinant $\triangle$ (see for example Theorem 4.C.5 in Takayama 1995). We have also shown above that $\partial \Omega_n / \partial B_n < \partial \Omega_n / \partial B_k < 0$ (from (80) and (83)). Therefore each element of $\boldsymbol{\Omega}_B$ is negative. It follows that the system of equations (40) satisfies the properties for unambiguous comparative statics in Proposition 17.G.3 of Section 17.G of Mas-Collel, Whinston, and Green (1995). Since $dB_n > 0$ and $dB_k < 0$ for all $k \neq n$, we have $dL_n > 0$ and $dL_k < 0$ for all $k \neq n$.

Consider an increase in location n's land supply ($dH_n > 0$) and a decrease in the land supply of all other locations ($dH_k < 0$ for $k \neq n$) such that expected utility ($\bar{W}$) remains constant. From the implicit function (78), we have:

$$\boldsymbol{dL} = -\boldsymbol{\Omega_L}^{-1}\,\boldsymbol{\Omega}_H\,\boldsymbol{dH},$$

$$\begin{pmatrix} dL_1 \\ \vdots \\ dL_n \\ \vdots \\ dL_N \end{pmatrix} = -\begin{pmatrix} \frac{\partial \Omega_1}{\partial L_1} & \cdots & \frac{\partial \Omega_1}{\partial L_n} & \cdots \frac{\partial \Omega_1}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial L_1} & \cdots & \frac{\partial \Omega_n}{\partial L_n} & \cdots \frac{\partial \Omega_n}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial L_1} & \cdots & \frac{\partial \Omega_N}{\partial L_n} & \cdots \frac{\partial \Omega_N}{\partial L_N} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Omega_1}{\partial H_1} & \cdots & \frac{\partial \Omega_1}{\partial H_n} & \cdots \frac{\partial \Omega_1}{\partial H_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_n}{\partial H_1} & \cdots & \frac{\partial \Omega_n}{\partial H_n} & \cdots \frac{\partial \Omega_n}{\partial H_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Omega_N}{\partial H_1} & \cdots & \frac{\partial \Omega_N}{\partial H_n} & \cdots \frac{\partial \Omega_N}{\partial H_N} \end{pmatrix} \begin{pmatrix} dH_1 \\ \vdots \\ dH_n \\ \vdots \\ dH_N \end{pmatrix}.$$

From Cramer's rule (see for example Samuelson 1953), we have:

$$\frac{\partial L_k}{\partial H_n} = -\frac{\sum_i \frac{\partial \Omega_i}{\partial H_n} \triangle_{ik}}{\triangle},$$

where $\triangle$ and $\triangle_{ik}$ are defined above.

We have shown above that $\partial \Omega_n / \partial L_k < 0 < \partial \Omega_n / \partial L_n$ (from (87) and (88)). Therefore the matrix $\boldsymbol{\Omega}_L$ satisfies gross substitution, with positive diagonal terms and negative off-diagonal terms. It follows that all the successive principal minors of $\boldsymbol{\Omega}_L$ are positive and the matrix $\boldsymbol{\Omega}_L$ is positive definite with a positive determinant $\triangle$ (see for example Theorem 4.C.5 in Takayama 1995). We have also shown above that $\partial \Omega_n / \partial H_n < \partial \Omega_n / \partial H_k < 0$ (from (81) and (84)). Therefore each element of $\boldsymbol{\Omega}_H$ is negative. It follows that the system of equations (41) satisfies the properties for unambiguous comparative statics in Proposition 17.G.3 of Section 17.G of Mas-Collel, Whinston, and Green (1995). Since $dH_n > 0$ and $dH_k < 0$ for all $k \neq n$, we have $dL_n > 0$ and $dL_k < 0$ for all $k \neq n$.

Consider a reduction in the common component of a location n's trade costs ($dD_n < 0$) and an increase in the common component of trade costs for all other locations ($dD_k > 0$ for $k \neq n$) such that expected utility ($\bar{W}$) remains constant. From the implicit function (78), we have:

$$\boldsymbol{dL} = -\boldsymbol{\Omega}_L{}^{-1}\, \boldsymbol{\Omega}_D\, \boldsymbol{dD},$$

$$\begin{pmatrix} dL_1 \\ \vdots \\ dL_n \\ \vdots \\ dL_N \end{pmatrix} = -\begin{pmatrix} \frac{\partial \Omega_1}{\partial L_1} & \cdots & \frac{\partial \Omega_1}{\partial L_n} & \cdots & \frac{\partial \Omega_1}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_n}{\partial L_1} & \cdots & \frac{\partial \Omega_n}{\partial L_n} & \cdots & \frac{\partial \Omega_n}{\partial L_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_N}{\partial L_1} & \cdots & \frac{\partial \Omega_N}{\partial L_n} & \cdots & \frac{\partial \Omega_N}{\partial L_N} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Omega_1}{\partial D_1} & \cdots & \frac{\partial \Omega_1}{\partial D_n} & \cdots & \frac{\partial \Omega_1}{\partial D_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_n}{\partial D_1} & \cdots & \frac{\partial \Omega_n}{\partial D_n} & \cdots & \frac{\partial \Omega_n}{\partial D_N} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial \Omega_N}{\partial D_1} & \cdots & \frac{\partial \Omega_N}{\partial D_n} & \cdots & \frac{\partial \Omega_N}{\partial D_N} \end{pmatrix} \begin{pmatrix} dD_1 \\ \vdots \\ dD_n \\ \vdots \\ dD_N \end{pmatrix}.$$

From Cramer's rule (see for example Samuelson 1953), we have:

$$\frac{\partial L_k}{\partial D_n} = -\frac{\sum_i \frac{\partial \Omega_i}{\partial D_n} \triangle_{ik}}{\triangle},$$

where $\triangle$ and $\triangle_{ik}$ are defined above.

We have shown above that $\partial \Omega_n / \partial L_k < 0 < \partial \Omega_n / \partial L_n$ (from (87) and (88)). Therefore the matrix $\boldsymbol{\Omega}_L$ satisfies gross substitution, with positive diagonal terms and negative off-diagonal terms. It follows that all the successive principal minors of $\boldsymbol{\Omega}_L$ are positive and the matrix $\boldsymbol{\Omega}_L$ is positive definite with a positive determinant $\triangle$ (see for example Theorem 4.C.5 in Takayama 1995). We have also shown above that $\partial \Omega_n / \partial D_n > 0$ and $\partial \Omega_n / \partial D_k > 0$ (from (85) and (86)). Therefore each element of $\boldsymbol{\Omega}_D$ is positive. It follows that the system of equations (42) satisfies the properties for unambiguous comparative statics in Proposition 17.G.3 of Section 17.G of Mas-Collel, Whinston, and Green (1995). Since $dD_n < 0$ and $dD_k > 0$ for all $k \neq n$, we have $dL_n > 0$ and $dL_k < 0$ for all $k \neq n$. ■

3.8 Recovering Location Fundamentals

Given values for the model's parameters $\{\alpha,\, \theta,\, \epsilon\}$, a parameterization of bilateral trade costs $\{d_{ni}\}$ and data on populations, wages and land supplies $\{L_n,\, w_n,\, H_n\}$, we now show that the solution to the general equilibrium of the model can be used to recover the unobserved location characteristics of amenities (B_n) and productivities (A_n).

Proposition 6 *Given the model parameters $\{\alpha, \sigma, \epsilon\}$, a parameterization of bilateral trade costs $\{d_{ni}\}$ and data on populations, wages and land supplies $\{L_n, w_n, H_n\}$, there exist unique values of amenities (B_n) and productivities (A_n) that are consistent with the data up to a normalization that corresponds to a choice of units in which to measure productivity and amenities.*

Proof. Given the model's parameters $\{\alpha,\, \sigma,\, \epsilon\}$ and a parameterization of trade costs $\{d_{ni}\}$, we first show how the population and wage data $\{L_n,\, w_n\}$ can be used to recover unique values of unobserved productivities $\{A_n\}$ up to a normalization that corresponds to a choice of units in which to measure productivity. The requirement that income in each location equals expenditure on goods produced by that location defines an excess demand system in unobserved productivities

$$D_i(\boldsymbol{A}) = w_i L_i - \sum_{n \in N} \frac{L_i \left(\frac{d_{ni} w_i}{A_i}\right)^{1-\sigma}}{\sum_{k \in N} L_k \left(\frac{d_{nk} w_k}{A_k}\right)^{1-\sigma}} w_n L_n = 0, \qquad i, n \in N, \tag{89}$$

where we again use bold math font to denote a vector or matrix. Following the same line of reasoning as in the proof of Proposition 3, the excess demand system (89) exhibits the same properties in $\boldsymbol{A}$ as exhibited by the excess demand system (43) in $\boldsymbol{A}$. Therefore, following the same line of reasoning as in the proof of Proposition 3, there exists a unique vector of productivities $\boldsymbol{A}^*$ such that $D(\boldsymbol{A}^*) = 0$. Using these solutions for productivities and the data on wages and populations $\{A_n, w_n, L_n\}$, we can solve for trade shares (π_{ni}) using (8), land rents (r_n) using (16), and price indices (P_n) using (10). Using these solutions for land rents and price indices and the data on wages $\{r_n, P_n, w_n\}$, we can recover unique values of unobserved amenities $\{B_n\}$ up to a normalization that corresponds to a choice of units in which to measure amenities. The residential choice probabilities imply the following excess demand system:

$$\mathbb{D}(\boldsymbol{B}) = \frac{L_n}{\bar{L}} - \frac{B_n \left(P_n^{\alpha} r_n^{1-\alpha}\right)^{-\epsilon} (w_n)^{\epsilon}}{\sum_{k \in N} B_k \left(P_k^{\alpha} r_k^{1-\alpha}\right)^{-\epsilon} (w_k)^{\epsilon}} = 0, \qquad n \in N, \tag{90}$$

which exhibits the same properties in amenities ($\boldsymbol{B}$) that the excess demand system (89) exhibits in productivities ($\boldsymbol{A}$). Therefore there exists a unique vector of amenities $\boldsymbol{B}^*$ such that $\mathbb{D}(\boldsymbol{B}^*) = 0$. ■

From Propositions 3 and 6, the constant and increasing returns models can be both calibrated to replicate the same data on populations, wages and land supplies $\{L_n, w_n, H_n\}$. In the constant returns model, the elasticity of trade with respect to variable trade costs is determined by the shape parameter of the productivity distribution ($\theta^N > \sigma^N - 1$), where the subscript N (neoclassical) indicates the constant returns model. In contrast, in the increasing returns model, the trade elasticity is dictated by the elasticity of substitution between variables ($\sigma^G - 1$), where the superscript G indicates the increasing returns to scale (new economic geography) model. Therefore calibrating both models to the same initial equilibrium requires different structural parameters for the elasticity of substitution ($\sigma^G - 1 = \theta^N > \sigma^N - 1$). Furthermore, population directly affects the trade shares in the increasing returns model (68), but does not directly affect the trade shares in the constant returns model (18). Therefore calibrating both models to the same initial equilibrium also requires assuming different unobserved productivities in the two models, as summarized in the following proposition.

Proposition 7 *Given the parameters $\{\alpha, \epsilon\}$ and $\theta^N = \sigma^G - 1$ and a parameterization of bilateral trade costs $\{d_{ni}\}$, the constant returns model (superscript N) and increasing returns model (superscript G) both can be calibrated to the same data on populations, wages and land supplies $\{L_n, w_n, H_n\}$ in an initial equilibrium. This calibration involves different structural parameters ($\sigma^N \neq \sigma^G$) and productivities ($A_n^N \neq A_n^G$) but the same amenities ($B_n^N = B_n^G$) in the two models.*

Proof. The proposition follows from Propositions 3 and 6 above. From the trade shares (18) and (68), we require $\theta^N = \sigma^G - 1$ to ensure the same trade shares for the same wages and parametrization of trade costs in the two models. This assumption implies $\sigma^G - 1 > \sigma^N - 1$, since we require $\theta^N > \sigma^N - 1$ to ensure a finite price index in the constant returns model. From the trade shares (18) and (68), the term $(A_i)^{\sigma-1} L_i$ enters the goods market clearing condition (67) in the increasing returns model in exactly the same way that the term A_i enters the goods market clearing condition (17) in the constant returns model. Therefore we require $A_i^N = \left(A_i^G\right)^{\sigma-1} L_i$ to ensure the same trade shares in the two models for the same wages, parameterization of trade costs and trade elasticity. It follows that the two models require different unobserved productivities ($A_i^N \neq A_i^G$) to explain the same data on populations, wages and land supplies. From the residential choice probabilities (19) and (69), under the assumptions that $A_i^N = \left(A_i^G\right)^{\sigma-1} L_i$ and $\theta^N = \sigma^G - 1$, the two models rationalize the same data on populations, wages and land supplies with the same unobserved amenities ($B_i^N = B_i^G$). ■

3.9 Counterfactuals

The system of equations for general equilibrium (67)-(69) again provides an approach for undertaking model-based counterfactuals that uses only parameters and the values of endogenous variables in an initial equilibrium. Denoting the relative value of variables in the counterfactual and initial equilibria by a hat ($\hat{x} = x'/x$), we can solve for the counterfactual effects of a change in trade frictions, productivity or amenities using:

$$\hat{w}_i \hat{\lambda}_i Y_i = \sum_{n \in N} \pi'_{ni} \hat{w}_n \hat{\lambda}_n Y_n, \tag{91}$$

$$\hat{\pi}_{ni} \pi_{ni} = \frac{\pi_{ni} \left(\hat{d}_{ni} \hat{w}_i / \hat{A}_i \right)^{1-\sigma} \hat{L}_i}{\sum_{k \in N} \pi_{nk} \left(\hat{d}_{nk} \hat{w}_k / \hat{A}_k \right)^{1-\sigma} \hat{L}_k}, \tag{92}$$

$$\hat{\lambda}_n \lambda_n = \frac{\hat{B}_n \hat{A}_n^{\alpha\epsilon} \hat{\pi}_{nn}^{-\frac{\alpha\epsilon}{\sigma-1}} \hat{\lambda}_n^{-\left(\epsilon(1-\alpha) - \frac{\alpha\epsilon}{\sigma-1}\right)} \lambda_n}{\sum_{k \in N} \hat{B}_k \hat{A}_k^{\alpha\epsilon} \hat{\pi}_{kk}^{-\frac{\alpha\epsilon}{\sigma-1}} \hat{\lambda}_k^{-\left(\epsilon(1-\alpha) - \frac{\alpha\epsilon}{\sigma-1}\right)} \lambda_k}. \tag{93}$$

where $Y_i = w_i L_i$ again denotes labor income and $\lambda_n = L_n / \bar{L}$ again denotes the population share within each country in the initial equilibrium. This system of equations can be solved for $\{\hat{\lambda}_n, \hat{w}_n, \hat{\pi}_{ni}\}$ given the observed variables in the initial equilibrium $\{\lambda_n, Y_n, \pi_{ni}\}$ and an assumed comparative static.

Comparing the counterfactual systems in the constant returns model ((45)-(47)) and the increasing returns model ((91)-(93)), the dependence of the measures of varieties on populations in the increasing returns model is reflected in both the trade shares (in the terms in $\hat{L}_i$ in (92)) and the residential choice probabilities (in the different exponents on $\hat{L}_i$ in (93)). This dependence of the measure of varieties on the endogenous populations of locations in the increasing returns model implies different counterfactual predictions for the impact of changes in trade costs from the constant returns model. These differences exist even if the two models are calibrated to the same initial equilibrium $\{w_n, L_n, \pi_{ni}\}$, the same elasticity of trade with respect to trade costs ($\theta^N = \sigma^G - 1$), and the same values of the other model parameters.

Proposition 8 *Suppose that the constant and increasing returns models are calibrated to the same data on populations, wages and land supplies $\{L_n, w_n, H_n\}$ in an initial equilibrium with the same trade elasticity $\theta^N = \sigma^G - 1$ and the same values of the other model parameters. Even when calibrated in this way, the two models imply different counterfactual predictions for the effects of a reduction in trade costs.*

Proof. The proposition follows immediately from the general equilibrium systems (45)-(47) in the constant returns model and (91)-(93) in the increasing returns model. The terms in $\hat{L}_i$ in the trade share (92) in the increasing returns model are absent in the trade share (46) in the constant returns model. The exponents on $\hat{L}_i$ in the residential choice probabilities (93) in the increasing returns model are different from those in the residential choice probabilities (47) in the constant returns model. Therefore the two models imply different counterfactual predictions for wages, trade shares and populations. ■

In an international trade context, in which population is immobile between locations, these two models imply the same counterfactual predictions for the effects of a reduction in trade costs on wages, trade shares and welfare (see Arkolakis, Costinot, and Rodriguez-Clare 2012). In contrast, in a setting in which labor is mobile across locations, the reallocation of population across locations in response to the reduction in trade costs leads to endogenous changes in the measures of varieties in the increasing returns model. These endogenous changes in the measure of varieties in turn lead to different counterfactual predictions for wages, trade shares and populations from the constant returns model.

3.10 Welfare Gains from Trade

We now examine the implications of the introduction of agglomeration forces for the welfare gains from trade. Using the residential choice probabilities (65) and expected utility (66), we can express expected utility in terms of income, goods price indices and land rents:

$$\left(\frac{\bar{U}_n}{\delta} \right)^{\epsilon} = \left(\frac{\bar{U}}{\delta} \right)^{\epsilon} = \frac{B_n v_n^{\epsilon}}{\left(P_n^{\alpha} r_n^{1-\alpha} \right)^{\epsilon} \left(L_n / \bar{L}^j \right)}, \qquad \forall \, n.$$

Using income equals expenditure (14), land market clearing (16) and the goods price index (64), expected utility for each location can be re-written solely in terms of its domestic trade share and population and model parameters:

$$\bar{U}_n = \bar{U} = \frac{\delta B_n^{\frac{1}{\epsilon}} A_n^{\alpha} \left(\frac{1}{\pi_{nn}}\right)^{\frac{\alpha}{\sigma-1}} H_n^{1-\alpha} L_n^{-\left(\frac{1}{\epsilon}+(1-\alpha)-\frac{\alpha}{\sigma-1}\right)}}{\alpha \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \left(\frac{\sigma}{\sigma-1}\right)^{\alpha} (\sigma F)^{\frac{\alpha}{\sigma-1}} \left(\bar{L}^j\right)^{-\frac{1}{\epsilon}}}, \qquad \forall\, n, \tag{94}$$

where the condition for the existence of a unique equilibrium $\sigma(1-\tilde{\alpha}) > 1$ implies that the expected utility for each location is decreasing in its population ($\frac{1}{\epsilon} + (1-\alpha) > \frac{\alpha}{\sigma-1}$). The domestic trade share (π_{nn}), population (L_n), the trade elasticity ($\sigma - 1$), the labor supply elasticity (ϵ) and the share of tradables in expenditure (α) are again sufficient statistics for the welfare gains from trade:

$$\frac{\bar{U}_n^T}{\bar{U}_n^A} = \frac{\bar{U}^T}{\bar{U}^A} = \left(\frac{1}{\pi_{nn}^T}\right)^{\frac{\alpha}{\sigma-1}} \left(\frac{L_n^A}{L_n^T}\right)^{\frac{1}{\epsilon}+(1-\alpha)-\frac{\alpha}{\sigma-1}}, \qquad \forall\, n, \tag{95}$$

where the exponent on relative populations now has an additional term relative to the constant returns model that captures the impact of labor supply on the endogenous measure of varieties.

One special case of the model is perfect labor mobility and no preference heterogeneity ($\epsilon \to \infty$), in which case the expression for welfare (94) simplifies to become:

$$\bar{U}_n = \bar{U} = \frac{A_n^{\alpha} \left(\frac{1}{\pi_{nn}}\right)^{\frac{\alpha}{\sigma-1}} H_n^{1-\alpha} L_n^{-\left((1-\alpha)-\frac{\alpha}{\sigma-1}\right)}}{\alpha \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \left(\frac{\sigma}{\sigma-1}\right)^{\alpha} (\sigma F)^{\frac{\alpha}{\sigma-1}}}, \qquad \forall\, n. \tag{96}$$

and the welfare gains from trade are:

$$\frac{\bar{U}_n^T}{\bar{U}_n^A} = \frac{\bar{U}^T}{\bar{U}^A} = \left(\frac{1}{\pi_{nn}^T}\right)^{\frac{\alpha}{\sigma-1}} \left(\frac{L_n^A}{L_n^T}\right)^{(1-\alpha)}, \qquad \forall n. \tag{97}$$

A second special case of the model is perfect labor immobility, in which case expected utility takes the same form as in (96), except that expected utility and the welfare gains from trade in general differ across locations because of the absence of labor mobility:

$$\frac{\bar{U}_n^T}{\bar{U}_n^A} = \left(\frac{1}{\pi_{nn}^T}\right)^{\frac{\alpha}{\sigma-1}} \neq \frac{\bar{U}_k^T}{\bar{U}_k^A}, \qquad n \neq k. \tag{98}$$

Intuitively, when labor is perfectly immobile, locations with better access to markets in the open economy experience larger welfare gains from trade, because population reallocations no longer provide a mechanism for utility equalization through changes in the price of land.

4 Quantitative Analysis

No further derivations required.

5 Regions and Countries

No further derivations required.

6 Conclusions

No further derivations required.

7 Commercial Land and Intermediate Inputs

In this section, we consider an extension of the model to allow land to be used commercially and to incorporate intermediate inputs in production. We develop this extension in the context of our baseline constant returns model from Section 2, but it is straightforward to instead consider our increasing returns model from Section 3 or our model with a distinction between regions and countries from Section 5.

7.1 Preferences, Endowments and Technology

Preferences are again defined over goods consumption (C_n) and residential land use (H_n) and take the same form as in (1). The goods consumption index (C_n) is defined over consumption of a fixed continuum of goods $j \in [0,1]$ as in (2). Each region draws an idiosyncratic productivity z_j for each good j as in (5) and goods are again homogeneous in the sense that one unit of a given good is the same as any other unit of that good. Goods are produced with labor, land and intermediate inputs under conditions of perfect competition according to a Cobb-Douglas production technology. Each good uses all other goods as intermediate inputs with the same CES aggregator as for consumer preferences, as in Krugman and Venables (1995) and Eaton and Kortum (2002). The cost to a consumer in region n of purchasing one unit of good j from region i is:

$$p_{ni}(j) = \frac{d_{ni} w_i^{\beta} r_i^{\eta} P_i^{1-\beta-\eta}}{z_i(j)}, \qquad 0 < \beta < 1,\ 0 < \eta < 1,\ 0 < \beta + \eta < 1, \tag{99}$$

where P_i is the dual price index.

7.2 Expenditure Shares and Price Indices

The representative consumer in region n sources each good from the lowest cost supplier to that region. Using equilibrium prices (99) and the properties of the Fréchet distribution, the share of expenditure of region n on goods produced by region i is:

$$\pi_{ni} = \frac{A_i \left(d_{ni} w_i^{\beta} r_i^{\eta} P_i^{1-\beta-\eta} \right)^{-\theta}}{\sum_{k \in N} A_k \left(d_{nk} w_k^{\beta} r_k^{\eta} P_k^{1-\beta-\eta} \right)^{-\theta}}, \tag{100}$$

while the price index for tradeable goods is:

$$P_n = \gamma \left[\sum_{i \in N} A_i \left(d_{ni} w_i^{\beta} r_i^{\eta} P_i^{1-\beta-\eta} \right)^{-\theta} \right]^{1/\theta}, \tag{101}$$

where $\gamma = \left[\Gamma\left(\frac{\theta+1-\sigma}{\theta} \right) \right]^{\frac{1}{1-\sigma}}$; $\Gamma(\cdot)$ denotes the Gamma function; and $\theta > \sigma - 1$.

7.3 Residential Choices and Income

Residential choices take a similar form as in section 2. Using the Fréchet distribution of idiosyncratic shocks to amenities, the probability that a worker chooses to live in location $n \in N$ is:

$$\frac{L_n}{\bar{L}} = \frac{B_n \left(v_n / P_n^{\alpha} r_n^{1-\alpha} \right)^{\epsilon}}{\sum_{k \in N} B_k \left(v_k / P_k^{\alpha} r_k^{1-\alpha} \right)^{\epsilon}}. \tag{102}$$

Expected worker utility is:

$$\bar{U} = \delta \left[\sum_{k \in N} B_k \left(v_k / P_k^{\alpha} r_k^{1-\alpha} \right)^{\epsilon} \right]^{\frac{1}{\epsilon}}, \tag{103}$$

where $\delta = \Gamma\left((\epsilon-1)/\epsilon\right)$; $\Gamma(\cdot)$ is the Gamma function; and $\epsilon > 1$.

Expenditure on land in each location is redistributed lump sum to the workers residing in that location, which implies that total income (v_n) equals labor income plus expenditure on commercial and residential land:

$$v_n L_n = \left(\frac{\beta+\eta}{\alpha\beta}\right) w_n L_n. \tag{104}$$

Land market clearing implies that the equilibrium land rent again can be determined from the equality of land income and expenditure:

$$r_n = \frac{(1-\alpha)\beta+\eta}{\alpha\beta}\frac{w_n L_n}{H_n}. \tag{105}$$

7.4 General Equilibrium

The general equilibrium of the model can be represented by the measure of workers (L_n), the trade share (π_{ni}), the wage (w_n) and the price index (P_n) for each location $n, i \in N$. Using labor income (104), the trade share (8), the price index (101), residential choice probabilities (102) and land market clearing (105), this equilibrium quadruple $\{L_n, \pi_{ni}, w_n, P_n\}$ solves the following system of equations for all $i, n \in N$. First, each location's income must equal expenditure on the goods produced in that location:

$$w_i L_i = \sum_{n\in N} \pi_{ni} w_n L_n. \tag{106}$$

Second, location expenditure shares are:

$$\pi_{ni} = \frac{A_i\left(d_{ni} w_i^{\beta+\eta} P_i^{1-\beta-\eta}\left(\frac{L_i}{H_i}\right)^{\eta}\right)^{-\theta}}{\sum_{k\in N} A_k\left(d_{nk} w_k^{\beta+\eta} P_k^{1-\beta-\eta}\left(\frac{L_k}{H_k}\right)^{\eta}\right)^{-\theta}}. \tag{107}$$

Third, price indices are:

$$P_n = \gamma\left[\sum_{i\in N} A_i\left(d_{ni} w_i^{\beta+\eta} P_i^{1-\beta-\eta}\left(\frac{(1-\alpha)\beta+\eta}{\alpha\beta}\frac{L_i}{H_i}\right)^{\eta}\right)^{-\theta}\right]^{1/\theta}, \tag{108}$$

Fourth, residential choice probabilities imply:

$$\frac{L_n}{\bar{L}} = \frac{B_n\left(P_n^{\alpha} w_n^{\alpha}\left(\frac{L_n}{H_n}\right)^{1-\alpha}\right)^{-\epsilon}}{\sum_{k\in N} B_k\left(P_k^{\alpha} w_k^{\alpha}\left(\frac{L_k}{H_k}\right)^{1-\alpha}\right)^{-\epsilon}}. \tag{109}$$

Therefore the general equilibrium of the model with commercial land and intermediate inputs can be analyzed using an analogous approach as for our baseline model.

References

Allen, T., and C. Arkolakis (2014): "Trade and the Topography of the Spatial Economy," *Quarterly Journal of Economics*, forthcoming.

Arkolakis, C., A. Costinot, and A. Rodriguez-Clare (2012): "New Trade Models, Same Old Gains," *American Economic Review*, 102(1), 94–130.

Davis, M. A., and F. Ortalo-Magné (2011): "Housing Expenditures, Wages, Rents," *Review of Economic Dynamics*, 14(2), 248–261.

Dekle, R., J. Eaton, and S. Kortum (2007): "Unbalanced Trade," *American Economic Review*, 97(2), 351–355.

Eaton, J., and S. Kortum (2002): "Technology, Geography, and Trade," *Econometrica*, 70(5), 1741–1779.

Fujimoto, T., and U. Krause (1985): "Strong Ergodicity for Strictly Increasing Nonlinear Operators," *Linear Algebra and its Applications*, 71, 101–112.

Fujita, M., P. Krugman, and A. J. Venables (1999): *The Spatial Economy: Cities, Regions and International Trade*. MIT Press, Cambridge.

Helpman, E. (1998): "The Size of Regions," in *Topics in Public Economics: Theoretical and Applied Analysis*, ed. by D. Pines, E. Sadka, and I. Zilcha. Cambridge University Press, Cambridge.

Krugman, P. (1991): "Increasing Returns and Economic Geography," *Journal of Political Economy*, 99(3), 483–499.

Krugman, P., and A. J. Venables (1995): "Globalisation and Inequality of Nations," *Quarterly Journal of Economics*, 60, 857–80.

Mas-Collel, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*. Oxford University Press, Oxford.

Samuelson, P. A. (1953): *Foundations of Economic Analysis*. Harvard University Press, Cambridge.

Takayama, A. (1995): *Mathematical Economics*. Cambridge University Press, Cambridge.