Physics 203, Fall 2003—Final Exam, January 17, 2004

Instructions:

1. Work the problems in the booklets provided with the exam. If you use more than one booklet, be sure to put your name on each one. Mark the booklets, "1 of 3","2 of 3", *etc.*

2. There are five problems, all of the same weight.

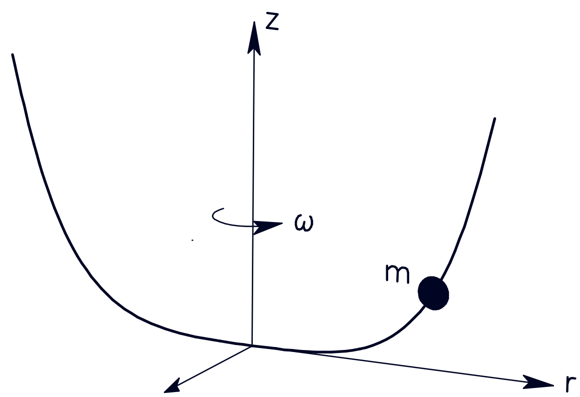
3. It is possible to do some parts of the problems without doing others. Partial credit will be given provided your work is shown and can be understood.

4. You may wish to refer to the attached list of useful formulae.

5. You will have three hours from the start of the exam.

6. Be sure to write and sign the pledge in the exam booklet: "I pledge my honor that I have not violated the Honor Code during this examination."

Problem 1



A bead of mass m slides without fiction on a wire whose shape is

$$z(r) = a(\frac{r}{a})^4$$

The wire rotates about the z axis with constant velocity ω. Earth's gravity causes acceleration g along the $-z$ axis.

1. By using cylindrical coordinates, write the Lagrangian of the system.

   $L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{z}^2 + \frac{1}{2}mr^2\omega^2 - mgz$

2. Find the equation of motion for the bead in terms of the coordinate r.

   $\dot{z} = 4\left(r/a\right)^3\dot{r};\quad L = \frac{1}{2}m\dot{r}^2 + 8m(r/a)^6\dot{r}^2 + \frac{1}{2}mr^2\omega^2 - mga(r/a)^4$

   $\frac{\partial L}{dr} = 48mr^5\dot{r}^2/a^6 + mr\omega^2 - 4mg(r/a)^3$

   $\frac{\partial L}{d\dot{r}} = m\dot{r} + 16m(r/a)^6\dot{r}$

   $\frac{d}{dt}\frac{\partial L}{d\dot{r}} = m\ddot{r} + 16m(r/a)^6\ddot{r} + 96mr^5\dot{r}^2/a^6$

   $\ddot{r} + 16(r/a)^6\ddot{r} + 48r^5\dot{r}^2/a^6 - r\omega^2 + 4g(r/a)^3 = 0$

3. Find the radius of the circular orbit. Check whether this orbit is stable.

   For circular orbit $\dot{r} = 0, \ddot{r} = 0;\quad -r_0\omega^2 + 4g(r/a)^3 = 0;\quad r_0 = \frac{\omega}{2}(a^3/g)^{1/2}$

4. If it is stable, find the frequency of small oscillations about this orbit.

   Take $r = r_0 + A\sin\Omega t$

   $-[1+16((r_0+A\sin\Omega t)/a)^6]A\Omega^2\sin\Omega t + 48(r_0+A\sin\Omega t)^5A^2\Omega^2\cos^2\Omega t/a^6 - (r_0 + A\sin\Omega t)\omega^2 + 4g[(r_0+A\sin\Omega t)/a]^3 = 0$

We need to get a solution to first order in A.

$-[1 + 16(r_0/a)^6]A\Omega^2 \sin \Omega t - A \sin \Omega t \omega^2 + 12gr_0^2 A \sin \Omega t/a^3 = 0$

$-[1 + 16(r_0/a)^6]\Omega^2 - 4gr_0^2/a^3 + 12gr_0^2/a^3 = 0$

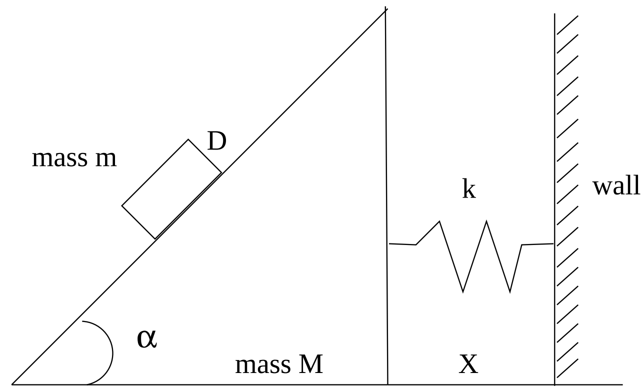
$\Omega^2 = 8gr_0^2/a^3[1 + 16(r_0/a)^6]$

Since Ω is real, the orbit is stable

5. Is the angular momentum conserved in this system?

   No, the wire applies a torgue to the particle forcing it to rotate at a contant angular velocity.

Problem 2



A block of mass m slides with no friction down a ramp of mass M and height L. The ramp is attached to the wall by a spring with spring constant k.

1. Write the Lagrangian of the system in terms of X, the distance of the ramp from the wall, and of D, the distance of the object from the top of the ramp.

   $L = T - U = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X} + \dot{D}\cos\alpha)^2 + \frac{1}{2}m\dot{D}^2\sin^2\alpha - \frac{1}{2}kX^2 + mgD\sin\alpha$

2. Write the coupled equations of motion for these generalized coordinates.

   $\frac{\partial L}{d\dot{X}} = (M+m)\dot{X} + m\dot{D}\cos\alpha$

   $\frac{\partial L}{dX} = -kX$

   $\frac{\partial L}{d\dot{D}} = m\dot{D} + m\dot{X}\cos\alpha$

   $\frac{\partial L}{dD} = mg\sin\alpha$

   $(M+m)\ddot{X} + m\ddot{D}\cos\alpha + kX = 0$

   $m\ddot{D} + m\ddot{X}\cos\alpha - mg\sin\alpha = 0$

3. Suppose that α is small, indeed take it to zero for the purpose of this calculation. For $\alpha = 0$, find the eigenfrequencies and the normal modes.

   [This exercise should make it plausible that, under appropriate conditions and at least for small α, the block can we made to slide up the ramp.]

   $(M+m)\ddot{X} + m\ddot{D} + kX = 0$

   $m\ddot{D} + m\ddot{X} = 0$

   $Take\ X = X_0e^{i\omega t},\ D = D_0e^{i\omega t}$

$$\begin{vmatrix} -(M+m)\omega^2 + k & -m\omega^2 \\ -m\omega^2 & -m\omega^2 \end{vmatrix} = 0$$

$$-m\omega^2[-(M+m)\omega^2 + k] - m^2\omega^4 = 0$$

Solutions $\omega = 0$ *and* $-(M+m)\omega^2 + k + m\omega^2 = -M\omega^2 + k = 0, \omega^2 = k/M$

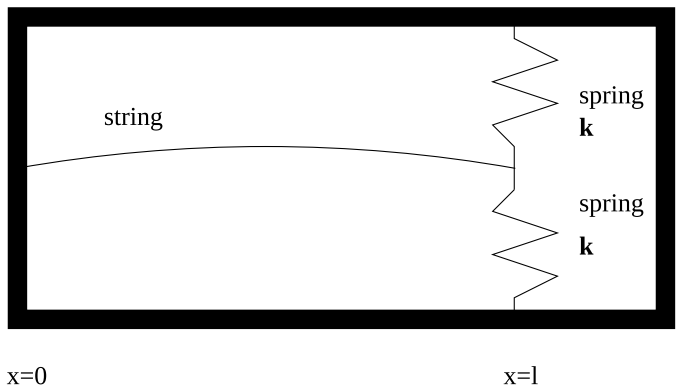
Eigenvectors for $\omega = 0$*:* $X = 0, D = D_0 + D_1 t$

Eigenvectors for $\omega^2 = k/M$*:* $X = X_0, D = -X_0$

As expected for horizontal slope, the first mode corresponds to a stationary mass M *and mass* m *moving with uniform velocity, the second mode corresponds to oscillating mass* M *and stationary mass* m*.*

Problem 3

A string of length l, linear mass density ρ and tension τ is fixed at one end $x = 0$. At $x = l$ there are two massless vertical springs, one above the other, as in the picture. The springs both have spring constant $\mathbf{k}$.



1. Write the one dimensional wave equation and show that

$$q(x,t) = [A \sin kx + B \cos kx]e^{i\omega t}$$

   is the general solution of this equation.

   (Obviously, k here is the wave number, not to be confused with the the spring constant $\mathbf{k}$.)

   Wave equation: $\frac{\partial^2 q}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 q}{\partial t^2} = 0$

   $-k^2[A \sin kx + B \cos kx]e^{i\omega t} + \frac{\omega^2}{v^2}[A \sin kx + B \cos kx]e^{i\omega t} = 0;\ \ v = \omega/k = (\tau/\rho)^{1/2}$

2. Find the solution (*i.e.* determine B and k) when $\mathbf{k} = \infty$, *i.e.* the string is fixed at $x = l$.

   The string is fixed at both ends: $B = 0,\ \ \sin kl = 0;\ k = n\pi/l,\ n = 1, 2, 3...$

3. Find the solution when $\mathbf{k} = 0$, *i.e.* when the string is free at $x = l$, with $\frac{dq}{dx}|_{x=l} = 0$.

   Now $\frac{dq}{dx}|_{x=l} = A\cos(kl) = 0,\ \ k = n\pi/2l,\ \ n = 1, 3, 5$

4. Write the boundary conditions for generic a $\mathbf{k}$, *i.e.* $0 < \mathbf{k} < \infty$.

If the string is displaced by distance q*,the force from the springs is equal to* $F = -2q\mathbf{k}$*. This force is compensated by the tension* $F = \tau \sin\theta$*, where* θ *is the slope of the string at* $x = l$ *and* $\frac{dq}{dx}|_{x=l} = \tan\theta$*. If the oscillations are small,* $\sin\theta = \tan\theta$*.*

$\frac{dq}{dx}|_{x=l} = -2q\mathbf{k}/\tau$

5. Find the condition on k for the solution to satisfy the boundary condition in (4).

   $A\cos(kl) = -2A\sin(kl)\mathbf{k}/\tau$

   $\tan(kl) = -\tau/2\mathbf{k}$

Problem 4

The three principal axes of a tennis racket are (1) along the handle, (2) perpendicular to the handle in the plane of the string and (3) perpendicular to the handle and strings. The moments of inertia are in the following relation

$$I_1 < I_2 < I_3$$

When a tennis racket is tossed in the air with a spin in the direction of either axis (1) or (3), the racket continues to spin uniformly about the initial axis and can be easily recaught. However, if the initial spin is around axis (2), the motion rapidly becomes irregular, and it is hard to catch the racket. Explain this behaviour by following the steps below:

1. Write Euler's equation for the torque-free motion of the racket, starting from the equation $\dot{\mathbf{L}} + \omega \times \mathbf{L} = 0$, where $\mathbf{L}$ is the angular momentum in the principal-axes coordinate system.
2. Show that the motion is stable around axis (1) and (3).
3. Show that the motion is unstable around axis (2).

   See section 11.12 of textbook for derivation

Problem 5

A rocket is fired from the ground toward East with initial velocity v_o, at an angle α above the horizontal and at a latitude λ.

Assume that the height of the rocket trajectory is much smaller than the radius of the Earth.

1. Ignoring earth's rotation, how long will the rocket be in the air, and how far does it land?

   Usual kinematics with $\hat{x}$ in the East direction and $\hat{y}$ in the local vertical direction

   $v_x = v_0 \cos\alpha$; $\;x = v_0 t \cos\alpha$

   $v_y = v_0 \sin\alpha - gt$; $\;y = v_0 t \sin\alpha - gt^2/2$

   The duration of the flight $T = 2v_0 \sin\alpha/g$ and the range $R = 2v_0^2 \sin\alpha \cos\alpha/g$

2. Now, taking into account Coriolis' force, answer the same questions as above.

   The Coriolis force is $F_c = -2m\omega \times \mathbf{v}$. The rotation is in the north direction. At latitude $\lambda, \omega_y = \omega \sin\lambda, \omega_z = -\omega\cos\lambda$

   We get $F_{cx} = 2mv_y\omega\cos\lambda, F_{cy} = -2mv_x\omega\cos\lambda, F_{cz} = -2mv_x\omega\sin\lambda$

   Modified equationsof motion:

   $\ddot{y} = -g - 2v_x\omega\cos\lambda = -g - 2v_0\omega\cos\alpha\cos\lambda$

   $y = v_0 t \sin\alpha - (g + 2v_0\omega\cos\alpha\cos\lambda)t^2/2$

   The total time in the air is $T = 2v_0 \sin\theta/(g + 2v_0\omega\cos\alpha\cos\lambda)$

   $\ddot{x} = 2v_y\omega\cos\lambda = 2(v_0\sin\alpha - gt)\omega\cos\lambda$

   $\dot{x} = v_0\cos\alpha + 2(v_0 t\sin\alpha - gt^2/2)\omega\cos\lambda$

   $R = v_0 T\cos\alpha + (v_0 T^2 \sin\alpha - gT^3/3)\omega\cos\lambda$

3. In which direction and by how much is the rocket deviated from the east direction?

   $\ddot{z} = -2v_x\omega\sin\lambda = -2v_0\omega\cos\alpha\sin\lambda$

   $z = -v_0\omega\cos\alpha\sin\lambda T^2$,deviated in the north direction.

Potentially Useful Relations

$$\tau = \dot{\mathbf{L}} + \omega \times \mathbf{L} \qquad \mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$$

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2 \qquad ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2$$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos n\omega t \cos(m\omega t) dt = \frac{\tau}{2}\delta_{n,m} \qquad \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t)\sin(m\omega) t dt = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \qquad \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = H \qquad L = T - V \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad \dot{q}_k = \frac{\partial H}{\partial p_k}$$

$$\Delta\phi = \int_{r_{\min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{2m}{\ell^2}(E - V) - \frac{1}{r^2}}} \qquad E_{\text{total}} = \frac{1}{2}m\dot{r}^2 + V(r) + \frac{\ell^2}{2mr^2}$$

$$x_p(t) = \frac{A\cos(\omega t - \delta)}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\omega^2\beta^2}}$$

$$F(t) = \frac{1}{2}a_o + \sum_n (a_n \cos n\omega t + b_n \sin n\omega t) \qquad \omega = \frac{2\pi}{\tau}$$

$$a_n = \frac{2}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} F(t)\cos(n\omega t) dt \qquad b_n = \frac{2}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} F(t)\sin(n\omega t) dt$$

$$r(\theta) = \frac{\alpha}{(1 + \epsilon\cos\theta)} \qquad \alpha \equiv \frac{\ell^2}{mk} \qquad \epsilon \equiv \sqrt{1 + \frac{2E\ell^2}{mk^2}}$$

$$F = -\frac{dV}{dr} \qquad V_{eff} = V(r) + \frac{\ell^2}{2mr^2}$$

$$\sin x = x - \frac{x^3}{3!} + ... \qquad \cos x = 1 - \frac{x^2}{2} + ... \qquad (1 + x)^n = 1 + nx + ...$$

$$r_{min} = a(1 - \epsilon) \qquad r_{max} = a(1 + \epsilon) \qquad E = -\frac{k}{2a} \qquad \frac{\tau^2}{a^3} = \frac{4\pi^2\mu}{k} \qquad k = GmM$$

$$\frac{d^2}{d\theta^2}(\frac{1}{r}) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$$\tau = r \times F; \qquad L = r \times p; \qquad \tau = \frac{dL}{dt}$$

$$F_{fict} = -2m\omega \times v - m\omega \times (\omega \times r)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = A\cos(\omega t) \qquad \delta = \tan^{-1}\frac{2\omega\beta}{\omega_o^2 - \omega^2}$$