Web Appendix to Theories of Heterogeneous Firms and Trade: Not for Publication

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1 Introduction

This appendix contains derivations of the expressions in the main text of the survey and is organized around the same section headings as in the main text of the survey.

2 Empirical Challenges

No derivations required.

3 The Melitz Model

The Melitz (2003) model combines a model of industry equilibrium featuring heterogeneous firm productivity with a model of trade based on love of variety preferences.

3.1 Preferences and Endowments

The world consists of many countries, such that each country trades with $n \geq 1$ foreign countries. We begin by considering the case of symmetric countries, before discussing below the implications of introducing country asymmetries. Labor is the sole factor of production in inelastic supply L for each country and is immobile across countries.

Consumer preferences are defined over consumption of a continuum of horizontally-differentiated varieties and take the Constant Elasticity of Substitution (CES) or Dixit and Stiglitz (1977) form:

$$C = \left[\int_{\omega \in \Omega} q(\omega)^{\rho} d\omega \right]^{\frac{1}{\rho}}, \qquad 0 < \rho < 1, \tag{1}$$

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where ω indexes varieties, Ω is the (endogenous) set of varieties, and the price index dual to (1) is:

$$P = \left[\int_{\omega \in \Omega} p(\omega)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}}, \qquad \sigma = \frac{1}{1-\rho} > 1,$$

where σ corresponds to the elasticity of substitution between varieties. The model can be interpreted as capturing an industry within an economy.

Given the above preferences, revenue for a variety supplied to the domestic market is:

$$r_d(\omega) = R \left(\frac{p_d(\omega)}{P} \right)^{1-\sigma}, \tag{2}$$

where $p_d(\omega)$ is the price of variety ω in the domestic market, R denotes aggregate revenue which equals aggregate income which equals aggregate expenditure, and the price index P summarizes the impact of the prices of competing varieties.

3.2 Production Technology

There is a competitive fringe of potential entrants that can enter by paying a sunk entry cost of f_e units of labor. Potential entrants face uncertainty about their productivity in the industry. Once the sunk entry cost is paid, a firm draws its productivity φ from a fixed distribution, $g(\varphi)$. As all firms with the same productivity behave symmetrically, we index firms from now on by φ alone. Productivity remains fixed after entry, but firms face a constant exogenous probability of death δ, which induces steady-state entry and exit of firms in the model.

The market structure is monopolistic competition. Production of each variety involves a fixed production cost of f_d units of labor and a constant variable cost that depends on firm productivity. The total labor required to produce $q(\varphi)$ units of a variety is therefore:

$$l(\varphi) = f_d + \frac{q(\varphi)}{\varphi}.$$

If firms decide to export, they also face a fixed exporting cost of f_x units of labor and iceberg variable costs of trade, such that $\tau > 1$ units of each variety must be exported in order for one unit to arrive in the foreign country.

3.3 Production and Exporting Decisions

As each firm supplies one of a continuum of varieties, it is of measure zero relative to the industry as a whole and hence takes the aggregate price index as given. The first-order condition for profit maximization yields the standard result that equilibrium prices are a mark-up over marginal cost that depends on the constant elasticity of demand. Given the same elasticity of demand in the domestic and export markets, equilibrium prices in the export market are a constant multiple of those in the domestic market due to the variable costs of trade:

$$p_x(\varphi) = \tau p_d(\varphi) = \tau \left(\frac{\sigma}{\sigma - 1} \right) \frac{w_d}{\varphi} = \frac{\tau}{\rho\varphi}, \tag{3}$$

where we choose the wage in one country as the numeraire and use country symmetry. Together these imply $w = 1$ for all countries.

Substituting the pricing rule into firm revenue (2), we obtain the following expression for equilibrium firm revenue in the export and domestic markets:

$$r_x(\varphi) = \tau^{1-\sigma} r_d(\varphi) = \tau^{1-\sigma} \left(\rho\varphi\right)^{\sigma-1} RP^{\sigma-1}. \tag{4}$$

Therefore the relative revenue of any two firms within the same market depends solely on their relative productivities:

$$\frac{r_d(\varphi'')}{r_d(\varphi')} = \frac{r_x(\varphi'')}{r_x(\varphi')} = \left(\frac{\varphi''}{\varphi'}\right)^{\sigma-1}, \tag{5}$$

while the relative revenue of a firm with a given productivity in the domestic and export markets depends solely on variable trade costs.

Consumer love of variety and a fixed production cost imply that no firm would ever export without also serving the domestic market. Therefore we can apportion the entire fixed production cost to the domestic market and the entire fixed exporting cost to the foreign market.[1] Adopting this convenient accounting device, the pricing rule (3) implies that variable profits in each market are proportional to revenue, while firm profits in each market equal variable profits minus the relevant fixed cost:

$$\pi_x(\varphi) = \frac{r_x(\varphi)}{\sigma} - f_x, \qquad \pi_d(\varphi) = \frac{r_d(\varphi)}{\sigma} - f_d. \tag{6}$$

The existence of a fixed production cost implies that there is a *zero-profit cutoff productivity* (φ_d^*) below which firms would make negative profits if they entered, and hence they exit immediately:

$$r_d(\varphi_d^*) = (\rho\varphi_d^*)^{\sigma-1} RP^{\sigma-1} = \sigma f_d. \tag{7}$$

Similarly, the existence of a fixed exporting cost implies that there is an *exporting cutoff productivity* (φ_x^*) below which surviving firms would make negative profits if they exported, and hence they serve only the domestic market:

$$r_x(\varphi_x^*) = \tau^{1-\sigma}(\rho\varphi_x^*)^{\sigma-1} RP^{\sigma-1} = \sigma f_x. \tag{8}$$

Now note that the relationship between variety revenues for firms with different productivities within the same market (5) implies:

$$r_d(\varphi_x^*) = \left(\frac{\varphi_x^*}{\varphi_d^*}\right)^{\sigma-1} r_d(\varphi_d^*),$$

which, using the relationship between variety revenue for firms with the same productivity in different markets (4), can be written as:

$$\tau^{\sigma-1} r_x(\varphi_x^*) = \left(\frac{\varphi_x^*}{\varphi_d^*}\right)^{\sigma-1} r_d(\varphi_d^*).$$

Using the zero-profit and exporting cutoff productivity conditions, (7) and (8) respectively, we obtain the rela-

[1] This is merely a convenient accounting device. Instead of analyzing the decision to export by comparing export profits to the fixed exporting cost, we could instead equivalently compare the sum of domestic and export profit profits to the sum of the fixed production and exporting costs.

tionship between the productivity cutoffs:

$$\varphi_x^* = \Lambda \varphi_d^*, \qquad \Lambda \equiv \tau \left(\frac{f_x}{f_d}\right)^{\frac{1}{\sigma-1}}. \tag{9}$$

For sufficiently high values of fixed and variable trade costs, we have $\Lambda > 1$, so that the model features selection into export markets, which is consistent with the empirical literature discussed above. Only the most productive firms export, while intermediate productivity firms serve only the domestic market, and low productivity firms exit.

3.4 Steady-state Industry Equilibrium

The steady-state industry equilibrium is characterized by constant masses of firms entering, producing and exporting as well as stationary *ex post* distributions of productivity among producing and exporting firms. With firm productivity fixed at entry and a constant independent probability of firm death, these stationary *ex post* distributions for productivity take a particularly tractable form. The *ex post* productivity distributions in the domestic and export markets, $\mu_d(\varphi)$ and $\mu_x(\varphi)$ respectively, are truncations of the *ex ante* productivity distribution, $g(\varphi)$, at the zero-profit and exporting cutoff productivities:

$$\mu_d(\varphi) = \begin{cases} \frac{g(\varphi)}{1-G(\varphi_d^*)} & \text{if } \varphi \geq \varphi_d^* \\ 0 & \text{otherwise} \end{cases}, \qquad \mu_x(\varphi) = \begin{cases} \frac{g(\varphi)}{1-G(\varphi_x^*)} & \text{if } \varphi \geq \varphi_x^* \\ 0 & \text{otherwise} \end{cases}. \tag{10}$$

In an equilibrium with positive firm entry, we require that the free entry condition holds, which equates the expected value of entry to the sunk entry cost. The expected value of entry depends on the value of a firm with each productivity φ, which equals the maximum of zero (if the firm exits) and the net present value of the stream of future profits (if the firm enters). Assuming, for simplicity, no time discounting, the value of a firm with productivity φ is:

$$v(\varphi) = \max\left\{0, \frac{\pi(\varphi)}{\delta}\right\}.$$

The free entry condition therefore takes the following form:

$$v_e = [1 - G(\varphi_d^*)] \frac{\bar{\pi}}{\delta} = [1 - G(\varphi_d^*)] \frac{[\bar{\pi}_d + \chi n \bar{\pi}_x]}{\delta} = f_e, \tag{11}$$

where the expected value of entry equals the probability of successful entry, $[1 - G(\varphi_d^*)]$, times expected profits conditional on successful entry, $\bar{\pi}$. Expected profits conditional on successful entry equal expected profits in the domestic market conditional on serving that market, $\bar{\pi}_d$, plus the probability of exporting, $\chi = [1 - G(\varphi_x^*)]/[1 - G(\varphi_d^*)]$, times the number of export markets, n, times expected profits in the export market conditional on serving that market, $\bar{\pi}_x$.

In a steady-state equilibrium with a constant mass of firms producing, we also require that the mass of successful entrants that draw a productivity above the zero-profit cutoff equals the mass of firms that die, which yields the following steady-state stability condition:

$$[1 - G(\varphi_d^*)]\, M_e = \delta M_d, \tag{12}$$

where M_e denotes the mass of entrants and M_d denotes the mass of producing firms.

While aggregate variables such as the price index and revenue depend on integrals of firm variables over values for productivity, the assumption of CES preferences greatly simplifies the analysis, because all aggregate variables can be written in terms of a weighted average of firm productivity. Therefore aggregate variables take the same value as in a model in which all firms have a common productivity equal to weighted-average productivity, but weighted average-productivity is itself endogenously determined by firm decisions.

Using the equilibrium pricing rule (3), the zero-profit and exporting cutoff productivities, (7) and (8) respectively, the *ex post* productivity distributions, (10), and country symmetry, we obtain the following expression for the price index:

$$\begin{aligned} P &= \left[\int_{\varphi_d^*}^{\infty} p_d(\varphi)^{1-\sigma} M_d \mu_d(\varphi)\, d\varphi + n\int_{\varphi_x^*}^{\infty} p_x(\varphi)^{1-\sigma} M_x \mu_x(\varphi)\, d\varphi\right]^{\frac{1}{1-\sigma}}, \\ &= \left[M_d \int_{\varphi_d^*}^{\infty} p_d(\varphi)^{1-\sigma} \mu_d(\varphi)\, d\varphi + \tau^{1-\sigma}\chi n M_d \int_{\varphi_x^*}^{\infty} p_d(\varphi)^{1-\sigma} \mu_x(\varphi)\, d\varphi\right]^{\frac{1}{1-\sigma}}, \\ &= \left[M_d \int_{\varphi_d^*}^{\infty} (\rho\varphi)^{\sigma-1} \frac{g(\varphi)}{1-G(\varphi_d^*)} d\varphi + \tau^{1-\sigma}\chi n M_d \int_{\varphi_x^*}^{\infty} (\rho\varphi)^{\sigma-1} \frac{g(\varphi)}{1-G(\varphi_x^*)} d\varphi\right]^{\frac{1}{1-\sigma}}, \\ &= \left[M_d(\rho\tilde{\varphi}_d)^{\sigma-1} + \chi n M_d \tau^{1-\sigma}(\rho\tilde{\varphi}_x)^{\sigma-1}\right]^{\frac{1}{1-\sigma}}, \\ &= \left[M_d p_d(\tilde{\varphi}_d)^{1-\sigma} + (\chi n M_d)\,\tau^{1-\sigma} p_d(\tilde{\varphi}_x)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \\ &= M_t^{\frac{1}{1-\sigma}} p(\tilde{\varphi}_t), \end{aligned} \tag{13}$$

where weighted average productivities in the domestic and export markets, $\tilde{\varphi}_d$ and $\tilde{\varphi}_x$, are defined as:

$$\tilde{\varphi}_d = \left[\int_{\varphi_d^*}^{\infty} \varphi^{\sigma-1} \frac{g(\varphi)}{1-G(\varphi_d^*)} d\varphi\right]^{\frac{1}{\sigma-1}}, \tag{14}$$

$$\tilde{\varphi}_x = \left[\int_{\varphi_x^*}^{\infty} \varphi^{\sigma-1} \frac{g(\varphi)}{1-G(\varphi_x^*)} d\varphi\right]^{\frac{1}{\sigma-1}}, \tag{15}$$

and depend solely on the productivity cutoffs in the domestic and export markets, φ_d^* and φ_x^*. Overall weighted average productivity, $\tilde{\varphi}_t$, is itself a weighted average of weighted average productivities in the domestic and export markets:

$$\tilde{\varphi}_t = \left\{\frac{1}{M_t}\left[M_d \tilde{\varphi}_d^{\sigma-1} + \chi n M_d \left(\tau^{-1}\tilde{\varphi}_x\right)^{\sigma-1}\right]\right\}^{\frac{1}{\sigma-1}}$$

where the mass of firms supplying varieties to a market, M_t, is given by:

$$M_t = (1+\chi n)\, M_d.$$

The mass of firms producing, M_d, can be determined from the ratio of aggregate revenue, R, to average revenue, $\bar{r}$:

$$M_d = \frac{R}{\bar{r}}. \tag{16}$$

To determine aggregate revenue, note that the steady-state stability condition implies:

$$[1 - G(\varphi_d^*)] = \frac{\delta M_d}{M_e}.$$

Substituting for $[1 - G(\varphi_d^*)]$ in the free entry condition yields:

$$v_e = \frac{M_d}{M_e}\bar{\pi} = f_e,$$

which implies that total payments to labor used in entry equal total profits: $L_e = M_e f_e = M_d \bar{\pi}$. Note that total payments to labor used in production equal aggregate revenue minus total profits: $L_p = R - M_d \bar{\pi}$. Therefore, using labor market clearing, we obtain:

$$L_e + L_p = L = R,$$

and aggregate revenue equals the economy's labor endowment.

Average revenue can be determined as follows:

$$\begin{aligned}
\bar{r} &= \int_{\varphi_d^*}^{\infty} \frac{r(\varphi)\, g(\varphi)\, d\varphi}{1 - G(\varphi_d^*)}, \\
&= \int_{\varphi_d^*}^{\infty} \frac{r_d(\varphi)\, g(\varphi)\, d\varphi}{1 - G(\varphi_d^*)} + \frac{1 - G(\varphi_x^*)}{1 - G(\varphi_d^*)} n \int_{\varphi_x^*}^{\infty} \frac{r_x(\varphi)\, g(\varphi)\, d\varphi}{1 - G(\varphi_x^*)}, \\
&= \sigma f_d \int_{\varphi_d^*}^{\infty} \left(\frac{\varphi}{\varphi_d^*}\right)^{\sigma-1} \frac{g(\varphi)\, d\varphi}{1 - G(\varphi_d^*)} + \frac{1 - G(\varphi_x^*)}{1 - G(\varphi_d^*)} n \sigma f_x \int_{\varphi_x^*}^{\infty} \left(\frac{\varphi}{\varphi_x^*}\right)^{\sigma-1} \frac{g(\varphi)\, d\varphi}{1 - G(\varphi_x^*)}, \\
&= \left(\frac{\tilde{\varphi}_d}{\varphi_d^*}\right)^{\sigma-1} \sigma f_d + \chi n \left(\frac{\tilde{\varphi}_x}{\varphi_x^*}\right)^{\sigma-1} \sigma f_x,
\end{aligned} \tag{17}$$

which depends solely on the productivity cutoffs, φ_d^* and φ_x^*, and parameters.

Similarly, average profit conditional on serving the domestic market is:

$$\begin{aligned}
\bar{\pi}_d &= \int_{\varphi_d^*}^{\infty} \frac{\pi_d(\varphi)\, g(\varphi)\, d\varphi}{1 - G(\varphi_d^*)}, \\
&= \int_{\varphi_d^*}^{\infty} \left(\frac{r_d(\varphi)}{\sigma} - f_d\right) \frac{g(\varphi)\, d\varphi}{1 - G(\varphi_d^*)}, \\
&= \int_{\varphi_d^*}^{\infty} \left[\left(\frac{\varphi}{\varphi_d^*}\right)^{\sigma-1} - 1\right] f_d \frac{g(\varphi)\, d\varphi}{1 - G(\varphi_d^*)},
\end{aligned} \tag{18}$$

while average profit conditional on serving the export market is:

$$\begin{aligned}
\bar{\pi}_x &= \int_{\varphi_x^*}^{\infty} \frac{\pi_x(\varphi)\, g(\varphi)\, d\varphi}{1 - G(\varphi_x^*)}, \\
&= \int_{\varphi_x^*}^{\infty} \left(\frac{r_x(\varphi)}{\sigma} - f_x\right) \frac{g(\varphi)\, d\varphi}{1 - G(\varphi_x^*)}, \\
&= \int_{\varphi_x^*}^{\infty} \left[\left(\frac{\varphi}{\varphi_x^*}\right)^{\sigma-1} - 1\right] f_x \frac{g(\varphi)\, d\varphi}{1 - G(\varphi_x^*)},
\end{aligned} \tag{19}$$

and average profit conditional on successful entry is:

$$\bar{\pi} = \bar{\pi}_d + \chi n \bar{\pi}_x, \tag{20}$$

where these expressions for $\bar{\pi}_d$, $\bar{\pi}_x$ and $\bar{\pi}$ again depend solely on the productivity cutoffs, φ_d^* and φ_x^*, and will prove useful in rewriting the free entry condition below.

3.5 Determination of Steady-State Industry Equilibrium

With symmetric countries, the steady-state industry equilibrium can be referenced by a quadruple $\{\varphi_d^*, \varphi_x^*, R, P\}$, in terms of which all other endogenous variables can be written. As the model has a recursive structure, it is straightforward to determine this steady-state equilibrium for a general continuous productivity distribution.

We begin by using the free entry condition to pin down the zero-profit cutoff productivity, φ_d^*. Using average profit in each market, (18) and (19), the probability of exporting, $\chi = [1 - G(\varphi_x^*)]/[1 - G(\varphi_d^*)]$, and the relationship between the productivity cutoffs (9), the free entry condition (11) can be re-written as:

$$\frac{f_d}{\delta} \int_{\varphi_d^*}^{\infty} \left[\left(\frac{\varphi}{\varphi_d^*} \right)^{\sigma-1} - 1 \right] g(\varphi) d\varphi + \frac{n f_x}{\delta} \int_{\Lambda \varphi_d^*}^{\infty} \left[\left(\frac{\varphi}{\Lambda \varphi_d^*} \right)^{\sigma-1} - 1 \right] g(\varphi) d\varphi = f_e, \tag{21}$$

where $\Lambda \equiv \tau \left(f_x / f_d \right)^{\frac{1}{\sigma-1}}$. The right-hand side of (21) is a constant. As $\varphi_d^* \to 0$, the left-hand side of (21) converges to infinity. As $\varphi_d^* \to \infty$, the left-hand side of (21) converges to zero. Additionally, the left-hand side of (21) is continuous and monotonically decreasing in φ_d^*. Therefore a simple fixed point argument establishes the existence of a unique equilibrium value of the zero-profit cutoff productivity, φ_d^*, independently of the other endogenous variables of the model. Having determined φ_d^*, the exporting cutoff productivity, φ_x^*, follows immediately from the relationship between the productivity cutoffs (9).

Aggregate revenue was solved for above and equals the economy's supply of labor, $R = L$, which leaves only the price index, P, to be characterized. The price index (13) has three components: the price of a variety with weighted average productivity in each market, $p_d(\tilde{\varphi}_d)$ and $p_x(\tilde{\varphi}_x)$, the probability of exporting, χ, and the mass of firms, M_d. The prices $p_d(\tilde{\varphi}_d)$ and $p_x(\tilde{\varphi}_x)$ depend solely on weighted average productivities in each market, $\tilde{\varphi}_d$ and $\tilde{\varphi}_x$, which in turn depend solely on the two productivity cutoffs, φ_d^* and φ_x^*, that were determined above (see (14) and (15)). The probability of exporting, $\chi = [1 - G(\varphi_x^*)]/[1 - G(\varphi_d^*)]$, also follows immediately from the two productivity cutoffs, φ_d^* and φ_x^*. Finally, the mass of firms, M_d, equals the ratio of aggregate revenue to average revenue, as shown in (16). Aggregate revenue was solved for above, while average revenue depends solely on the already-determined productivity cutoffs, φ_d^* and φ_x^*, as evident from (17). This completes the characterization of the equilibrium quadruple.

3.6 Trade Liberalization and Intra-industry Reallocation

As shown in the main text of the survey, the opening of the closed economy to trade raises the zero-profit cutoff productivity below which firms exit, φ_d^*. As long as there is selection into export markets, reductions in variable trade costs in the open economy equilibrium also raise the zero-profit cutoff productivity, as can be shown by applying the implicit function theorem to (21).

The rise in the zero-profit cutoff productivity, φ_d^*, induced by trade liberalization leads to within-industry reallocations of resources across firms. There is exit by low-productivity firms with productivities above the old but below the new zero-profit cutoff. Additionally, intermediate-productivity firms that serve only the domestic market experience a contraction in revenue as a result of the rise in the zero-profit cutoff, since $r_d(\varphi) = (\varphi/\varphi_d^*)^{\sigma-1} \sigma f_d$ from the relationship between variety revenues (5).

To determine the impact of trade liberalization on the revenue of exporting firms, note that the total revenue of an exporting firm can be written as:

$$r(\varphi) = \left(1 + n\tau^{1-\sigma}\right) r_d(\varphi).$$

Using the relationship between variety revenues (5), the impact of trade liberalization on the total revenue of an exporting firm with a given productivity φ depends on the direction of the change in $\left(1 + n\tau^{1-\sigma}\right) / (\varphi_d^*)^{\sigma-1}$. As shown in Melitz (2003), reductions in variable trade costs raise $\left(1 + n\tau^{1-\sigma}\right) / (\varphi_d^*)^{\sigma-1}$. Since the autarky equilibrium is obtained as the limiting equilibrium as τ increases to infinity, we have $r_a(\varphi) = \lim_{\tau\to\infty} r_d(\varphi) = \lim_{\tau\to\infty} \left(1 + n\tau^{1-\sigma}\right) r_d(\varphi)$. It follows that $r_a(\varphi) < \left(1 + n\tau^{1-\sigma}\right) r_d(\varphi)$ for any finite value of variable trade costs τ. Therefore both the opening of the closed economy to trade and reductions in variable trade costs in the open economy equilibrium raise the revenue of exporting firms.

The above reallocations of resources across firms raise standard revenue-weighted measures of aggregate industry productivity based on output measured at the factory gate. To demonstrate this, define:

$$\Phi = h^{-1}\left(\frac{1}{R}\int_0^\infty r(\varphi)\, h(\varphi)\, g(\varphi)\, d\varphi\right)$$

as such a measure of aggregate industry productivity where $h(\cdot)$ is any increasing function. Let $\Phi_a = h^{-1}\left((1/R)\int_0^\infty r_a(\varphi)\, h(\varphi)\, g(\varphi)\, d\varphi\right)$ correspond to this measure of aggregate industry productivity in autarky. Then Φ is necessarily greater than Φ_a, for any increasing function $h(\cdot)$, since the distribution $r(\varphi)\, g(\varphi)/R$ first-order stochastically dominates the distribution $r_a(\varphi)\, g(\varphi)/R$ from the analysis above.

3.7 Welfare

Given our choice of numeraire, and with symmetric countries, indirect utility is:

$$\mathbb{V} = \frac{1}{P},$$

where the price index can be expressed as:

$$\begin{aligned} P &= \left[M_d p_d(\tilde{\varphi}_d)^{1-\sigma} + \chi n M_d \tau^{1-\sigma} p_d(\tilde{\varphi}_x)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \\ &= M_t^{\frac{1}{1-\sigma}} p(\tilde{\varphi}_t), \\ &= M_t^{\frac{1}{1-\sigma}} \frac{1}{\rho\tilde{\varphi}_t}, \end{aligned}$$

Therefore indirect utility can be written as:

$$\mathbb{V} = M_t^{\frac{1}{\sigma-1}} \rho\tilde{\varphi}_t. \tag{22}$$

Recall from the relationship between relative revenues of varieties within the same market (5):

$$\left(\frac{\tilde{\varphi}_t}{\varphi_d^*}\right)^{\sigma-1} = \frac{r_d\left(\tilde{\varphi}_t\right)}{r_d\left(\varphi_d^*\right)} = \frac{R/M_t}{\sigma f_d} = \frac{L/M_t}{\sigma f_d}.$$

Using this relationship in indirect utility (22), we obtain the expression for welfare in the main text of the survey:

$$\mathbb{V} = \rho\left(\frac{L}{\sigma f}\right)^{1/(\sigma-1)} \varphi_d^* \tag{23}$$

The closed economy corresponds to the limiting case of the open economy with infinitely high trade costs, where $\varphi_x^* \to \infty$, and it is straightforward to show that the same expression for welfare holds in the closed economy. As established above, in an open economy equilibrium where only some firms export, the zero-profit productivity cutoff is strictly higher than that in the closed economy. Therefore, from welfare (23), there are necessarily welfare gains from trade in the Melitz model.

3.8 Pareto Distribution

While the above analysis was undertaken for any continuous productivity distribution, the case of a Pareto productivity distribution has received particular attention in the literature and will be considered in a later section. In this case:

$$g(\varphi) = k\varphi_{\min}^k \varphi^{-(k+1)}, \qquad G(\varphi) = 1 - \left(\frac{\varphi_{\min}}{\varphi}\right)^k, \tag{24}$$

where $\varphi_{\min} > 0$ is the lower bound of the support of the productivity distribution; lower values of the shape parameter k correspond to greater dispersion in productivity; and we require $k > 1$ for the productivity distribution to have a finite mean.

Average firm revenue in the domestic market with a Pareto productivity distribution can be evaluated as:

$$\begin{aligned} \bar{r}_d &= \sigma f_d \int_{\varphi_d^*}^{\infty} \left(\frac{\varphi}{\varphi_d^*}\right)^{\sigma-1} \frac{g\left(\varphi\right) d\varphi}{1 - G\left(\varphi_d^*\right)}, \\ &= \sigma f_d \left(\varphi_d^*\right)^{1-\sigma} \left(\frac{\varphi_d^*}{\varphi_{\min}}\right)^k \int_{\varphi_d^*}^{\infty} k\varphi_{\min}^k \varphi^{-(k-(\sigma-1)+1)} d\varphi, \\ &= \frac{\sigma k}{k-(\sigma-1)} f_d, \end{aligned} \tag{25}$$

Average revenue in the export market can be determined analogously. For average firm revenue in the domestic and export market to have a finite mean, we require $k > \sigma - 1$.

4 Integrated Equilibrium

No derivations required.

5 Trade and Market Size

The Melitz and Ottaviano (2008) model combines firm heterogeneity with quasi-linear preferences to generate endogenous mark-ups that vary with firm productivity, market size and trade integration.

5.1 Preferences and Endowments

Labor is the sole factor of production and each country i is endowed with L^i workers. The representative consumer's preferences are defined over consumption of a continuum of differentiated varieties, q^c_ω, and consumption of a homogeneous good, q^c_0:

$$U = q^c_0 + \alpha \int_{\omega\in\Omega} q^c_\omega d\omega - \frac{1}{2}\gamma \int_{\omega\in\Omega} \left(q^c_\omega\right)^2 d\omega - \frac{1}{2}\eta \left(\int_{\omega\in\Omega} q^c_\omega d\omega\right)^2 . \tag{26}$$

The representative consumer's budget constraint is:

$$\int_{\omega\in\Omega} p_\omega q^c_\omega d\omega + q^c_0 = w, \tag{27}$$

where we have chosen the homogeneous good as the numeraire and hence $p^c_0 = 1$. Each country's labor endowment is assumed to be sufficiently large that it both consumes and produces the homogeneous good. Using the budget constraint (27) to substitute for consumption of the homogeneous good, q^c_0, in utility (26), the representative consumer's first-order conditions for utility maximization imply the following inverse demand curve for a differentiated variety:

$$p_\omega = \alpha - \gamma q^c_\omega - \eta Q^c, \qquad Q^c = \int_{\omega\in\Omega} q^c_\omega d\omega, \tag{28}$$

where demand for a variety is positive if:

$$p_\omega \leq \alpha - \eta Q^c,$$

which defines a "choke price" above which demand is zero. Total output of differentiated varieties, Q^c, can be expressed as follows:

$$\begin{aligned} Q^c &= \int_{\omega\in\Omega} q^c_\omega d\omega, \\ &= N\left(\frac{\alpha - \eta Q^c}{\gamma}\right) - \int_{\omega\in\Omega} \frac{p_\omega}{\gamma} d\omega, \\ &= \frac{N\left(\alpha - \bar{p}\right)}{\eta N + \gamma}, \end{aligned}$$

where:

$$\bar{p} = \frac{1}{N}\int_{\omega\in\Omega} p_\omega d\omega. \tag{29}$$

Substituting for Q^c in demand (28) yields:

$$
\begin{aligned}
q_\omega^c &= \frac{\alpha}{\gamma} - \frac{p_\omega}{\gamma} - \frac{\eta}{\gamma}\left(\frac{N\alpha - N\bar{p}}{\eta N+\gamma}\right), \\
&= \frac{\alpha}{\eta N+\gamma} - \frac{p_\omega}{\gamma} + \frac{\eta N}{\eta N+\gamma}\frac{\bar{p}}{\gamma},
\end{aligned}
$$

where demand for a variety is positive if:

$$p_\omega \leq \frac{1}{\eta N+\gamma}\left(\alpha\gamma + \eta N\bar{p}\right).$$

Total demand for each differentiated variety across all consumers is:

$$q_\omega = Lq_\omega^c = \frac{\alpha L}{\eta N+\gamma} - \frac{p_\omega L}{\gamma} + \frac{\eta N}{\eta N+\gamma}\frac{\bar{p}L}{\gamma}. \tag{30}$$

5.2 Production and Exporting Decisions

The homogeneous good is produced under conditions of perfect competition and constant returns to scale with a unit labor requirement. Differentiated varieties are produced under conditions of monopolistic competition and constant returns to scale. To enter the differentiated sector, a firm must incur a sunk entry cost of f_e units of labor, after which its unit labor requirement or cost, c, is drawn from a cumulative distribution function $G(c)$ with support on $[0, c_M]$. As firms with the same cost, c, behave symmetrically, we index firms by c alone from now onwards. If a firm decides to export, it faces iceberg variable costs of trade, such that $\tau^j > 1$ units of a variety must be exported to country j in order for one unit to arrive.

As long as the homogeneous good is produced, productivity in this sector pins down the wage in each country as equal to one. As markets are assumed to be segmented and the production technology exhibits constant returns to scale, the supplier of each differentiated variety maximizes independently the profits earned from domestic and export sales.[2] Profits from domestic sales in country i are:

$$\pi_d^i(c) = p_d^i(c)\, q_d^i(c) - cq_d^i(c).$$

The first-order condition for profit maximization implies:

$$q_d^i(c) + p_d^i(c)\frac{\partial q_d^i(c)}{\partial p_d^i(c)} - c\frac{\partial q_d^i(c)}{\partial p_d^i(c)} = 0.$$

Using demand (28) together with this first-order condition yields:

$$q_d^i(c) = \frac{L}{\gamma}\left[p_d^i(c) - c\right]. \tag{31}$$

Thus the cost cutoff in the domestic market is defined by:

$$p_d^i\left(c_d^i\right) = c_d^i.$$

[2]Using the pricing rules derived below, it can be shown that equilibrium variety prices are such that there exist no profitable arbitrage opportunities across markets.

At the cost cutoff, we have:

$$q_d^i\left(c_d^i\right)=\frac{L}{\gamma}\left[p_d^i\left(c_d^i\right)-c_d^i\right]=0.$$

Solving for $q_d^i\left(c_d^i\right)=0$ in demand (30) using $p_d^i\left(c_d^i\right)=c_d^i$ yields the following expression for average prices:

$$\bar{p}^i=\frac{\left(\eta N^i+\gamma\right)c_d^i-\alpha\gamma}{\eta N^i}.$$

Using this expression average prices in demand (30), we obtain:

$$q_d^i\left(c\right)=-\frac{L}{\gamma}p_d^i\left(c\right)+\frac{L}{\gamma}c_d^i. \tag{32}$$

Combining this equation for demand (32) with the firm's first-order condition (31), we have the following solution for the equilibrium pricing rule for the domestic market in the main text of the survey:

$$p_d^i\left(c\right)=\frac{1}{2}\left(c_d^i+c\right). \tag{33}$$

Profits from export sales from country i to country j are:

$$\pi_x^i\left(c\right)=\left[p_x^i\left(c\right)-\tau^j c\right]q_x^i\left(c\right).$$

Following the same line of reasoning as for the domestic market above, yields the equilibrium pricing rule for the export market in the main text of the survey:

$$p_x^i\left(c\right)=\frac{\tau^j}{2}\left(c_x^i+c\right). \tag{34}$$

where c_x^i is the cost cutoff for firms exporting from country i to j; and $c_x^i=c_d^j/\tau^j$.

6 Gravity

Arkolakis et al. (2008) consider a static version of the Melitz model with a single differentiated sector, in which there are many (potentially asymmetric) countries and productivity follows the Pareto distribution (24). There are fixed costs in terms of labor of supplying each market, f_{ij}, which are incurred in the country of consumption j rather than the country of production i. Values of fixed and variable trade costs are assumed to be sufficiently large that there is selection into export markets: $\varphi_{ij}^*>\varphi_{ii}^*$ for all countries $j\neq i$.

6.1 Production and Exporting Decisions

CES preferences imply the following demand for a variety produced in country i and consumed in country j:

$$x_{ij}\left(\varphi\right)=\frac{p_{ij}\left(\varphi\right)^{-\sigma}}{P_j^{1-\sigma}}w_jL_j,$$

where the CES price index is:

$$P_j^{1-\sigma} = \sum_s \int_0^\infty p_{sj}(\varphi)^{1-\sigma} M_{sj}\mu_{sj}(\varphi)\, d\varphi,$$

where $\mu_{sj}(\varphi)$ is the *ex post* distribution of productivity conditional on a variety being produced in country i and consumed in country j.

With CES preferences, constant marginal costs and fixed costs of supplying each market, the firm's profit maximization problem reduces to choosing the price for each market separately to maximize its profits derived from supplying that market:

$$\pi_{ij}(\varphi) = \max_{p_{ij}} \left\{ \frac{p_{ij}^{1-\sigma}}{P_j^{1-\sigma}} w_j L_j - w_j L_j \tau_{ij} \frac{p_{ij}^{-\sigma} w_i}{P_j^{1-\sigma}\varphi} - w_j f_{ij}, 0 \right\}.$$

The first-order condition for profit maximization implies:

$$p_{ij}(\varphi) = \frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}.$$

Given this pricing rule, equilibrium firm revenue from market j for a firm based in country i is:

$$r_{ij}(\varphi) = \left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{ij}\right)^{1-\sigma} \frac{w_j L_j}{P_j^{1-\sigma}}.$$

Equilibrium firm profits from market j for a firm based in country i are:

$$\pi_{ij}(\varphi) = \frac{r_{ij}(\varphi)}{\sigma} - w_j f_{ij},$$

The cutoff productivity for market j for a firm based in country i is therefore defined by:

$$\pi_{ij}\left(\varphi_{ij}^*\right) = \left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi_{ij}^*}\tau_{ij}\right)^{1-\sigma} \frac{1}{\sigma}\frac{w_j L_j}{P_j^{1-\sigma}} - w_j f_{ij} = 0$$

$$\left(\varphi_{ij}^*\right)^{\sigma-1} = \frac{f_{ij}}{\left(\frac{\sigma}{\sigma-1}\tau_{ij} w_i\right)^{1-\sigma} \frac{1}{\sigma}\frac{L_j}{P_j^{1-\sigma}}} \tag{35}$$

6.2 Free Entry

The free entry condition equating the expected value of entry to the sunk entry cost is:

$$v_e = \left[1 - G\left(\varphi_{ii}^*\right)\right]\bar{\pi}_i = w_i f_e,$$

where with a Pareto productivity distribution:

$$\left[1 - G\left(\varphi_{ii}^*\right)\right] = \left(\frac{\varphi_{i\min}}{\varphi_{ii}^*}\right)^k,$$

and where expected profits conditional on successful entry are:

$$\bar{\pi}_i = \sum_s \int_{\varphi_{is}^*}^{\infty} \left(\frac{1 - G(\varphi_{is}^*)}{1 - G(\varphi_{ii}^*)} \right) \frac{r_{is}(\varphi)}{\sigma} \frac{g(\varphi)\, d\varphi}{1 - G(\varphi_{is}^*)} - \sum_s \int_{\varphi_{is}^*}^{\infty} \left(\frac{1 - G(\varphi_{is}^*)}{1 - G(\varphi_{ii}^*)} \right) w_s f_{is} \frac{g(\varphi)\, d\varphi}{1 - G(\varphi_{is}^*)}.$$

Using these results, the free entry condition can be written as:

$$v_e = \sum_s \int_{\varphi_{is}^*}^{\infty} \frac{\left(\frac{\sigma}{\sigma-1} \frac{\tau_{is} w_i}{\varphi} \right)^{1-\sigma}}{P_s^{1-\sigma} \sigma} w_s L_s \left(\frac{\varphi_{ii}^*}{\varphi_{is}^*} \right)^k k \frac{(\varphi_{is}^*)^k}{\varphi^{k+1}} d\varphi$$
$$- \sum_s w_s f_{is} \left(\frac{\varphi_{ii}^*}{\varphi_{is}^*} \right)^k \int_{\varphi_{is}^*}^{\infty} k \frac{(\varphi_{is}^*)^k}{\varphi^{k+1}} d\varphi = \frac{w_i f_e}{\varphi_{i\min}^k / (\varphi_{ii}^*)^k},$$

where simplification has been avoided to clarify the derivation. Evaluating the integrals and using the export cutoff condition (35) yields:

$$v_e = \sum_s w_s f_{is} \frac{k}{k - \sigma + 1} \left(\frac{\varphi_{ii}^*}{\varphi_{is}^*} \right)^k - \sum_s w_s f_{is} \left(\frac{\varphi_{ii}^*}{\varphi_{is}^*} \right)^k = \frac{w_i f_e}{\varphi_{i\min}^k / (\varphi_{ii}^*)^k},$$

which can be re-arranged to obtain the following free entry relationship that will be used below:

$$\sum_s \frac{w_s}{w_i} f_{is} \frac{(\varphi_{ii}^*)^k}{(\varphi_{is}^*)^k} \left(\frac{\sigma - 1}{k - \sigma + 1} \right) = \frac{f_e}{\varphi_{i\min}^k / (\varphi_{ii}^*)^k}, \tag{36}$$

where simplification has been avoided to facilitate later derivations.

6.3 Price Index and Mass of Firms

To solve for the mass of firms, Arkolakis et al. (2008) begin by deriving a relationship from the price index, which will be later combined with labor market clearing and the free entry relationship (36). The price index can be expressed as:

$$P_j^{1-\sigma} = \sum_s M_s \frac{1 - G(\varphi_{sj}^*)}{1 - G(\varphi_{ss}^*)} \int_{\varphi_{sj}^*}^{\infty} p_{sj}(\varphi)^{1-\sigma} \frac{g(\varphi)}{1 - G(\varphi_{sj}^*)} d\varphi$$

$$P_j^{1-\sigma} = \sum_s M_s \left(\frac{\varphi_{ss}^*}{\varphi_{sj}^*} \right)^k \int_{\varphi_{sj}^*}^{\infty} \left(\frac{\sigma}{\sigma - 1} \frac{\tau_{sj} w_s}{\varphi} \right)^{1-\sigma} \frac{k (\varphi_{sj}^*)^k}{\varphi^{k+1}} d\varphi$$

$$P_j^{1-\sigma} = \sum_s M_s \left(\frac{\varphi_{ss}^*}{\varphi_{sj}^*} \right)^k \left(\varphi_{sj}^* \right)^{\sigma-1} \left(\frac{\sigma}{\sigma - 1} \tau_{sj} w_s \right)^{1-\sigma} \frac{k}{k - \sigma + 1} \tag{37}$$

Using the cutoff productivity condition (35), one obtains the following relationship from the price index that will be used below:

$$L_j = \sum_s M_s \left(\frac{\varphi_{ss}^*}{\varphi_{sj}^*} \right)^k f_{sj} \frac{\sigma k}{k - \sigma + 1}, \tag{38}$$

Now use labor market clearing to determine the mass of firms in country i:

$$M_i\left(\sum_s\int_{\varphi_{is}^*}^{\infty}l_{is}^{var}\left(\varphi\right)\frac{1-G\left(\varphi_{is}^*\right)}{1-G\left(\varphi_{ii}^*\right)}\frac{g\left(\varphi\right)d\varphi}{1-G\left(\varphi_{is}^*\right)}\right)+M_i\frac{f_e}{1-G\left(\varphi_{ii}^*\right)}+\sum_s M_s\frac{1-G\left(\varphi_{si}^*\right)}{1-G\left(\varphi_{ss}^*\right)}f_{si}=L_i,$$

where the destination fixed costs f_{si} are incurred in country i by firms that are located in countries s, and where variable labor input is:

$$l_{is}^{var}\left(\varphi\right)=\left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{is}\right)^{-\sigma}\frac{w_sL_s}{P_s^{1-\sigma}}\frac{\tau_{is}}{\varphi}.$$

Using this expression for variable labor input and the Pareto productivity distribution, the labor market clearing condition can be re-written as:

$$M_i\left(\sum_s\int_{\varphi_{is}^*}^{\infty}\left(\frac{\sigma}{\sigma-1}\frac{w_i}{\varphi}\tau_{is}\right)^{-\sigma}\frac{w_sL_s}{P_s^{1-\sigma}}\frac{\tau_{is}}{\varphi}\left(\frac{\varphi_{ii}^*}{\varphi_{is}^*}\right)^k\frac{k\left(\varphi_{is}^*\right)^k}{\varphi^{k+1}}d\varphi\right)$$
$$+M_i\frac{f_e}{\left(\varphi_{i\min}/\varphi_{ii}^*\right)^k}+\sum_s M_s\left(\frac{\varphi_{ss}^*}{\varphi_{si}^*}\right)^k f_{si}=L_i.$$

Evaluating the integral, using the cutoff productivity condition (35) and simplifying, one obtains:

$$M_i\left[\sum_s\left(\sigma-1\right)\frac{w_s}{w_i}f_{is}\left(\frac{\varphi_{ii}^*}{\varphi_{is}^*}\right)^k\frac{k}{k-\sigma+1}\right]+M_i\frac{f_e}{\left(\varphi_{i\min}/\varphi_{ii}^*\right)^k}$$
$$+\sum_s M_s\left(\frac{\varphi_{ss}^*}{\varphi_{si}^*}\right)^k f_{si}=L_i,$$

which can be re-written using the free entry relationship (36) as:

$$M_i\left(k\frac{f_e}{\left(\varphi_{i\min}/\varphi_{ii}^*\right)^k}+\frac{f_e}{\left(\varphi_{i\min}/\varphi_{ii}^*\right)^k}\right)+\sum_s M_s\left(\frac{\varphi_{ss}^*}{\varphi_{si}^*}\right)^k f_{si}=L_i. \tag{39}$$

where again simplification has been avoided to clarify the derivation. Now note that the relationship (38) from the price index implies:

$$\sum_s M_s\left(\frac{\varphi_{ss}^*}{\varphi_{si}^*}\right)^k f_{si}=\left(\frac{k-\sigma+1}{k\sigma}\right)L_i.$$

Using this result in the labor market clearing condition (39) yields:

$$M_i\left(k+1\right)\frac{f_e}{\left(\varphi_{i\min}/\varphi_{ii}^*\right)^k}=L_i-L_i\left(\frac{k-\sigma+1}{k\sigma}\right),$$

which implies the following solution for the mass of firms as a function of the domestic productivity cutoff:

$$M_i=\frac{\left(\sigma-1\right)\left(\varphi_{i\min}/\varphi_{ii}^*\right)^k}{k\sigma f_e}L_i. \tag{40}$$

In contrast to Chaney (2008), the mass of firms (40) is endogenously determined in Arkolakis et al. (2008), as a function of the domestic productivity cutoff, φ_{ii}^*, and the labor endowment, L_i.

6.4 Intensive and Extensive Margins of Exports

The probability of exporting from country i to country j is:

$$\chi_{ij} = \frac{1 - G(\varphi_{ij}^*)}{1 - G(\varphi_{ii}^*)} = \left(\frac{\varphi_{ii}^*}{\varphi_{ij}^*} \right)^k. \tag{41}$$

Average exports conditional on exporting from country i to country j are:

$$\begin{aligned}
\bar{r}_{ij} &= \int_{\varphi_{ij}^*}^{\infty} p_{ij}(\varphi)^{1-\sigma} \frac{w_j L_j}{P_j^{1-\sigma}} \frac{g(\varphi)}{1 - G(\varphi_{ij}^*)} d\varphi, \\
&= \int_{\varphi_{ij}^*}^{\infty} \left(\frac{\sigma}{\sigma - 1} \frac{w_i \tau_{ij}}{\varphi} \right)^{1-\sigma} \frac{w_j L_j}{P_j^{1-\sigma}} k \frac{(\varphi_{ij}^*)^k}{\varphi^{k+1}} d\varphi.
\end{aligned}$$

Using the cutoff productivity condition (35), average exports conditional on exporting from country i to country j can be written as:

$$\bar{r}_{ij} = w_j f_{ij} \frac{\sigma k}{k - \sigma + 1}, \tag{42}$$

which is analogous to equation (25) above.

Therefore, combining the probability of exporting (41) and average exports conditional on exporting (42), one obtains the decomposition for total exports from country i to country j in the main text of the survey:

$$X_{ij} = \underbrace{\left(\frac{\varphi_{ii}^*}{\varphi_{ij}^*} \right)^k M_i}_{\text{extensive}} \underbrace{w_j f_{ij} \frac{\sigma k}{k - \sigma + 1}}_{\text{intensive}} \tag{43}$$

Using this decomposition (43), the share of country j's expenditure on goods produced in country i, ϑ_{ij}, can be written as:

$$\begin{aligned}
\vartheta_{ij} &= \frac{X_{ij}}{\sum_s X_{sj}}, \\
&= \frac{\left(\frac{\varphi_{ii}^*}{\varphi_{ij}^*} \right)^k M_i f_{ij} \frac{\sigma k}{k - \sigma + 1},}{\sum_s \left(\frac{\varphi_{ss}^*}{\varphi_{sj}^*} \right)^k M_s f_{sj} \frac{\sigma k}{k - \sigma + 1}},
\end{aligned}$$

which using the mass of firms (40) becomes:

$$\vartheta_{ij} = \frac{(\varphi_{ij}^*)^{-k} L_i f_{ij} \varphi_{i\min}^k}{\sum_s (\varphi_{sj}^*)^{-k} L_s f_{sj} \varphi_{s\min}^k}.$$

Finally, substituting for φ_{ij}^* and φ_{sj}^* using the productivity cutoff condition (35), and simplifying, Arkolakis et al. (2008) obtain the expression for the trade share in the main text of the survey:

$$\vartheta_{ij} = \frac{L_i \varphi_{i\min}^k (\tau_{ij} w_i)^{-k} f_{ij}^{1-k/(\sigma-1)}}{\sum_s L_s \varphi_{s\min}^k (\tau_{sj} w_s)^{-k} f_{sj}^{1-k/(\sigma-1)}}. \tag{44}$$

6.5 Welfare

Indirect utility in country j is given by:

$$\mathbb{V}_j = \frac{w_j}{P_j},$$

where the CES price index can be written as in (37).

To derive an expression for welfare in terms of the trade share, Arkolakis et al. (2008) begin by solving for the CES price index in terms of wages and parameters. Note that the cutoff productivity condition (35) implies the following relationship:

$$\left(\frac{\sigma}{\sigma-1}\tau_{sj}w_s\right)^{1-\sigma} = \frac{f_{sj}}{(\varphi_{sj}^*)^{\sigma-1}\frac{1}{\sigma}\frac{L_j}{P_j^{1-\sigma}}}$$

Substituting for the left-hand side of the above expression in the CES price index (37), one obtains the following relationship:

$$1 = \sum_s M_s\left(\frac{\varphi_{ss}^*}{\varphi_{sj}}\right)^k \frac{f_{sj}}{\frac{1}{\sigma}L_j}\frac{k}{k-\sigma+1} \tag{45}$$

Now note that the cutoff productivity condition (35) also implies:

$$\varphi_{sj}^* = f_{sj}^{\frac{1}{\sigma-1}}\left(\frac{\sigma}{\sigma-1}\tau_{sj}w_s\right)\left(\frac{1}{\sigma}\right)^{-\frac{1}{\sigma-1}} P_j^{-1}L_j^{-\frac{1}{\sigma-1}}.$$

Substituting for φ_{sj}^* in the relationship (45) derived from the CES price index yields:

$$P_j^{-k} = \sum_s M_s\left(\varphi_{ss}^*\right)^k f_{sj}^{1-\frac{k}{\sigma-1}}\left(\frac{\sigma}{\sigma-1}\tau_{sj}w_s\right)^{-k}\left(\frac{1}{\sigma}\right)^{-\left(1-\frac{k}{\sigma-1}\right)} L_j^{-\left(1-\frac{k}{\sigma-1}\right)}\frac{k}{k-\sigma+1}$$

Using the mass of firms (40) in the above expression, one arrives at:

$$P_j^{-k} = \frac{L_j^{-\left(1-\frac{k}{\sigma-1}\right)}}{f_e}\sigma^{-\frac{k}{\sigma-1}}\left(\frac{\sigma}{\sigma-1}\right)^{-k}\frac{\sigma-1}{k-\sigma+1}\sum_s L_s\varphi_{s\min}^k\left(\tau_{sj}w_s\right)^{-k} f_{sj}^{1-\frac{k}{\sigma-1}}, \tag{46}$$

which determines the price index as a function of wages in each country and parameters.

Now note that the wage can be re-written in terms of a country's trade share with itself from (44):

$$w_j^k = \frac{1}{\vartheta_{jj}}\frac{L_j\varphi_{j\min}^k f_{jj}^{1-k/(\sigma-1)}}{\sum_s L_s\varphi_{s\min}^k\left(\tau_{sj}w_s\right)^{-k} f_{sj}^{1-k/(\sigma-1)}}, \tag{47}$$

where we have used $\tau_{jj} = 1$.

Combining the CES price index (46) and wage (47), Arkolakis et al. (2008) obtain the following expression for welfare in terms of a country's trade share with itself, as reported in the main text of the survey:

$$\mathbb{V}_j = \frac{w_j}{P_j} = \vartheta_{jj}^{-1/k}L_j^{1/(\sigma-1)}\left[\frac{\varphi_{j\min}^k f_{jj}^{1-k/(\sigma-1)}}{f_e\left(\frac{\sigma}{\sigma-1}\right)^k\sigma^{k/(\sigma-1)}}\frac{\sigma-1}{k-\sigma+1}\right]^{1/k}. \tag{48}$$

6.6 Mass of Varieties Available for Consumption

To determine the mass of varieties available for consumption, use the definition of the trade share:

$$\vartheta_{ij} = \frac{X_{ij}}{w_j L_j}.$$

Now use the decomposition of bilateral trade into the intensive and extensive margins (43) to write the trade share as:

$$\vartheta_{ij} = \frac{\left(\frac{\varphi_{ii}^*}{\varphi_{ij}^*}\right)^k M_i \, w_j f_{ij} \frac{\sigma k}{k-\sigma+1}}{w_j L_j} \tag{49}$$

Now note that the mass of firms selling from country i to country j relative to country i's share of the market in country j is:

$$\frac{M_{ij}}{\vartheta_{ij}} = \frac{\left(\frac{\varphi_{ii}^*}{\varphi_{ij}^*}\right)^k M_i}{\vartheta_{ij}} = \frac{\left(\frac{\varphi_{ii}^*}{\varphi_{ij}^*}\right)^k M_i}{\left(\frac{\varphi_{ii}^*}{\varphi_{ij}^*}\right)^k \frac{M_i f_{ij}}{L_j} \frac{\sigma k}{k-\sigma+1}},$$

where the second equality uses (49) and again simplification has been avoided to clarify the derivation. One thus arrives at the following expression for the mass of firms selling from country i to country j:

$$M_{ij} = \vartheta_{ij} \frac{L_j}{f_{ij} \frac{\sigma k}{k-\sigma+1}}$$

Summing across source countries, Arkolakis et al. (2008) obtain the following expression for the mass of varieties available for consumption in country j, as reported in the main text of the survey:

$$\sum_s M_{sj} = \frac{L_j}{f_{jj} \frac{\sigma k}{k-\sigma+1}} + \frac{L_j}{\frac{\sigma k}{k-\sigma+1}} \sum_{s \neq j} \vartheta_{sj} \left(\frac{1}{f_{sj}} - \frac{1}{f_{jj}} \right),$$

where $\vartheta_{jj} = 1 - \sum_{s \neq j} \vartheta_{sj}$ has been used.

7 Quantitative Analysis

No derivations required.

8 Labor Markets

When wages vary with revenue across firms, within-industry reallocations across firms provide a new channel through which trade can affect income distribution. As shown in Helpman et al. (2010), the opening of the closed economy to trade raises wage inequality within industries when the following three conditions are satisfied: (a) wages and employment are power functions of productivity, (b) only some firms export and exporting raises the wage paid by a firm with a given productivity, and (c) productivity is Pareto distributed. Under these conditions, the wage and employment of firms can be expressed in terms of their productivity, φ, a term capturing whether or not a firm exports, $\Upsilon(\varphi)$, the zero-profit cutoff productivity, φ_d^*, and parameters:

$$l(\varphi) = \Upsilon(\varphi)^{\psi_l} l_d \left(\frac{\varphi}{\varphi_d^*}\right)^{\zeta_l}, \tag{50}$$

$$w(\varphi) = \Upsilon(\varphi)^{\psi_w} w_d \left(\frac{\varphi}{\varphi_d^*}\right)^{\zeta_w}, \tag{51}$$

where l_d and w_d are the employment and wage of the least productive firm, respectively, and:

$$\Upsilon(\varphi) = \begin{cases} \Upsilon_x > 1 & \text{for } \varphi \geq \varphi_x^* \\ 1 & \text{for } \varphi < \varphi_x^* \end{cases},$$

where $\Upsilon_x^{\psi_w}$ and $\Upsilon_x^{\psi_l}$ are, respectively, the wage and employment premia from exporting for a firm of a given productivity.

Using the Pareto productivity distribution, Helpman et al. (2010) show that the distribution of wages across workers within the industry, $G_w(w)$, can be evaluated as:

$$G_w(w) = \begin{cases} S_{l,d} G_{w,d}(w) & \text{for } \; w_d \leq w \leq w_d \left(\varphi_x^*/\varphi_d^*\right)^{\zeta_w}, \\ S_{l,d} & \text{for } \; w_d \left(\varphi_x^*/\varphi_d^*\right)^{\zeta_w} \leq w \leq w_x, \\ S_{l,d} + (1 - S_{l,d}) G_{w,x}(w) & \text{for } \; w \geq w_x, \end{cases} \tag{52}$$

where $w_d \left(\varphi_x^*/\varphi_d^*\right)^{\zeta_w}$ is the highest wage paid by a domestic firm; $w_x = w_d \Upsilon_x^{\psi_w} \left(\varphi_x^*/\varphi_d^*\right)^{\zeta_w}$ is the lowest wage paid by an exporting firm; $\Upsilon_x^{\psi_w}$ is the increase in wages at the productivity threshold for entry into export markets; $S_{l,d}$ is the employment share of domestic firms.

Given that wages (50) and employment (51) are power functions of productivity and productivity has a Pareto distribution (24), the employment share of domestic firms can be evaluated as:

$$\begin{aligned} S_{l,d} &= 1 - \frac{\int_{\varphi_x^*}^{\infty} \Upsilon_x^{\psi_l} l_d \left(\frac{\varphi}{\varphi_d^*}\right)^{\zeta_l} k \left(\varphi_d^*\right)^k \varphi^{-(k+1)} d\varphi}{\int_{\varphi_d^*}^{\varphi_x^*} l_d \left(\frac{\varphi}{\varphi_d^*}\right)^{\zeta_l} k \left(\varphi_d^*\right)^k \varphi^{-(k+1)} d\varphi + \int_{\varphi_x^*}^{\infty} \Upsilon_x^{\psi_l} l_d \left(\frac{\varphi}{\varphi_d^*}\right)^{\zeta_l} k \left(\varphi_d^*\right)^k \varphi^{-(k+1)} d\varphi}, \\ &= 1 - \frac{\Upsilon_x^{\psi_l} \left(\frac{\varphi_d^*}{\varphi_x^*}\right)^{k-\zeta_l}}{\left[1 - \left(\frac{\varphi_d^*}{\varphi_x^*}\right)^{k-\zeta_l}\right] + \Upsilon_x^{\psi_l} \left(\frac{\varphi_d^*}{\varphi_x^*}\right)^{k-\zeta_l}}, \\ &= \frac{1 - \left(\frac{\varphi_d^*}{\varphi_x^*}\right)^{k-\zeta_l}}{1 + \left[\Upsilon_x^{\psi_l} - 1\right] \left(\frac{\varphi_d^*}{\varphi_x^*}\right)^{k-\zeta_l}}. \end{aligned} \tag{53}$$

To characterize the distribution of wages across workers employed by exporters, note that the share of workers employed by exporters who are employed by an exporter with productivity less than φ is:

$$\begin{aligned} Z_x(\varphi) &= 1 - \frac{\int_{\varphi}^{\infty} \Upsilon_x^{\psi_l} l_d \left(\frac{\xi}{\varphi_d^*}\right)^{\zeta_l} k \left(\varphi_x^*\right)^k \xi^{-(k+1)} d\xi}{\int_{\varphi_x}^{\infty} \Upsilon_x^{\psi_l} l_d \left(\frac{\xi}{\varphi_d^*}\right)^{\zeta_l} k \left(\varphi_x^*\right)^k \xi^{-(k+1)} d\xi}, \\ &= 1 - \left(\frac{\varphi_x^*}{\varphi}\right)^{k-\zeta_l}. \end{aligned}$$

Now note that relative firm productivities and relative firm wages in (51) are related as follows:

$$\frac{\varphi_x^*}{\varphi} = \left(\frac{w_x}{w}\right)^{\frac{1}{\zeta_w}}.$$

Therefore the share of workers employed by exporters who are employed by an exporter with a wage less than w – i.e. the cumulative distribution function of wages for workers employed by exporters – is:

$$G_{w,x}(w) = 1 - \left(\frac{w_x}{w}\right)^{\zeta_g}, \qquad \zeta_g \equiv \frac{k - \zeta_l}{\zeta_w}, \qquad \text{for } w \geq w_x, \tag{54}$$

The distribution of wages across workers employed by domestic firms can be determined by a similar line of reasoning and follows a truncated Pareto distribution with the same shape parameter as the distribution of wages across workers employed by exporters:

$$G_{w,d}(w) = \frac{1 - \left(\frac{w_d}{w}\right)^{\zeta_g}}{1 - \left(\frac{w_d}{w_x}\right)^{\zeta_g}}, \qquad \zeta_g \equiv \frac{k - \zeta_l}{\zeta_w}, \qquad \text{for } w_d \leq w \leq w_d \left(\varphi_x^*/\varphi_d^*\right)^{\zeta_w}. \tag{55}$$

When the three conditions discussed above are satisfied, and hence the wage distribution within industries is characterized by (52), (53), (54) and (55), Helpman et al. (2010) prove the following results:

Proposition 1 *(i) Sectoral wage inequality in the open economy when some but not all firms export is strictly greater than in the closed economy; and (ii) Sectoral wage inequality in the open economy when all firms export is the same as in the closed economy.*

Proof. See Helpman et al. (2010). ■

Corollary 1 ***(to Proposition 1)*** *An increase in the fraction of exporting firms raises sectoral wage inequality when the fraction of exporting firms is sufficiently small and reduces sectoral wage inequality when the fraction of exporting firms is sufficiently large.*

Proof. See Helpman et al. (2010). ■

Both results hold for any measure of inequality that respects second-order stochastic dominance, including all standard measures of inequality, such as the Theil Index and the Gini Coefficient.

9 Endogenous Firm Productivity

9.1 Multi-product Firms

As shown in Bernard et al. (2006), the Melitz model in Section 3 can be extended to allow heterogeneous firms to optimally choose the range of products to produce and export.

9.1.1 Consumer Preferences

The world consists of a continuum of symmetric countries, such that each country is of measure zero and trades with a measure n of foreign countries. Suppose that the representative consumer derives utility from the consumption of a continuum of symmetric products h defined on the interval $[0, 1]$. There is a constant elasticity of substitution across products so that the utility function takes the following form:

$$U = \left[\int_0^1 C_h^{\nu} dh\right]^{\frac{1}{\nu}}, \qquad 0 < \nu < 1. \tag{56}$$

Within each product market, a continuum of firms produce differentiated varieties of the product, so that C_h is a consumption index which also takes the constant elasticity of substitution form:

$$C_h = \left[\int_{\omega \in \Omega_h} c_h(\omega)^{\rho} d\omega\right]^{\frac{1}{\rho}}, \qquad 0 < \rho < 1. \tag{57}$$

where ω indexes varieties and Ω_h is the (endogenous) set of varieties.

9.1.2 Production Technology

There is a competitive fringe of potential firms who are identical prior to entry. In order to enter, firms must incur a sunk entry cost of $f_e > 0$ units of labor. Incurring the sunk entry cost creates a firm brand and a blueprint for one horizontally differentiated variety of each product that can be supplied using this brand. After incurring the sunk entry cost, a firm observes two components of productivity: "ability" $\varphi \in (0, \infty)$ that is common to all products and drawn from a distribution $g(\varphi)$, and "expertise" $\lambda_h \in (0, \infty)$ that is specific to each product and drawn from a distribution $z(\lambda)$. The two distributions $g(\varphi)$ and $z(\lambda)$ are independent of one another and common to all firms, and the product expertise distribution is independently and identically distributed across products.

Once the sunk entry cost has been incurred, and productivity and consumer tastes are observed, firms decide whether to enter and what products to produce and export. The market structure is monopolistic competition and labor is the sole factor of production. The firm faces fixed production and exporting costs for each product, f_d and f_x, in the domestic and export market. Similarly, there is a fixed cost to becoming a firm and to becoming an exporter, F_d and F_x, which are independent of the number of products supplied. In addition to these fixed costs, there is also a constant marginal cost of production for each firm and product that depends on productivity, such that $q_{jh}(\varphi, \lambda_h) / (\varphi\lambda_h)$ units of labor are required to produce $q_{jh}(\varphi, \lambda_h)$ units of output of product h for market $j \in \{d, x\}$. As products are symmetric, the implicit dependence on h is suppressed from now onwards, except where otherwise indicated.

9.1.3 Production and Export Decisions

Demand for a product variety depends upon the own variety price, the price index for the product and the price indices for all other products. If a firm is active in a product market, it manufactures one of a continuum of varieties and so is unable to influence the price indices. As a result, the firm's profit maximization problem reduces to choosing the price of each product variety separately to maximize the profits derived from that product variety. This optimization problem yields the standard result that the equilibrium price of a product variety is a constant mark-up over marginal cost, with prices in the export market a constant multiple of those in the domestic market due to the variable costs of trade:

$$p_x(\varphi, \lambda) = \tau p_d(\varphi, \lambda) = \frac{\tau}{\rho} \frac{w}{\varphi \lambda}, \tag{58}$$

where we choose the wage in one country as the numeraire and use country symmetry. Together these imply $w = 1$ for all countries.

Substituting for the pricing rule, equilibrium firm revenue and profits from a product variety in the domestic and export markets are:

$$r_d(\varphi, \lambda) = R(\rho P \varphi \lambda)^{\sigma-1}, \qquad r_x(\varphi, \lambda) = \tau^{1-\sigma} R(\rho P \varphi \lambda)^{\sigma-1}, \tag{59}$$

where R denotes aggregate expenditure on a product and P is the price index dual to the product consumption index (57).[3] Therefore the relative revenues of two varieties of the same product within the same market depend solely on their relative productivities, and hence on their relative values of ability and expertise:

$$\frac{r_d(\varphi'', \lambda'')}{r_d(\varphi', \lambda')} = \frac{r_x(\varphi'', \lambda'')}{r_x(\varphi', \lambda')} = \left(\frac{\varphi''}{\varphi'}\right)^{\sigma-1} \left(\frac{\lambda''}{\lambda'}\right)^{\sigma-1}. \tag{60}$$

Consumer love of variety and a fixed production cost for each product imply that no firm would ever export a product without also serving the domestic market. As a result, following a similar convention to that adopted in Section 3, we can apportion the entire product fixed production cost to the domestic market and the entire fixed exporting cost to the export market:

$$\pi_d(\varphi, \lambda) = \frac{r_d(\varphi, \lambda)}{\sigma} - f_d, \qquad \pi_x(\varphi, \lambda) = \frac{r_x(\varphi, \lambda)}{\sigma} - f_x \tag{61}$$

A firm with a given ability φ and expertise λ decides whether or not to supply a product to the domestic market based on a comparison of variable profits and fixed production costs for the product. For each firm ability φ, there is a **zero-profit cutoff for expertise,** $\lambda_d^*(\varphi)$, such that a firm supplies the product domestically if it draws a value of λ equal to or greater than $\lambda_d^*(\varphi)$. This value of $\lambda_d^*(\varphi)$ is defined by the following zero-profit cutoff condition:

$$r_d(\varphi, \lambda_d^*(\varphi)) = \sigma f_d = r_d(\varphi_d^*, \lambda_d^*(\varphi_d^*)). \tag{62}$$

Using this product zero-profit cutoff condition for each firm ability together with relative variety revenues (60), $\lambda_d^*(\varphi)$ can be expressed relative to its value for the lowest-ability supplier to the domestic market, $\lambda_d^*(\varphi_d^*)$:

$$\lambda_d^*(\varphi) = \left(\frac{\varphi_d^*}{\varphi}\right) \lambda_d^*(\varphi_d^*), \tag{63}$$

As there is a unit continuum of symmetric products and expertise is independently and identically distributed, the law of large numbers implies that the fraction of products supplied to the domestic market by a firm with

[3] In the specification of consumer preferences, firm varieties were assumed to enter utility symmetrically, but it is straightforward to allow asymmetric weights for firm varieties to capture differences in demand. With CES preferences and monopolistic competition, these demand parameters would enter firm revenue in exactly the same way as firm productivity. Therefore an alternative and equivalent interpretation of λ is as a firm demand parameter that varies across products, and could also in principle also vary across countries, as in Bernard, Redding and Schott (2009).

a given ability φ equals the probability of that firm drawing an expertise above $\lambda_d^*(\varphi)$, that is $[1 - Z(\lambda_d^*(\varphi))]$. From the relationship between the zero-profit expertise cutoffs (63), higher ability firms have lower expertise cutoffs, and hence supply a wider range of products to the domestic market.

Following a similar line of reasoning, a firm with a given ability φ and expertise λ decides whether or not to export the product based on a comparison of variable profits and fixed exporting costs for the product. For each firm ability φ, there is an **exporting cutoff for expertise,** $\lambda_x^*(\varphi)$, such that the firm exports the product if it draws a value of λ equal to or greater than $\lambda_x^*(\varphi)$. This value of $\lambda_x^*(\varphi)$ is defined by the following exporting cutoff condition:

$$r_x(\varphi, \lambda_x^*(\varphi)) = \sigma f_x = r_x(\varphi_x^*, \lambda_x^*(\varphi_x^*)) . \tag{64}$$

Using this product exporting cutoff condition for each firm ability together with relative variety revenues (60), $\lambda_x^*(\varphi)$, can be expressed relative to its value for the lowest-ability exporter, $\lambda_x^*(\varphi_x^*)$:

$$\lambda_x^*(\varphi) = \left(\frac{\varphi_x^*}{\varphi}\right) \lambda_x^*(\varphi_x^*) , \tag{65}$$

As there is a unit continuum of symmetric products and expertise is independently and identically distributed, the law of large numbers implies that the fraction of products exported by a firm with a given level of ability φ equals the probability of that firm drawing an expertise above $\lambda_x^*(\varphi)$, that is $[1 - Z(\lambda_x^*(\varphi))]$. From the relationship between the exporting expertise cutoffs (65), higher ability firms have lower expertise cutoffs, and hence export a wider range of products.

9.1.4 Steady-state Industry Equilibrium

The determination of steady-state industry equilibrium is directly analogous to that in Section 3 above. As shown in Bernard et al. (2006), the open economy equilibrium is referenced by a sextuple $\{\varphi_d^*, \varphi_x^*, \lambda_d^*(\varphi_d^*), \lambda_x^*(\varphi_x^*), P, R\}$, in terms of which all other endogenous variables can be written.

As in Section 3, the opening of the closed economy to trade leads to exit by low productivity firms, as the zero-profit ability cutoff below which firms exit, φ_d^*, rises. Unlike Section 3, the opening of the closed economy to trade also leads surviving firms to drop low expertise products and focus on their core competencies in higher expertise products, since the rise in φ_d^* raises $\lambda_d^*(\varphi)$ for firms of all abilities in (63). As a result, trade liberalization not only leads to reallocations of resources across firms, which raise weighted-average industry productivity, but also leads to reallocations of resources within firms, which raise weighted-average firm productivity. Therefore, reallocation may be even more important than hitherto thought, in so far as it occurs across products within firms as well as across firms.

While the discussion here interprets expertise, λ, as a component of productivity that is specific to a firm and product, it can also be interpreted as a component of demand that is specific to a firm, product and country. As shown in Bernard, Redding and Schott (2009), the resulting framework provides a natural explanation for a number of key features of the distribution of exports across firms, products and countries. While higher trade costs reduce exports of a given firm and product, they also lead to changes in export composition towards higher export value firms and products. Consistent with this, distance is negatively and statistically significantly related to exports of a given firm and product, the number of exporters and the number of exported products, but has a positive and statistically insignificant effect on average exports per firm and product. More generally, models of multi-product firms expand the number extensive margins of trade to include firms, products and destinations.

As shown in Bernard, Jensen, Redding and Schott (2009), these extensive margins together account for most of the cross-country variation in bilateral trade.

9.2 Technology and Skills

No derivations required.

9.3 International Production Networks

No derivations required.

10 Firm and Aggregate Dynamics

No derivations required.

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