# PHY 203: Solutions to Problem Set 4

October 24, 2006

## 1 Buoy Dropped from an Airplane

From energy conservation the buoy hits the surface of the water with a velocity $v_0 = \sqrt{2gh}$. The equation of motion that determines its subsequent behaviour is

$$F = m\dot{v} = mg - \rho V g - kv. \tag{1}$$

Writing $\alpha \equiv \rho V g - mg$ we can integrate this equation and find

$$\int dt = -\int \frac{mdv}{\alpha + kv} \qquad \Rightarrow \qquad v = Ce^{-kt/m} - \frac{\alpha}{k}, \tag{2}$$

where the integration constant is fixed by the initial velocity: $C - \alpha/k = v_0$. The velocity of the buoy is zero at time $t_0$ given by

$$t_0 = \frac{m}{k} \ln\left(\frac{kv_0 + \alpha}{\alpha}\right). \tag{3}$$

Integrating the velocity again and fixing the integration constant such that $y(0) = 0$, we find the depth as a function of time:

$$y(t) = -\frac{m}{k}Ce^{-kt/m} - \frac{\alpha}{k}t + \frac{m}{k}C. \tag{4}$$

Thus the maximum depth is

$$y(t_0) = \frac{mv_0}{k} - \frac{\alpha m}{k^2} \ln\left(\frac{kv_0 + \alpha}{\alpha}\right) \simeq 127\text{m}. \tag{5}$$

## 2 Disk Rolling down Inclined Plane

The Lagrangian for this problem is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + mgx\sin\alpha - mgy\cos\alpha, \tag{6}$$

with the following constraints that capture the normal and frictional forces at the point of contact:

$$g_1 = \lambda_1 y = 0, \qquad g_2 = \lambda_2(x - R\theta) = 0. \tag{7}$$

Note that we have written down the kinetic and potential energy for the $y$-direction even though we know these terms are zero. Nevertheless, we can obtain the normal force in this way. The three equations of motion from varying the Lagrangian with respect to $x, y$ and $\theta$, respectively, are:

$$\begin{aligned} mg\sin\alpha - m\ddot{x} + \lambda_2 &= 0 & (8) \\ -mg\cos\alpha - m\ddot{y} + \lambda_1 &= 0 & (9) \\ -I\ddot{\theta} - \lambda_2 R &= 0. & (10) \end{aligned}$$

Combining the first and third equation, using the second constraint and $I = mR^2/2$ we find

$$\ddot{x} = \frac{2}{3}g\sin\alpha, \qquad \lambda_2 = -\frac{1}{3}mg\sin\alpha. \tag{11}$$

Similarly from the second equation and the first constraint we have

$$\ddot{y} = 0, \qquad \lambda_1 = mg\cos\alpha. \tag{12}$$

Thus the critical value of $\mu$ at which the disk just starts slipping is given by $|\lambda_2| = \mu\lambda_1$, or

$$\mu = \frac{1}{3}\tan\alpha. \tag{13}$$

# 3 Problem 7.28

This is just a simple application of Hamiltonian dynamics. The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}, \tag{14}$$

in polar coordinates. The generalized momenta are given by

$$p_r = m\dot{r}, \qquad p_\theta = mr^2\dot{\theta}, \tag{15}$$

and the Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}. \tag{16}$$

Therfore Hamilton's equation of motion are

$$\begin{aligned} \dot{r} &= \frac{p_r}{2m}, & \dot{p_r} &= \frac{p_\theta^2}{mr^3} - \frac{k}{r^2}, & (17) \\ \dot{\theta} &= \frac{p_\theta}{mr^2}, & \dot{p_\theta} &= 0. & (18) \end{aligned}$$

# 4 Problem 5.16

The gravitational potential generated by the sphere (outside its volume) is that of a point mass (make sure you can show this!)

$$\Phi(d) = -\frac{GM}{d}. \tag{19}$$

By symmetry the force on the plane must be vertical and considering a small ring of mass $\rho r d\phi dr$ subtending an angle $\theta$ with the vertical we have

$$dF_z = \frac{GM\rho\cos\theta r d\phi dr}{d^2}. \tag{20}$$

Using $\cos\theta = h/d$ and $d^2 = h^2 + r^2$ it is easy to integrate this expression over all $r$, which leads to

$$F_z = 2\pi GM\rho. \tag{21}$$

# 5 The Falling Rod

The coordinates of the center of mass $(x, y)$ are related to the angle $\theta$ of the rod with the vertical by

$$x = \frac{l}{2}\sin\theta, \qquad y = \frac{l}{2}\cos\theta. \tag{22}$$

The frictional and normal forces are easily found by considering the forces on the center of mass:

$$F_f = m\ddot{x} = \frac{ml}{2}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right), \tag{23}$$

$$F_n = mg + m\ddot{y} = mg - \frac{ml}{2}\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right). \tag{24}$$

Either by writing down a simple pendulum type Lagrangian or directly from energy conservation we find the equation of motion

$$\ddot{\theta} = \frac{3g}{2l}\sin\theta \quad \Leftrightarrow \quad \dot{\theta}^2 = \frac{3g}{l}(1 - \cos\theta). \tag{25}$$

The condition for the onset of slipping is $|F_f| = \mu F_n$ and substituting in from above we find

$$\mu(1 - 3\cos\theta)^2 - |9\sin\theta\cos\theta - 6\sin\theta| = 0. \tag{26}$$

For small $\mu$ this equation has three solutions in the range $0 \leq \theta < \pi/2$. This can easily be understood by considering how the rod would fall if its pivot were clamped down. If it fell to the left, its center of mass would have to accelerate to the left first, but then accelerate to the right as $\theta$ approaches $\pi/2$. Thus the pivot would want to slip right for small angles, but to the left for larger angles.

We are interested in when it first slips of course (since our equations of motion are no longer valid after that), so we care for the solution at the smallest possible angle. For $\mu$ smaller than some critical $\mu_c \sim 0.37$ there are three solutions and the relevant one corresponds to slipping to the right. For $\mu$ larger than $\mu_c$ there is only one solution which corresponds to slipping to the left. The rod always slips before the normal force vanishes at an angle of $\theta = \cos^{-1}(1/3)$.

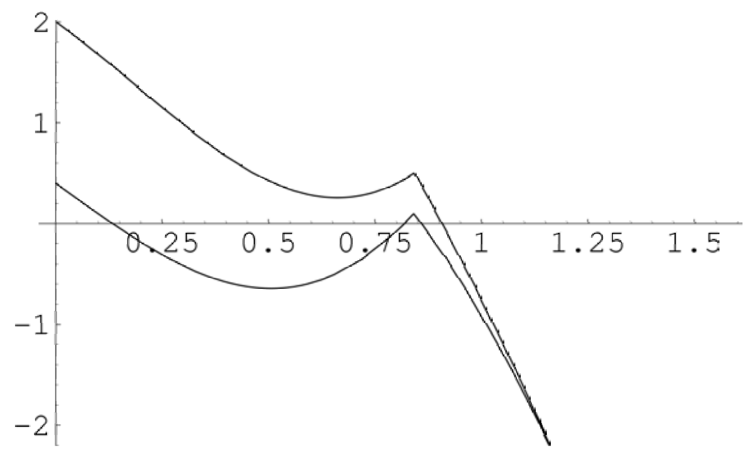


Figure 1: Plot of the left hand side of (26) vs. $\theta$ for $\mu = 0.1$ and $\mu = 0.5$.

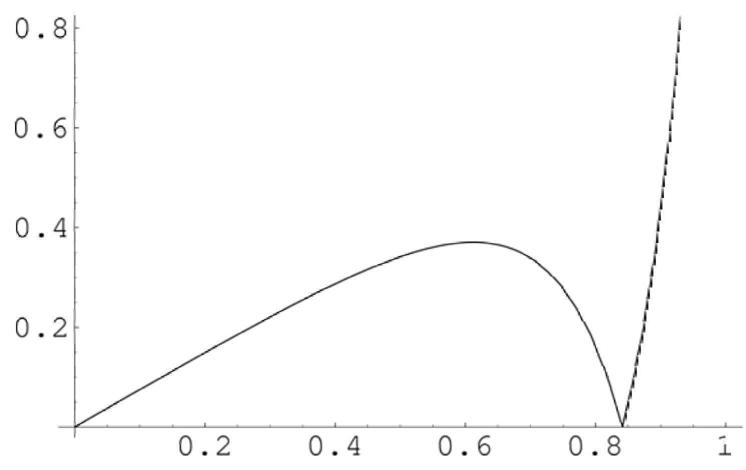


Figure 2: Plot of $\mu$ versus the critical angle $\theta$ that solves (26).