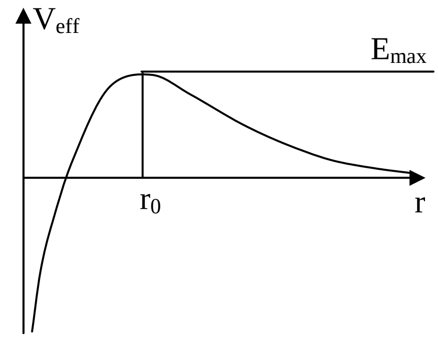
PHYS 203 **Princeton University** **Fall 2006**

Final Exam

1. **Death Star** (20 pt) A particle of mass *m* is subject to a central force given by the potential

$$V = -\frac{k}{r^3} \qquad (k > 0)$$

a) What is the effective potential? Make a sketch.



$$V_{eff} = -\frac{k}{r^3} + \frac{l^2}{2mr^2}$$

b) Determine the radius of the circular orbit for a particle with angular momentum *l*. Is the orbit stable?

$$\frac{\partial V_{eff}}{\partial t} = \frac{3k}{r^4} - \frac{l^2}{mr^3} = 0$$

$$r_0 = \frac{3km}{l^2}$$

It is clear from the figure that the effective potential has a negative curvature and the orbit is unstable

c) If the particle is approaching from infinity with energy *E*, what is the total cross-section for the particle to fall into the force center?

If the particle gets over the peak of the potential it will spiral into the star. So we need to find the maximum energy it can have before reaching the top, which is equal to

$$E_{\max} = V_{eff}(r_0) = -\frac{kl^6}{(3km)^3} + \frac{l^6}{2m(3km)^2} = -\frac{l^6}{27k^2m^3} + \frac{l^6}{18k^2m^3} = \frac{1}{54}\frac{l^6}{k^2m^3}$$

The angular momentum is related to the impact parameter and the velocity at infinity $l = mbV_\infty$. *Now we can find b,*

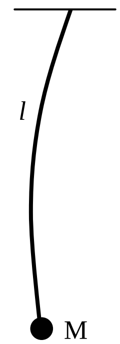
$$E = \frac{1}{54}\frac{m^3b^6V_\infty^6}{k^2} = \frac{4b^6E^3}{27k^2}; \qquad b = \left(\frac{27k^2}{4E^2}\right)^{1/6}$$

The total cross-section is the area that would be swept out by the star if a uniform stream of particles was impinging on it:

$$\sigma_{tot} = \pi b^2 = 3\pi\left(\frac{k}{2E}\right)^{2/3}$$

Now check the dimensions, which work out because k has units (J m^3*)*

2. **A string with mass** (20 pt) A ball of mass M is suspended from a string of length l which has a finite mass m. You can assume that $M>>m$, but their ratio is finite. Consider small oscillations on the string.



a) What is the general solution $y(x,t)$ for the standing waves on the string?

The solution has to satisfy the wave equation, it can be written as $y(x,t) = (A\cos(kx) + B\sin(kx))\sin(\omega t + \phi)$, *where A, B and ϕ are arbitrary constants and k and ω are related by $k = \omega/v$, where* $v = \sqrt{\frac{\tau}{\lambda}} = \sqrt{\frac{Mgl}{m}}$

b) What are the boundary condition on $y(x,t)$ at $x = 0$ and $x = l$?

$y(0,t) = 0,\ \ at\ \ x = l:\ \ \tau\frac{\partial y}{\partial x} = -M\frac{\partial^2 y}{\partial t^2}$ *which comes from the condition that the tension of the wire times its slope provides the acceleration of the mass.*

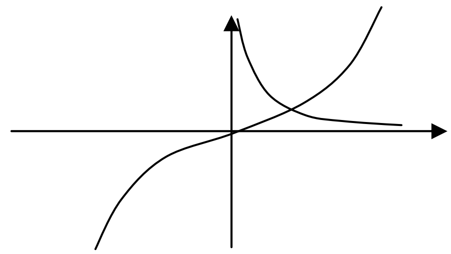
c) What is the equation for the frequencies of the normal modes on the string? Sketch both sides of the equation and indicate graphically the solutions.

The boundary condition at x = 0 makes A = 0 in the general solution. The phase ϕ can be also set to zero by the shift of the origin of time. To find k we use the boundary condition at x = l.

$$\tau Bk\cos(kl)\sin(\omega t) = MB\omega^2\sin(kl)\sin(\omega t)$$

$$\cos(kl) = kl\frac{M}{m}\sin(kl)$$

$$\frac{m}{xM} = \tan(x),\quad where\quad x = kl$$



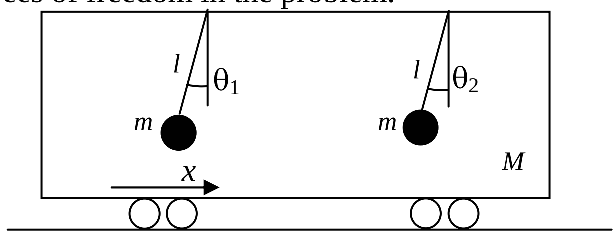
d) Show that for $M>>m$ the frequency of the slowest normal mode is equal to the frequency of the pendulum with a massless string of length l.

For m/M<<1 the intersection of the two curves will happen in a region where tan(x)~x,

$$\frac{m}{M} = x^2,\quad kl = \sqrt{\frac{m}{M}} = \omega l\sqrt{\frac{m}{Mgl}},\quad \omega = \sqrt{\frac{g}{l}}$$

3. **Something inside** (20 pt) Two heavy pendula of mass *m* are suspended from strings of length *l* inside a railroad car of mass *M* that can roll freely on horizontal tracks (ignore the moment of inertia of the wheels).

a) Write down the Lagrangian for the three degrees of freedom in the problem.



$$x_1 = x + l\sin\theta_1 \approx x + l\theta_1$$
$$x_2 = x + l\sin\theta_2 \approx x + l\theta_2$$
$$T = \frac{M}{2}\dot{x}^2 + \frac{m}{2}(\dot{x} + l\dot{\theta}_1)^2 + \frac{m}{2}(\dot{x} + l\dot{\theta}_2)^2$$
$$U = mgl(1-\cos\theta_1) + mgl(1-\cos\theta_1) = \frac{mgl}{2}\theta_1^2 - \frac{mgl}{2}\theta_2^2 + const$$
$$L = T - U = \frac{M}{2}\dot{x}^2 + m\dot{x}^2 + ml\dot{x}\dot{\theta}_1 + ml\dot{x}\dot{\theta}_2 + \frac{ml^2}{2}\dot{\theta}_1^2 + \frac{ml^2}{2}\dot{\theta}_2^2 - \frac{mgl}{2}\theta_1^2 - \frac{mgl}{2}\theta_1^2$$

b) Use momentum conservation to eliminate one of the degrees of freedom.

$\frac{\partial L}{\partial x} = 0,\quad p_x = \frac{\partial L}{\partial \dot{x}} = (M+2m)\dot{x} + ml(\dot{\theta}_1 + \dot{\theta}_2) = const$. Set this constant to zero in CM frame

$$\dot{x} = -\frac{ml}{M+2m}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$L = \frac{m^2l^2}{2(M+2m)}(\dot{\theta}_1 + \dot{\theta}_2)^2 - \frac{m^2l^2}{(M+2m)}(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{ml^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{mgl}{2}(\theta_1^2 + \theta_2^2)$$
$$L = \frac{ml^2}{2}\frac{M+m}{M+2m}(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{m^2l^2}{(M+2m)}\dot{\theta}_1\dot{\theta}_2 - \frac{mgl}{2}(\theta_1^2 + \theta_2^2)$$

c) Find the frequencies of the normal modes.

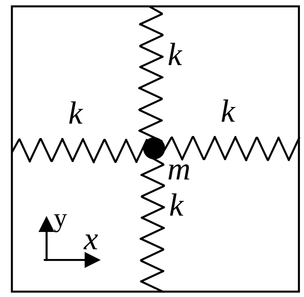
$$A = mgl\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};\qquad m = ml^2\begin{pmatrix}\frac{M+m}{M+2m} & -\frac{m}{M+2m}\\ -\frac{m}{M+2m} & \frac{M+m}{M+2m}\end{pmatrix} = ml^2\begin{pmatrix}\alpha & -\beta\\ -\beta & \alpha\end{pmatrix},\qquad \alpha = \frac{M+m}{M+2m},\ \beta = \frac{m}{M+2m}$$

$$|A - m\omega^2| = ml^2\begin{vmatrix}\omega_0^2 - \alpha\omega^2 & \beta\omega^2\\ \beta\omega^2 & \omega_0^2 - \alpha\omega^2\end{vmatrix} = 0,\quad \omega_0^2 = \frac{g}{l}$$
$$\left(\omega_0^2 - \alpha\omega^2\right)^2 - \beta^2\omega^4 = 0$$
$$\omega_0^4 - 2\alpha\omega_0^2\omega^2 + \alpha^2\omega^4 - \beta^2\omega^4 = 0$$
$$\left(\alpha^2 - \beta^2\right)\omega^4 - 2\alpha\omega_0^2\omega^2 + \omega_0^4 = 0$$
$$\omega^2 = \frac{2\alpha\omega_0^2 \pm \sqrt{4\alpha^2\omega_0^4 - 4\left(\alpha^2 - \beta^2\right)\omega_0^4}}{2\left(\alpha^2 - \beta^2\right)} = \frac{\alpha\omega_0^2 \pm \beta\omega_0^2}{\left(\alpha^2 - \beta^2\right)};\quad \omega_1 = \frac{\omega_0}{\sqrt{\alpha - \beta}},\ \omega_2 = \frac{\omega_0}{\sqrt{\alpha + \beta}}$$
$$\omega_1 = \sqrt{\frac{g}{l}\frac{M+2m}{M}},\quad \omega_2 = \sqrt{\frac{g}{l}}$$

One can show that ω_1 corresponds to in-phase oscillations of the masses and ω_2 to out of phase oscillations.

4. **Toy testing at Santa workshop (**20 pt) Mass *m* is attached to a rectangular frame by 4 springs as shown in the figure and is constrained to move in the horizontal plane. The springs are not stretched in their equilibrium position. As everyone knows, the Santa workshop is located at the North pole and the Earth is rotating around its axis.

   a) Write down the system of differential equations for small oscillations of the mass in the x-y plane.



The forces come from springs and the Coriolis force. It can be shown that for small oscillations the forces from tilting of the springs are higher order in displacement than the forces from compression of the springs.

$$m\ddot{x} = -2kx + 2m\omega\dot{y}$$
$$m\ddot{y} = -2ky - 2m\omega\dot{x}$$

   b) Introduce a variable $s = x + iy$ and combine the two equations. Write down the general solution for the resulting differential equation.

$$m\ddot{s} = -2ks - 2m\omega i\dot{s}$$
$$\ddot{s} + 2i\omega\dot{s} + 2\omega_0^2 s = 0; \quad \omega_0^2 = k/m$$
$$s = Ae^r; \quad r^2 + 2i\omega r + 2\omega_0^2 = 0$$
$$r = \frac{-2i\omega \pm \sqrt{-4\omega^2 - 8\omega_0^2}}{2} = -i\omega \pm i\sqrt{2\omega_0^2 + \omega^2}$$
$$s = A\exp(-i\omega + i\sqrt{2\omega_0^2 + \omega^2}) + B\exp(-i\omega - i\sqrt{2\omega_0^2 + \omega^2})$$

   c) Can you guess what will be the motion of the mass?

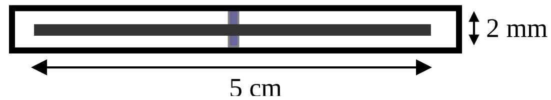
The plane of oscillation of the pendulum will precess with frequency ω similar to the precession of Foucault pendulum.

5. **Oops!** (20 pt) Your laptop slid off your deck and after making one complete flip hit a hard floor. If you are a physics major, your first thought upon discovering that the hard drive no longer works might be "Did it die before or after hitting the floor?"

   a) If the desk is 1 meter high and the dent left on the laptop is 3 mm deep, estimate the kinematic parameters. How long did it take the laptop to hit the floor? What was its velocity just before the impact? What was its deceleration (assumed uniform) during impact with the floor? You can ignore the rotation of the laptop for this part.

$$h = gt^2/2; \quad t_f = \sqrt{\frac{2h}{g}} = 0.45\sec; \qquad v_0 = gt_f = 4.4\, m/\sec$$
$$v_0 - at_{im} = 0; \qquad d = v_0 t_{im} - at_{im}^2/2 = \frac{v_0^2}{a} - \frac{v_0^2}{2a} = \frac{v_0^2}{2a}; \quad a = \frac{v_0^2}{2d} = 3226\, m/\sec^2$$

b) Estimate the forces acting on the bearings during the tumble. The hard drive platter has a mass of 50 g, diameter of 5 cm, and is spinning at 7500 rpm (revolutions per minute). The distance between suspension points is 2 mm. Assume that while falling the laptop made one revolution around an axis parallel to its plane.



Since the rotation angular velocity is much greater than the tumble angular velocity one can assume that the total angular momentum is parallel to the axis of the disk. The equation for the torque is applied in the inertial rest frame.

$$\tau = \frac{dL}{dt} = \Omega L = \Omega I \omega = \frac{mr^2}{2}\Omega\omega$$

$$\Omega = \frac{2\pi}{t_f} = 14\ \sec^{-1}$$

$$\omega = 2\pi 7500/60 = 785 \sec^{-1}$$

$$\tau = 2F(d/2)$$

$$F = \frac{mr^2\Omega\omega}{2d} = 85\,N$$

c) Which force in the bearings is greater, during the deceleration or during the tumble?

Linear deceleration gives a force on each bearing of $3226 * 0.05/2 = 80\,N$ *, so the forces are comparable in size.*