Physics 203—First Midterm, Fall 2003

Problem 1 *(10 points)*

A roller-coaster designer believes in speedy rides. Find the differential equation for the shape of the track on which a cart released from point A (at height h) reaches point B in the minimum possible time. Assume that gravity is the only force in the game, and that the cart starts at A with zero velocity. (This is the classic brachistochrone problem. The solution is a cycloid, but you do not need to solve the equation here.)

Problem 2 *(20 points)*

On April 1, 1977, four members of the Oxford University's Dangerous Sports Club performed the first ever bungee jump from the Clifton suspension bridge. They had just taken the Oxford version of Death Mechanics, and wanted to check their calculation of the tension in the rope as function of the angle θ. As they made sure that the rope could withstand the maximum tension they had calculated, they survived the jump but were promptly arrested.

- Solve for the tension in the rope, assuming for simplicity that the length of the rope stays unchanged during the experiment.

Problem 3 *(25 points)*

To explain the puzzle of the "missing mass" of the universe, physicists hypothesize the existence of "dark matter", a cloud of unknown particles surrounding the center of galaxies. Assume that a galaxy is made (mostly) of dark matter, with uniform density (mass per unit volume) ρ.

- What are the gravitational force and potential experienced by a star of mass M at a distance r from the center of the galaxy? (Assume the dimensions of the galaxy is bigger than r.)
- Assume a non-zero angular momentum l, and show that the system admits a circular orbit. Find its radius and energy.
- Show that the circular orbit is stable, and find the frequency of small oscillations.

Potentially Useful Relations

$$R_E = 6.37 \times 10^6 m \qquad G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2} \qquad M_E = 5.98 \times 10^{24} kg$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \qquad \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = H \qquad L = T - V \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad \dot{q}_k = \frac{\partial H}{\partial p_k}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_l \lambda_l a_{lj} \quad \text{where } a_{lj} = \frac{\partial g_l}{\partial q_j}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \qquad \frac{\partial f}{\partial x} - \frac{d}{dx}(f - y'\frac{\partial f}{\partial y'}) = 0$$

$$\Delta\phi = \int_{r_{\min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{2m}{\ell^2}(E-V) - \frac{1}{r^2}}} \qquad \theta \simeq -\frac{b}{E}\int_b^{\infty} \frac{\partial V}{\partial r}\frac{dr}{\sqrt{r^2 - b^2}}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$r(\theta) = \frac{\alpha}{(1 + \epsilon\cos\theta)} \qquad \alpha \equiv \frac{\ell^2}{mk} \qquad \epsilon \equiv \sqrt{1 + \frac{2E\ell^2}{mk^2}} \qquad \epsilon = \frac{\sqrt{a^2 - b^2}}{a} \qquad a = \frac{\alpha}{1 - \epsilon^2}$$

$$b = \frac{\alpha}{\sqrt{1 - \epsilon^2}} \qquad r_{min} = a(1 - \epsilon) \qquad r_{max} = a(1 + \epsilon) \qquad E = -\frac{k}{2a} \qquad \frac{\tau^2}{a^3} = \frac{4\pi^2\mu}{k} \qquad k = GmM$$

$$E_{\text{total}} = \frac{1}{2}m\dot{r}^2 + U(r) + \frac{\ell^2}{2mr^2} \qquad v = v_o + u\ln(\frac{m_o}{m}) \qquad v = -gt + u\ln(\frac{m_o}{m})$$

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sin 2\theta = 2\sin\theta\cos\theta \qquad \sin^2\theta/2 = \frac{1}{2}(1 - \cos\theta)$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B; \qquad \cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan^2\frac{\theta}{2} = \frac{1 - \cos\theta}{1 + \cos\theta} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}}; \qquad \frac{d}{dx}\cos^{-1}\frac{x}{a} = \frac{-1}{\sqrt{a^2 - x^2}}; \qquad \frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2 + x^2}$$

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d\tan\theta}{d\theta} = \frac{1}{\cos^2\theta} \qquad \frac{d\cot\theta}{d\theta} = \frac{-1}{\sin^2\theta} \qquad \frac{d}{dx}\log(x) = \frac{1}{x}$$

$$\int \frac{xdx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2} \qquad \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} \qquad \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\cos^{-1}\left|\frac{a}{x}\right|$$