PHY 203: Solutions to Problem Set 9

December 12, 2006

1 Problem 10.9

We want to calculate the deflection on a projectile undergoing (almost) parabolic motion due to the inertial forces of the rotating frame of the Earth. In order to solve this problem we are going to consider a series of approximations.

- We consider the motion to take place in a much shorter scale than that of the curvature of the Earth. That means that we are zooming in on a region where the Earth looks flat.
- The way to understand how small inertial forces are is to compare them to the force that dominates the motion: gravity. For centrifugal forces the relevant parameter is $\frac{\omega^2 R_{Earth}}{g} \sim 3.5 \times 10^{-3} \ll 1$, where ω is the angular frequency of Earth's rotation. For Coriolis forces the important parameter is $\frac{\omega V_0}{g} \sim 10^{-2} \ll 1$, where V_0, the characteristic speed involved, was taken to be 5000km/h (which is more than enough for all kinds of missiles and jets). The conclusion is that our results will be expressed with great accuracy to first order in those parameters.
- Finally, since our paths are short, we will use the uncorrected parabolic trajectory when we calculate the forces. Then given these forces we will compute the deviation from the original path.

Having discussed the approximations, we turn to the problem. We pick Cartesian coordinates on the surface of the Earth where we perform the experiment. The z unit vector $\hat{k}$ will point upwards. To first order, it is not important whether we correct this up direction by the centrifugal force or not. So we can think of it as coinciding with the radial vector $\hat{r}$ pointing normal to the surface of the Earth. We let the y unit vector $\hat{\jmath}$ point East. Finally, $\hat{\imath}$ is directed South, in the x-direction (as in T&M Fig. 10-9).

In these coordinates the angular velocity of the Earth and the velocity of the particle on the uncorrected parabolic trajectory are

$$\vec{\omega} = (-\omega \cos\lambda, 0, \omega \sin\lambda), \tag{1}$$

$$\vec{v_r} = (0, v_0 \cos\alpha, v_0 \sin\alpha - gt). \tag{2}$$

The Coriolis force is given by

$$\vec{F}_{cor} = -2m\vec{\omega} \times \vec{v_r}, \tag{3}$$

and from above we have

$$\begin{aligned}
\vec{\omega} \times \vec{v_r} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & v_0\cos\alpha & v_0\sin\alpha - gt \end{vmatrix} \\
&= (-v_0\omega\cos\alpha\sin\lambda, \omega\cos\lambda(v_0\sin\alpha - gt), -v_0\omega\cos\alpha\cos\lambda).
\end{aligned}$$

Therefore

$$\vec{a} = (2v_0\omega\cos\alpha\sin\lambda, -2\omega\cos\lambda(v_0\sin\alpha - gt), -g + 2v_0\omega\cos\alpha\cos\lambda), \tag{4}$$

where the corrections to the y and z components are not important to first order; they basically just deform the parabola, changing the time of impact slightly. The deviation from the original direction of motion towards the South is given by

$$x = v_0 t^2 \omega \cos\alpha \sin\lambda, \tag{5}$$

and the time of flight is

$$t = \frac{2v_0\sin\alpha}{g - 2v_0\omega\cos\alpha\cos\lambda} \approx \frac{2v_0\sin\alpha}{g}. \tag{6}$$

Thus the deflection due to the Coriolis force to leading order is

$$d_{cor} = \frac{4\omega v_0^3}{g^2}\sin^2\alpha\cos\alpha\sin\lambda. \tag{7}$$

The same procedure can be applied to calculate the deflection due to the centrifugal force (in the same direction), which is not negligible unless v_0 is very large:

$$d_{cen} = \frac{2\omega^2 R_{Earth} v_0^2}{g^2}\sin^2\alpha\cos\lambda\sin\lambda. \tag{8}$$

2 Problem 10.19

Once again, the Coriolis force is given by the expression:

$$\vec{F} = -2m\vec{\omega} \times \vec{v}, \tag{9}$$

where m is the mass of the particle, $\vec{\omega}$ the angular velocity of the non-inertial frame and $\vec{v}$ the velocity in that frame. In our case the data is:

$$\begin{aligned}
m &= 1300kg \quad (10) \\
\vec{\omega} &= \frac{2\pi}{24}h^{-1}\hat{z} \quad (11) \\
\vec{v} &= -100\frac{km}{h}\hat{\theta} \quad \text{(North)} \quad (12) \\
\lambda &= 65^\circ N, \quad (13)
\end{aligned}$$

where λ is the latitude. The direction of the force is given by the right hand rule; it points East. Its magnitude is

$$F = 2m\omega v \sin\lambda = 4.78N. \tag{14}$$

3 Frictionless Puck on Turntable

Taking into account Coriolis and centrifugal effects, the inertial force in the rotating frame of the turntable is

$$\vec{F}_{eff} = m\vec{a_r} = -m(\vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v_r}. \tag{15}$$

The coordinate system is defined with respect to the centre of the turntable, with the z-axis pointing out of the page. Thus

$$\vec{\omega} = (0, 0, \omega), \tag{16}$$
$$\vec{r} = (x, y, 0), \tag{17}$$
$$\vec{v_r} = (\dot{x}, \dot{y}, 0), \tag{18}$$

and hence

$$\begin{aligned} \vec{\omega} \times \ (\vec{\omega} \times \vec{r}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & w \\ -y\omega & x\omega & 0 \end{vmatrix} \\ &= (-x\omega^2, -y\omega^2, 0). \end{aligned}$$

Also,

$$\vec{\omega} \times \vec{v_r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ \dot{x} & \dot{y} & 0 \end{vmatrix} \tag{19}$$
$$= (-\dot{y}\omega, \dot{x}\omega, 0). \tag{20}$$

Therefore we find the acceleration

$$\vec{a_r} = -\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \vec{v_r} = (x\omega^2 + 2\dot{y}\omega, y\omega^2 - 2\dot{x}\omega, 0). \tag{21}$$

Splitting this up into components, we see that

$$\begin{aligned} \ddot{x} &= x\omega^2 + 2\dot{y}\omega, \\ \ddot{y} &= y\omega^2 - 2\dot{x}\omega. \end{aligned} \tag{22}$$

Now consider the complex coordinate $q \equiv x + iy$, such that these two real equations can be written as

$$\ddot{q} - \omega^2 q + 2i\omega\dot{q} = 0. \tag{23}$$

We make the ansatz

$$q = Ae^{i\gamma t} \quad \Rightarrow \quad \gamma = \frac{-2\omega \pm \sqrt{4\omega^2 - 4\omega^2}}{2} = -\omega,$$

and since both roots are the same the solution is

$$q = Ae^{-i\omega t} + Bte^{-i\omega t}, \tag{24}$$

where A and B are complex. Writing $A = C + Di$ and $B = E + Fi$, the initial conditions show that $C = x_0, D = 0, E = v_x$ and $F = v_y + x_0\omega$. Thus the final answer is

$$\begin{aligned} x &= (x_0 + v_x t)\cos\omega t + (v_y + x_0\omega)t\sin\omega t, \\ y &= (v_y + x_0\omega)t\cos\omega t - (x_0 + v_x t)\sin\omega t. \end{aligned} \tag{25}$$

4 Compton Generator

We define our coordinate axes as in the first problem such that

$$\vec{\omega} = (-\omega\cos\lambda, 0, \omega\sin\lambda). \tag{26}$$

The position of a point along the water-filled tube is

$$\vec{r} = (R\cos\alpha\cos\theta, R\sin\theta, R\sin\alpha\cos\theta), \tag{27}$$

where θ is the angle along the ring (defined with respect to the x-axis), and α is the angle of rotation. Therefore, since α is changes with time

$$\vec{v_r} = \dot{\vec{r}} = (-R\cos\theta\dot{\alpha}\sin\alpha, 0, R\cos\theta\dot{\alpha}\cos\alpha). \tag{28}$$

Thus we find

$$\begin{aligned} \vec{\omega} \times \vec{v_r} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ -R\cos\theta\dot{\alpha}\sin\alpha & 0 & R\cos\theta\dot{\alpha}\cos\alpha \end{vmatrix} \\ &= -\hat{j}(-R\omega\dot{\alpha}\cos\lambda\cos\theta\cos\alpha + R\omega\dot{\alpha}\sin\lambda\cos\theta\sin\alpha). \end{aligned}$$

Hence the acceleration due to the Coriolis force is

$$\vec{a} = -2R\omega\dot{\alpha}\cos\theta\cos(\lambda + \alpha)\hat{j}. \tag{29}$$

It points in the y-direction, though we really need the tangential acceleration, which we obtain by multiplying by $\cos\theta$. We then average over the ring to find the mean acceleration (which is what matters for an incompressible fluid)

$$\langle a \rangle = \frac{1}{2\pi}\int_0^{2\pi} -2R\omega\dot{\alpha}\cos^2\theta\cos(\lambda + \alpha)d\theta = -R\omega\dot{\alpha}\cos(\lambda + \alpha).$$

Now integrating over half a period of revolution we find the final average tangential velocity. Though it is clear that it does not make a difference, we may assume we rotate the ring at a constant angular velocity $\alpha = \frac{2\pi}{\tau} t$.

$$v_t = \int_0^{\frac{\tau}{2}} -R\omega\dot{\alpha}\cos(\lambda + \alpha)dt = 2R\omega\sin\lambda. \tag{30}$$

Therefore, the velocity of the water in the tube is given by $v_t = 2R\omega\sin\lambda$, where λ is defined from the equator. It is interesting to note that a rotation from the vertical to the vertical position (i.e. from -$\frac{\tau}{2}$ to $\frac{\tau}{2}$) produces a $\cos\lambda$ dependence, while a full rotation by 2π does not produce any motion.