PHY 203: Solutions to Problem Set 8

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1 Problem 3.35

We would like to solve for the motion of a damped harmonic oscillator that is driven sinusoidally with angular frequency ω and acceleration a for a finite time interval $0 < t < \pi/\omega$. For $t < 0$ the oscillator is at rest: $x(t) = 0$. This trivially solves the homogeneous differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0. \tag{1}$$

The general solution of this equation is the complementary function

$$x_c(t) = e^{-\beta t}\left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right], \tag{2}$$

where $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ and the two integration constants A and B are to be determined. For $0 < t < \pi/\omega$ we have to solve the inhomogeneous equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = a\sin\omega t, \tag{3}$$

whose the solutions are the sum of the complementary function $x_c(t)$ given by (2) and the particular integral

$$x_p(t) = \frac{a}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}\sin(\omega t - \delta), \tag{4}$$

where

$$\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right). \tag{5}$$

This can easily be derived by substituting the ansatz $x_p(t) \propto \sin(\omega t - \delta)$ into (3). Matching boundary conditions at $t = 0$ we find that the equations $x_c(0) + x_p(0) = 0$ and $\dot{x}_c(0) + \dot{x}_p(0) = 0$ are satisfied if

$$A = \frac{2a\omega\beta}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}, \tag{6}$$

$$B = -\frac{a\omega(\omega_0^2 - \omega^2 - 2\beta^2)}{\omega_1[(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2]}. \tag{7}$$

This specifies the solution for as long as the system is driven. For $t > \pi/\omega$ the system is undriven and obeys (1), thus

$$x(t) = x_c(t) = e^{-\beta t} \left[C \cos(\omega_1 t) + D \sin(\omega_1 t)\right]. \tag{8}$$

Again matching x and $\dot{x}$ and $t = \pi/\omega$ we find after a little bit of algebra

$$C = A + e^{\beta\pi/\omega} \left[A \cos\left(\frac{\omega_1}{\omega}\pi\right) - B \sin\left(\frac{\omega_1}{\omega}\pi\right)\right], \tag{9}$$

$$D = B + e^{\beta\pi/\omega} \left[A \sin\left(\frac{\omega_1}{\omega}\pi\right) + B \cos\left(\frac{\omega_1}{\omega}\pi\right)\right]. \tag{10}$$

With these constants equation (8) describes the behaviour for all times $t > \pi/\omega$.

An equivalent, if more elegant solution would be to use the method of Green's functions. This would save you the trouble of matching boundary conditions, at the cost of having to do an integral. Here either method works fine, since it was easy to find the particular integral in our case, but in general the machinery of Green's functions is much more powerful.

2 Problem 3.40

This is a simple application of the forced undamped harmonic oscillator. The car drives at speed $v = 20$ km/h over a road with sinusoidal profile, so that its springs are compressed by an amount $y(t) = A\sin(\omega t)$, where $A = 0.05$ m and $\omega = 2\pi v/\lambda = 174.5\,\mathrm{s}^{-1}$. You are told that the car simply oscillates vertically, which is consistent with the separation between front and rear axle being an integer multiple of the wavelength of the bumps λ.

We treat the car as a mass m attached to a spring characterized by k, which leads to the resonance frequency $\omega_0^2 = k/m = 98.1\,\mathrm{s}^{-2}$. The system satisfies the equation

$$\ddot{x} + \omega_0^2 x = A\omega_0^2 \sin(\omega t). \tag{11}$$

We can then directly read off the amplitude of oscillations from the particular solution (4) we found in the previous problem:

$$\max[x_p(t)] = \frac{A\omega_0^2}{|\omega_0^2 - \omega^2|} = 1.6\ 10^{-4}\,\mathrm{m}. \tag{12}$$

3 Beam with Springs Attached

Let θ be the angle of the beam with respect to the horizontal and z_1, z_2 the extensions of the springs from their equilibrium lengths.

In the small angle approximation the Lagrangian for the system is given by

$$L = \frac{m}{2}[2\dot{\theta}^2 l^2 + 2\dot{\theta} l(\dot{z}_1 - \dot{z}_2) + \dot{z}_1^2 + \dot{z}_2^2] - \frac{k}{2}(z_1^2 + z_2^2). \tag{13}$$

Note that there are no gravitational terms in the potential energy, since gravity just determines the equilibrium extension of the springs, and rotation of the beam does not change the height of the center of mass. This leads to the equations of motions encoded by the matrices

$$\mathbf{m} = m \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad \mathbf{A} = k \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{14}$$

and setting $\det(\omega^2 \mathbf{m} - \mathbf{A}) = 0$ we find the frequencies $\omega_1 = 0$ and $\omega_2 = \sqrt{k/m}$. The corresponding eigenvectors are

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{15}$$

The first mode is a so-called zero-mode. The corresponding frequency vanishes which means there are no oscillations. The beam is in neutral equilibrium for any angle θ, which is apparent from the fact that the potential energy of the system does not depend on θ. This is often referred to as a 'flat direction' of the potential.

The second normal consists of an in-phase oscillation of the two masses, with the beam remaining at rest. Hence its frequency is just that of a simple mass on a spring.

You might ask yourself at this point what happened to the third normal mode, given that the system seems to have three degrees of freedom. It is intuitively clear that it would correspond to the two masses oscillating in antiphase, with the beam oscillating at the same frequency, yet there is no such eigenvector. To find out why go through the steps above again for a beam with non-zero moment of inertia I and then take the limit $I \to 0$. You will find that the frequency of the third normal mode diverges. Hence there is no third eigenmode in this limit. Another way to reach the same conclusion is to note that because the beam is massless we know that the total torque acting on it has to vanish, which allows us to eliminate one degree of freedom from the problem.